Hokkaido Mathematical Journal Vol. 13 (1984) p. 119–122

# The weak Behrens' property and the corona

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(Received March 9, 1983)

**Abstract:** In this paper we study a class of infinitely connected domains larger than the one considered by Behrens [1] and prove that the corona problem has an affirmative answer.

**Introduction.** Let $D$ be a bounded domain in the complex plane and $H^{\infty}(D)$ be the algebra of bounded analytic functions on $D$. The corona problem asks whether $D$ is weak* dense in the space $\mathcal{M}(D)$ of maximal ideals of $H^{\infty}(D)$. Carleson [3] proved that the open unit disc $\Delta$ is dense in $\mathcal{M}(\Delta)$. In [7] Stout extended Carleson's result to finitely connected domains. In [1] Behrens found a class of infinitely connected domains for which the corona problem has an affirmative answer. In this paper we will use Behrens' idea to extend the results to more general domains. See [4] and [5] for other extensions.

By a $\Delta$-domain we mean a domain $D$ obtained from the open unit disc $\Delta$ by deleting the origin and a sequence of disjoint closed discs $\Delta_{n}=\Delta(c_{n}, r_{n})=\{z\in C: |z-c_{n}|\leq r_{n}\}$ with $c_{n}\rightarrow 0$. Under the additional hypothesis $\sum\frac{r_{n}}{|c_{n}|}<\infty$, Zalcman showed in [8] that there is a distinguished homomorphism in $\mathcal{M}(D)$ defined by

$$\varphi_{0}(f)=\frac{1}{2\pi i}\int_{\partial D}\frac{f(z)}{z}dz.$$

The distinguished homomorphism $\varphi_{0}$, if it exists, is always adherent to $D$ [6]. Behrens showed that if there are numbers $R_{n}>r_{n}$ such that $\sum\frac{r_{n}}{R_{n}}<\infty$ and the discs $D_{n}=\Delta(c_{n}, R_{n})$ are disjoint, then $D$ is dense in $\mathcal{M}(D)$. Such a domain is called a Behrens' domain.

**Notations and terminology:** Throughout we assume that $D$ is a $\Delta$-domain and that there exists numbers $R_{n}>r_{n}$ such that the discs $D_{n}=\Delta(c_{n}, R_{n})$ are disjoint and $\frac{r_{n}}{R_{n}}\rightarrow 0$. Let $E_{n}=\frac{r_{n}}{z-c_{n}}$ for $z\in\Delta_{n}^{c}=C\backslash\Delta_{n}$, $n=1,2,\cdots$. Let $s_{n}=\sqrt{r_{n}R_{n}}$, so $\frac{r_{n}}{s_{n}}\rightarrow 0$ and $\frac{s_{n}}{R_{n}}\rightarrow 0$, and let $B_{n}=\Delta(c_{n}, s_{n})$. Let $H^{\infty}(\Delta\times N)$ be the algebra of bounded functions which are analytic on each slice of $\Delta\times N$

and $\mathcal{M}(\varDelta\times N)$ its maximal ideal space. Let $X=\mathcal{M}(\varDelta\times N)\setminus\bigcup_n \mathcal{M}(\varDelta)\times\{n\}$ see [2], for details. Each $f\in H^\infty(D)$ can be written in $D_n\setminus\varDelta_n$ as $f(z)=(P_nf)(z)+a_n(f)+F_n(z)$ where $(P_nf)(z)=\frac{1}{2\pi i}\int_{\partial\varDelta_n}\frac{f(\zeta)}{\zeta-z}d\zeta$, $a_n(f)=\frac{1}{2\pi i}\int_{\partial\varDelta_n}\frac{f(z)}{z-c_n}dz$ and $F_n(z)$ analytic in $D_n$, see [4] for more details. Let $\mathcal{M}_0$ denote the fiber in $\mathcal{M}(D)$ at the origin and $A_0=H^\infty(D)_{|\mathcal{M}_0}$. If $D'$ is a domain containing $D$, then $H^\infty(D')\subset H^\infty(D)$ and $\mathcal{M}(D)\subset\mathcal{M}(D')$ by restriction. We will say that $D$ has the weak Behrens' property if for every non-distinguished $\varphi\in\mathcal{M}_0$ there exists a domain $D^*=\varDelta\setminus(\bigcup\varDelta_{n_j}\cup\{0\})$ such that $\{\varDelta_{n_j}\}$ is a subsequence of $\varDelta_n$, $\sum\frac{r_{n_j}}{R_{n_j}}<\infty$ and $\varphi_{|H^\infty(D^*)}$ is not distinguished in $\mathcal{M}(D^*)$. We write $\varphi^*$ for $\varphi_{|H^\infty(D^*)}$.

LEMMA 1: *Suppose that $D^*$ is a domain as described above. Then $a_n(f)\to\varphi_0^*(f)$ for all $f\in H^\infty(D^*)$, where $\varphi_0^*$ is the distinguished homomorphism in $\mathcal{M}(D^*)$.*

PROOF: Consider first the subsequence $a_m$, $m\notin\{n_j: j=1,2,\cdots\}$. Since $f\in H^\infty(D^*)$ then $f$ is analytic in $\varDelta_m$ for all $m$, so $a_m(f)=\frac{1}{2\pi i}\int_{\partial\varDelta_m}\frac{f(z)}{z-c_m}dz=f(c_m)$. Now $D^*$ is a Behrens' domain and $\{c_m\}$ is a sequence of points in $D^*\setminus\bigcup D_{n_j}$ so $f(c_m)\to\varphi_0^*(f)$ [1]. It is known that $a_{n_j}(f)\to\varphi_0^*(f)$ [4] and this proves the lemma.

Define $\phi$: $H^\infty(D)\to H^\infty(\varDelta\times N)$ by $\phi(f)(z,n)=(P_nf)\circ E_n^{-1}(z)+a_n(f)$.

LEMMA 2 [4, Proposition 1]: *The map $\phi$ induces a map $\bar{\phi}$: $A_0\to H^\infty(\varDelta\times N)_{|X}$, which is an algebra isometric isomorphism of $A_0=H^\infty{}_{|\mathcal{M}_0}$ with a closed subalgebra $B$ of $H^\infty(\varDelta\times N)|_X$.*

Using lemma 2, the map $\bar{\phi}$ induces a homomorphism $\theta$: $\mathcal{M}(B)\to\mathcal{M}_0$ defined by $\theta(\phi)(f)=\phi(\bar{\phi}(f))$, *for* $\phi\in\mathcal{M}(B)$ and $f\in A_0$.

LEMMA 3. *Suppose that D has the weak Behrens' property. If $\varphi\in\mathcal{M}_0$ and $D^*$ is the corresponding domain, then $\theta^{-1}(\varphi)$ can be extended to a homomorphism on $H^\infty(\varDelta\times N)$.*

PROOF: Fix $\varphi$ and $D^*$ as above. Let

$$L^*(z)=\sum E_{n_j}(z)-E_{n_j}(0)\,,\qquad z\in D^*.$$ Clearly

$$\phi(L^*)(z,n)=\begin{cases}a_n(L^*), & n\neq n_j\\ z+a_{n_j}(L^*), & n=n_j\end{cases}$$

and $\bar{\phi}(L^*)(z,n)=\langle\alpha_n z\rangle_{|X}$ by lemma 1, where $\alpha_n=0$ if $n\neq n_j$ and $\alpha_{n_j}=1$, denote this function by $Z^*$. We will show that $Z^*H^\infty(\varDelta\times N)\subset B$. Let

$F=\langle f_n\rangle\in H^\infty(\Delta\times N)$, then $Z^*F=\langle \alpha_n g_n\rangle$ where $g_n=zf_n$. Clearly $g=\sum\alpha_n g_n\circ E_n^{-1}\in H^\infty(D^*)$ and $\bar{\phi}(g-\varphi_0(g))=Z^*F$. Let $\tilde{\varphi}=\theta^{-1}(\varphi)$, now

$$\tilde{\varphi}(Z^*)=\tilde{\varphi}\left(\bar{\phi}(L^*)\right)=\varphi(L^*)=\varphi^*(L^*)\neq 0$$

because $\varphi_0^*$ is the only homomorphism in $\mathscr{M}(D^*)$ which vanishes on $L^*$ [1]. Now for $F\in H^\infty(\Delta\times N)$ define

$$\tilde{\varphi}(F)=\frac{\tilde{\varphi}(Z^*F)}{\tilde{\varphi}(Z^*)}\,. \qquad \text{Clearly } \tilde{\varphi}\in\mathscr{M}(\Delta\times N)\,.$$

We are now in a position to prove the main result of this paper. The proof is essentially the same as the one given in [1].

THEOREM: *If $D$ has the weak Behrens' property then $D$ is dense in $\mathscr{M}(D)$.*

PROOF: Let $\varphi\in\mathscr{M}(D)$, if $\varphi\notin\mathscr{M}_0$ then $\varphi\in\bar{D}$ [3], also if $\varphi=\varphi_0$ then $\varphi\in\bar{D}$ [6]. So suppose $\varphi\in\mathscr{M}_0$, $\varphi\neq\varphi_0$. Let $D^*$ be the corresponding domain to $\varphi$. Let $\tilde{\varphi}=\theta^{-1}(\varphi)$ then $\tilde{\varphi}$ can be extended to $H^\infty(\Delta\times N)$ by Lemma 3. For each $p\in N$ define $I_P\in H^\infty(\Delta\times N)$ by $I_p(\lambda, n)=1$ if $n\geq p$ and $I_p(\lambda, n)=0$ if $n<p$. Clearly $\tilde{\varphi}(I_p)=1$ and $\tilde{\varphi}(Z^*)\neq 0$ by Lemma 3. Let $f_1, f_2, \cdots, f_k\in H^\infty(D)$, $\varepsilon>0$ be such that $|\tilde{\varphi}(Z^*)|>2\varepsilon$. Choose $p\in N$ such that $\frac{r_n}{s_n}<\varepsilon$ if $p\leq n$. Since $\Delta\times N$ is dense in $\mathscr{M}(\Delta\times N)$ [2] then there exists $(\lambda, n)\in\Delta\times N$ such that

$$\left|\phi(f_i)(\lambda, n)-\tilde{\varphi}\left(\phi(f_i)\right)\right|<\varepsilon \qquad 1\leq i\leq k\,,$$

$$\left|Z(\lambda, n)-\tilde{\varphi}(Z)\right|<\varepsilon \qquad \text{and}$$

$$\left|I_p(\lambda, n)-\tilde{\varphi}(I_p)\right|<\varepsilon$$

From (1) we get $|\lambda|>\frac{|\tilde{\varphi}(Z)|}{2}>\varepsilon$ because $\tilde{\varphi}(Z)=\tilde{\varphi}(Z^*)$, and from 2 we get $I_p(\lambda, n)=0$ so $p<n$. Clearly $E_n^{-1}(\lambda)\in B_n\backslash\Delta_n$. Choosing $p$ large enough as in Lemma 1 of [4] we get

$$\left|f_i\left(E_n^{-1}(\lambda)\right)-\varphi(f_i)\right|\leq\left|f_i\left(E_n^{-1}(\lambda)\right)-(P_nf_i)\,E_n^{-1}(\lambda)\right.$$
$$\left.-a_n(f_i)\right|+\left|\phi(f_i)(\lambda, n)-\tilde{\varphi}\left(\phi(f_i)\right)\right|<2\varepsilon$$

for $1\leq i\leq k$, hence $\varphi\in\bar{D}$ which completes the proof.

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