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# Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation

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## § 1. Introduction

In this paper we continue the study of linear symmetric systems of the form

$$(1.1)\qquad A^{0}w_{t}+\sum_{j=1}^{n}A^{j}w_{x_{j}}-\sum_{j,k=1}^{n}B^{jk}w_{x_{j}x_{k}}+Lw=0,$$

where $t\geq 0$, $x=(x_{1}, \cdots, x_{n})\in \boldsymbol{R}^{n}$ and $w$ is a function of the variables $t$ and $x$, valued in $\boldsymbol{R}^{m}$. $A^{0}$, $A^{j}(j=1, \cdots, n)$, $B^{jk}=B^{kj}(j, k=1, \cdots, n)$ and $L$ are $m\times m$ constant matrices. For notational convenience, we set

$$(1.2)\qquad \begin{aligned} A(\omega)&=\sum_{j=1}^{n}A^{j}\omega_{j},\\ B(\omega)&=\sum_{j,k=1}^{n}B^{jk}\omega_{j}\omega_{k}, \end{aligned}$$

where $\omega=(\omega_{1}, \cdots, \omega_{n})$ is a unit vector in $\boldsymbol{R}^{n}$. The first assumptions on the coefficient matrices can be stated as follows.

CONDITION 1.1. (i) $A^{j}(j=1, \cdots, n)$ and $B^{jk}(j, k=1, \cdots, n)$ are real symmetric matrices and for each $\omega\in S^{n-1}$, $B(\omega)$ is nonnegative definite.

(ii) $A^{0}$ and $L$ are real symmetric matrices. Furthermore, $A^{0}$ is positive definite and $L$ is nonnegative definite.

The above condition gives a stable nature to the system (1.1) but it is not strong enough to guarantee the decay of solutions. We look for nontrivial solutions of the linear homogeneous equation

$$(1.3)\qquad \lambda A^{0}\phi+\{L+\zeta A(\omega)-\zeta^{2}B(\omega)\}\phi=0,$$

for $\zeta\in i\boldsymbol{R}$ and $\omega\in S^{n-1}$. The admissible values of $\lambda$ are the zeros of $\det(\lambda A^{0}+L+\zeta A(\omega)-\zeta^{2}B(\omega))$. We write $\lambda=\lambda(\zeta, \omega)$ and define what we call the strict dissipativity.

DEFINITION 1.1. The system (1.1) is said to be strictly dissipative, if the real part of $\lambda(\zeta, \omega)$ is negative for each $\zeta\in i\boldsymbol{R}\backslash\{0\}$ and $\omega\in S^{n-1}$.

The main purpose of the present paper is to prove that the strict dissipativity brings about the decay of solutions. The result seems to be new, because the rotational invariance is not assumed to hold for (1.1). We note that, in the previous works ([8], [6]), the decay estimates were obtained under CONDITION 1.1 and an additional condition which is as follows.

CONDITION 1.2. There exists a set of $m\times m$ real matrices $K^j(j=1, \cdots, n)$ satisfying the following properties:

(i) $K^jA^0(j=1, \cdots, n)$ is a real skew-symmetric matrix.

(ii) Let $[K^jA^k]'$ be the symmetric part of $K^jA^k$, namely, $[K^jA^k]'=\{K^jA^k+{}^t(K^jA^k)\}/2$. Then, for each $\omega\in S^{n-1}$,

$$\sum_{j,k=1}^{n}([K^jA^k]'+B^{jk})\omega_j\omega_k+L$$

is a positive definite matrix.

The meaning of the above condition does not seem to be intuitively clear, because it comes from computational necessities. We remined the reader that CONDITION 1.2, together with CONDITION 1.1, enabled us to use an energy method and establish the desired decay estimates. The existence of such matrices was shown each time for concrete problems ([8], [6]). It seems quite plausible that the matrices $K^j(j=1, \cdots, n)$ exist for any system (1.1) whose solutions obey the decay estimates. Unfortunately we are unable to prove this conjecture. So we are led to introduce the notion of what we call the compensating function. Let $K(\omega)$ be a $m\times m$ real matrix for each $\omega\in S^{n-1}$.

DEFINITION 1.2. $K(\omega)$ is called a compensating function for the system (1.1), if the following properties are satisfied:

(i) $K(\omega)$ is a $C^\infty$-function on $S^{n-1}$ and $K(-\omega)=-K(\omega)$ for each $\omega\in S^{n-1}$.

(ii) $K(\omega)A^0$ is a skew-symmetric matrix for each $\omega\in S^{n-1}$.

(iii) $[K(\omega)A(\omega)]'+B(\omega)+L$ is a positive definite matrix for each $\omega\in S^{n-1}$.

It is easily seen that we can get a compensating function by putting $K(\omega)=\sum_{j=1}^{n}K^j\omega_j$, if CONDITION 1.2 is assumed to hold. Hence the existence of a compensating function follows from CONDITION 1.2. The computations needed for obtaining the decay estimates in [8], [6] remain valid provided that a compensating function exists for (1.1). We note that a compensating function can be regarded as a symbol of a singular integral operator of homogeneous degree zero. See, for example, [7]. Observe also that $\sum_{j=1}^{n}K^j\xi_j$ corresponds to a certain differential operator. Now our aim is to prove the equivalence of the existence of a compensating function and the strict dissipativity of the system.

THEOREM 1.1. *Assume* CONDITION 1.1. *Let $A(\omega)$ and $B(\omega)$ be as in* (1.2) *and let $\lambda(\zeta, \omega)$ be the value of $\lambda$ corresponding to a nontrivial* *solution of* (1.3). *Then each of the following four conditions implies the other three*:

(1) *There exists a compensating function* $K(\omega)$ *for* (1.1).

(2) (1.1) *is strictly dissipative.*

(3) *Let* $\psi\in\boldsymbol{R}^m\setminus\{0\}$ *and let* $B(\omega)\psi=L\psi=0$ *for some* $\omega\in S^{n-1}$. *Then we have* $\mu A^0\psi+A(\omega)\psi\neq 0$ *for any* $\mu\in\boldsymbol{R}$.

(4) *Let* $\rho(r)=r^2/(1+r^2)$ *for* $r\geq 0$. *Then, there exists a positive constant* $\delta$ *such that for any* $\zeta\in i\boldsymbol{R}$ *and* $\omega\in S^{n-1}$, *we have* $\mathrm{Re}\,\lambda(\zeta,\omega)\leq-\delta\rho(|\zeta|)$.

REMARK 1.1. CONDITION (3) will turn out to be a quite useful criterion for some concrete problems. The decay estimates are established on the basis of CONDITION (4).

REMARK 1.2. The explicit forms (1.2) for $A(\omega)$ and $B(\omega)$ are not employed in the proof. The properties we use are:

(i) $A(\omega)$ and $B(\omega)$ are $m\times m$ matrices for each $\omega\in S^{n-1}$. $A(\omega)$ and $B(\omega)$ are $C^\infty$ functions on $S^{n-1}$ and we have $A(-\omega)=-A(\omega)$, $B(-\omega)=B(\omega)$.

(ii) For each $\omega\in S^{n-1}$, $A(\omega)$ and $B(\omega)$ are real symmetric matrices. Moreover, $B(\omega)$ is nonnegative definite.

(iii) $A^0$ and $L$ are real symmetric matrices. Furthermore, $A^0$ is positive definite and $L$ is nonnegative definite.

These conditions (i), (ii) and (iii) can replace CONDITION 1.1 in THEOREM 1.1.

THEOREM 2.1 in [8] and THEOREM 3.A.2 in [6] can be improved by THEOREM 1.1 as follows.

THEOREM 1.2. *Suppose* CONDITION 1.1 *and one of the four conditions stated in* THEOREM 1.1. *Write* $w(0,x)=w_0(x)$ *for the initial condition to* (1.1) *and let* $w_0\in H^s(\boldsymbol{R}^n)\cap L^p(\boldsymbol{R}^n)$, *where s is nonnegative integer and* $p\in[1,2]$. *Then, there exist positive constants* $\delta'$ *and* $C$ *such that the solution* $w=w(t,x)$ *of the initial value problem for* (1.1) *satisfies*

$$(1.4)\qquad \|D_x^l w(t)\|\leq C\{e^{-\delta' t}\|D_x^l w_0\|+(1+t)^{-(\gamma+l/2)}\|w_0\|_{L^p}\}$$

*for any integer* $l$ *with* $0\leq l\leq s$, *where* $D_x^l=\{(\partial/\partial x)^\alpha;\ |\alpha|=l\}$, $\gamma=n(1/2p-1/4)$ *and* $\|\cdot\|$ *denotes the* $L^2(\boldsymbol{R}^n)$*-norm.*

We give in § 2 some preliminary lemmas in linear algebra which will be needed in the next section. THEOREM 1.1 is proved in § 3. In § 4, we discuss the global solutions to the initial value problem of a certain quasilinear symmetric system of hyperbolic-parabolic type. The results in [6] are reformulated by using THEOREM 1.1. The equations for the three dimensional viscous compressible fluid are mentioned as an example. In § 5, discrete velocity models for the Boltzmann equation are studied and the results of

[5], namely, the existence of solutions in the large are obtained again. See THEOREM 5.1. We deal with the two dimensional 8-velocity model in § 6, which was considered in [5]. The three dimensional 14-velocity model by Cabannes is treated in § 7. It is shown that CONDITION (3) in THEOREM 1.1 holds true for these models. Hence THEOREM 5.1 can be applied.

## § 2. Lemmas from linear algebra

In this section, we give several lemmas needed in the proof of THEOREM 1.1. Most of the materials are found in [2], but here we give the proofs from a different viewpoint.

We work in the space $\mathscr{X}$ of $m\times m$ matrices with complex entries. $\mathscr{X}$ is a Hilbert space under the inner product

$$\{X, Y\}=\mathrm{tr}(XY^*), \tag{2.1}$$

where $Y^*$ denotes the conjugate transpose of $Y$ and $\mathrm{tr}(XY^*)$ denotes the trace of $XY^*$. For given $A\in\mathscr{X}$, we define a linear transformation $\Phi_A$ acting in $\mathscr{X}$ by

$$\Phi_A(X)=[A, X]\equiv AX-XA,\quad X\in\mathscr{X}. \tag{2.2}$$

LEMMA 2.1. *Let $A\in\mathscr{X}$ be a normal (resp. hermitian) matrix. Suppose that $A$ has the complete set of linearly independent eigenvectors $\psi_j\in\boldsymbol{C}^m(j=1,\cdots,m)$ corresponding to eigenvalues $\mu_j\in\boldsymbol{C}$ (resp. $\mu_j\in\boldsymbol{R}$)$(j=1,\cdots,m)$, respectively (not necessarily distinct). Suppose in addition that $(\psi_j, \psi_k)=\delta_{jk}$, where $(\ ,\ )$ denotes the Euclidean innerproduct in $\boldsymbol{C}^m$. Then we have:*

*(i) $\Phi_A$ is a normal (resp. hermitian) transformation in $\mathscr{X}$.*

*(ii) The eigenvalues of $\Phi_A$ are the $m^2$ numbers $\mu_j-\mu_k$ $(j, k=1,\cdots,m)$. The corresponding eigenvectors are $X_{jk}\equiv\psi_j\psi_k^*$ $(j, k=1,\cdots,m)$, respectively.*

PROOF. We have

$$\begin{aligned}\{\Phi_A(X), Y\}&=\mathrm{tr}((AX-XA)Y^*)=\mathrm{tr}(X(Y^*A-AY^*))\\&=\mathrm{tr}(X(A^*Y-YA^*)^*)=\{X, \Phi_{A^*}(Y)\},\end{aligned}$$

for $X, Y\in\mathscr{X}$. Let $A$ be a hermitian matrix. Then the above equality gives $\{\Phi_A(X), Y\}=\{X, \Phi_A(Y)\}$, which implies that $\Phi_A$ is a hermitian transformation in $\mathscr{X}$. If $A$ is a normal matrix, we have

$$\begin{aligned}\Phi_{A^*}(\Phi_A(X))&=[A^*, \Phi_A(X)]=A^*(AX-XA)-(AX-XA)A^*\\&=A(A^*X-XA^*)-(A^*X-XA^*)A\\&=[A, \Phi_{A^*}(X)]=\Phi_A(\Phi_{A^*}(X)),\end{aligned}$$

for $X\in\mathscr{X}$. These two equalities show that $\Phi_A$ is a normal transformation in $\mathscr{X}$. This completes the proof of (i). Secondly,

$$\begin{aligned}\Phi_A(X_{jk})&=AX_{jk}-X_{jk}A=A\psi_j\psi_k^*-\psi_j\psi_k^*A\\&=(\mu_j-\mu_k)\psi_j\psi_k^*=(\mu_j-\mu_k)X_{jk},\end{aligned}$$

where $\psi_k^* A = \mu_k \psi_k^*$ was used. This proves (ii).

We set

$$(2.3)\qquad \begin{aligned} &\mathscr{C}(A) = \{X \in \mathscr{X};\ [A, X] = 0\}, \\ &\mathscr{G}(A) = \{[A, X];\ X \in \mathscr{X}\}. \end{aligned}$$

These subspaces of $\mathscr{X}$ are the kernel and the range of $\Phi_A$, respectively. Define $\Pi_A$ to be the orthogonal projection onto $\mathscr{C}(A)$. Then the next lemma gives a recipe for computing $\Pi_A$.

LEMMA 2.2. *(Ellis-Pinsky [2]) Let $A \in \mathscr{X}$ be diagonable, namely, let $A$ be similar to a diagonal matrix. Suppose that $A = \lambda_1 P_1 + \cdots + \lambda_r P_r$ is the spectral resolution of $A$, where $\lambda_1, \cdots, \lambda_r$ are the distinct eigenvalues of $A$ corresponding to the eigenprojections $P_1, \cdots, P_r$, respectively (not necessarily orthogonal). Define a linear transformation $\tilde{\Pi}_A$ in $\mathscr{X}$ by*

$$(2.4)\qquad \tilde{\Pi}_A(X) = \sum_{j=1}^{r} P_j X P_j, \quad X \in \mathscr{X}.$$

*Then we have:*

(i) *$\tilde{\Pi}_A$ is a projection onto $\mathscr{C}(A)$.*

(ii) *If $A$ is a normal matrix, then $\tilde{\Pi}_A$ is the orthogonal projection onto $\mathscr{C}(A)$. Hence $\tilde{\Pi}_A = \Pi_A$. $\mathscr{C}(A)$ and $\mathscr{G}(A)$ are the orthogonal complements of each other. The kernel of $\tilde{\Pi}_A$ coincides with $\mathscr{G}(A)$.*

PROOF. For $X \in \mathscr{X}$, we have

$$\tilde{\Pi}_A^2(X) = \sum_{j=1}^{r} P_j \tilde{\Pi}_A(X) P_j = \sum_{j,k=1}^{r} P_j P_k X P_k P_j = \sum_{j=1}^{r} P_j X P_j = \tilde{\Pi}_A(X).$$

Hence $\tilde{\Pi}_A$ is a projection. Since $AP_j = P_j A = \lambda_j P_j (j = 1, \cdots, r)$, we obtain

$$A\tilde{\Pi}_A(X) = \sum_{j=1}^{r} AP_j X P_j = \sum_{j=1}^{r} \lambda_j P_j X P_j = \sum_{j=1}^{r} P_j X P_j A = \tilde{\Pi}_A(X) A.$$

It follows that $[A, \tilde{\Pi}_A(X)] = 0$, namely, the range of $\tilde{\Pi}_A$ is contained in $\mathscr{C}(A)$. But, if $X \in \mathscr{C}(A)$, then $X$ commutes with $P_j (j = 1, \cdots, r)$. Therefore,

$$\tilde{\Pi}_A(X) = \sum_{j=1}^{r} P_j X P_j = X \sum_{j=1}^{r} P_j^2 = X \sum_{j=1}^{r} P_j = X,$$

which implies that $\mathscr{C}(A)$ is contained in the range of $\tilde{\Pi}_A$. This completes the proof of (i). Next we prove (ii). Since $P_j^* = P_j (j = 1, \cdots, r)$, we have

$$\begin{aligned} \{\tilde{\Pi}_A(X),\ Y\} &= \mathrm{tr}\Big(\sum_{j=1}^{r} P_j X P_j \cdot Y^*\Big) = \mathrm{tr}\Big(X \cdot \sum_{j=1}^{r} P_j Y^* P_j\Big) \\ &= \mathrm{tr}(X \tilde{\Pi}_A(Y)^*) = \{X, \tilde{\Pi}_A(Y)\}, \end{aligned}$$

for $X, Y \in \mathscr{X}$. Hence $\tilde{\Pi}_A$ is a hermitian transformation in $\mathscr{X}$. Combining this with (i), we conclude that $\tilde{\Pi}_A$ is the orthogonal projection onto $\mathscr{C}(A)$. The fact that $\tilde{\Pi}_A = \Pi_A$ is a consequence of the uniqueness of the orthogonal projection onto a subspace. In order to prove the last statement of (ii), we note that $\Phi_A$ is a normal transformation in $\mathscr{X}$ (see (i) of LEMMA 2.1). Since $\mathscr{C}(A)$ and $\mathscr{G}(A)$ are the kernel and the range of a normal transformation $\Phi_A$, respectively, these subspaces are the orthogonal complements of each other. The proof of LEMMA 2.2 is completed.

LEMMA 2.3. (*Ellis-Pinsky* [2]) *Let $A$ be a normal matrix. If $B$ is hermitian, then $\Pi_A(B)$ is also hermitian. Furthermore, if $B$ is nonnegative definite, then $\Pi_A(B)$ is nonnegative definite also.*

PROOF. Since $A$ is normal, $\Pi_A=\tilde{\Pi}_A$ by (ii) of Lemma 2.2. Hence we have

$$\Pi_A(B)^*=(\sum_{j=1}^{r}P_jBP_j)^*=\sum_{j=1}^{r}P_jBP_j=\Pi_A(B).$$

For $x\in\boldsymbol{C}^m$,

$$(\Pi_A(B)x,\ x)=\sum_{j=1}^{r}(P_jBP_jx,\ x)=\sum_{j=1}^{r}(BP_jx,\ P_jx)\geq 0.$$

This completes the proof of LEMMA 2.3.

LEMMA 2.4. *Suppose that $A$ and $B$ are hermitian matrices. Then there exists a skew-hermitian matrix $K$ such that*

$$B=\Pi_A(B)+[A,\ K]. \tag{2.5}$$

*If, in addition, $A$ and $B$ are real matrices, then $K$ can be chosen so as to be a real skew-symmetric matrix.*

PROOF. We see that the kernel of $\Pi_A$ coincides with $\mathscr{G}(A)$ by (ii) of LEMMA 2.2. The existence of $K$ satisfying (2.5) then follows at once. Since both $A$ and $B$ are hermitian, $\Pi_A(B)$ is hermitian by LEMMA 2.3. Hence $[A,\ K]$ is also a hermitian matrix. We obtain, therefore,

$$[A,\ K]=[A,\ K]^*=[K^*,\ A].$$

Setting $K_1=(K+K^*)/2$ and $K_2=(K-K^*)/2$, we have $K=K_1+K_2$. The above equality then reduces to $[A,\ K_1]=0$. Hence,

$$[A,\ K]=[A,\ K_1]+[A,\ K_2]=[A,\ K_2].$$

This means that the skew-symmetric matrix $K_2$ also satisfies (2.5). Now we assume that both $A$ and $B$ are real matrices. Then $\Pi_A(B)$ is a real matrix and hence $[A,\ K]$ is real also. Denoting by bar the complex conjugate of a matrix, we get

$$[A,\ K]=\overline{[A,\ K]}=[A,\ \bar{K}].$$

Set $K'=(K+\bar{K})/2$, $K''=(K-\bar{K})/2$. Then $K=K'+K''$. Taking into account of the above equality which implies $[A,\ K'']=0$, we have

$$[A,\ K]=[A,\ K']+[A,\ K'']=[A,\ K'].$$

Therefore, the real matrix $K'$ satisfies (2.5). The proof of LEMMA 2.4 is complete.

The following theorem is a prototype of THEOREM 1.1.

THEOREM 2.5. *Suppose that $A$ and $B$ are real hermitian matrices. Suppose furthermore that $B$ is nonnegative definite. Then each of the following two conditions implies the other*:

(1) *There exists a real skew-symmetric matrix $K$ such that $B+[K, A]$ is positive definite.*

(2) *Let $\psi \in \boldsymbol{R}^m \backslash \{0\}$ and let $B\psi=0$. Then, for any $\mu \in \boldsymbol{R}$, We have $\mu\psi+A\psi \neq 0$.*

PROOF. We prove first that (1) implies (2). Suppose for a moment that $\psi \neq 0$, $B\psi=0$ and $\mu\psi+A\psi=0$ for some $\mu \in \boldsymbol{R}$. Then for any real skew-symmetric matrix $K$, we have

$$\begin{aligned}((B+[K, A])\psi, \psi) &= (KA\psi, \psi)-(AK\psi, \psi)=2(KA\psi, \psi)\\ &= -2\mu(K\psi, \psi)=0.\end{aligned}$$

This implies that (1) is not true.

Conversely, assume that (2) holds true. By LEMMA 2.4, we have the decomposition (2.5) for $B$ with a real skew-symmetric matrix $K$. As a consequence of LEMMA 2.3, $\Pi_A(B)$ is real symmetric and nonnegative definite. To prove further the positive definiteness of $\Pi_A(B)$, it suffices to show that $x=0$ follows from $(\Pi_A(B)x, x)=0$. We recall that $A=\lambda_1 P_1+\cdots+\lambda_r P_r$, where $\lambda_j \in \boldsymbol{R}$ are the distinct eigenvalues of $A$ and $P_j$ are the corresponding eigenprojections. Let

$$(\Pi_A(B)x, x)=\sum_{j=1}^{r}(BP_jx, P_jx)=0.$$

Setting $x_j=P_jx$, we have $Bx_j=0$ and $-\lambda_j x_j+Ax_j=0$ for $j=1, \cdots, r$. Hence, $x_j=0$ $(j=1, \cdots, r)$ by (2). Therefore, $x=\sum_{j=1}^{r} x_j=0$. This implies that (1) follows from (2). The proof of the theorem is completed.

## § 3. Proof of Theorem 1.1

First of all, we reduce the system to a standard type by suitable transformation.

LEMMA 3.1. *It suffices to prove the theorem in the case of $A^0=I$ (unit matrix).*

PROOF. Let us consider a system (1.1) satisfying CONDITION 1.1 where $A(\omega)$ and $B(\omega)$ are defined by (1.2). This implies that we are given a quadruplet $\{A^0, A(\omega), B(\omega), L\}$ satisfying (i), (ii) and (iii) in REMARK 1.2. Making use of the positive definiteness of $A^0$, we set

$$\tilde{A}(\omega)=(A^0)^{-1/2}A(\omega)(A^0)^{-1/2}, \quad \tilde{B}(\omega)=(A^0)^{-1/2}B(\omega)(A^0)^{-1/2},$$
$$\tilde{L}=(A^0)^{-1/2}L(A^0)^{-1/2},$$

and obtain a new system, i. e., a quadruplet $\{I, \tilde{A}(\omega), \tilde{B}(\omega), \tilde{L}\}$. We assume first that CONDITION (1) holds for the latter. This means that there exists a compensating function $\tilde{K}(\omega)$ for the quadruplet $\{I, \tilde{A}(\omega), \tilde{B}(\omega), \tilde{L}\}$. Then, setting $K(\omega)=(A^0)^{1/2}\tilde{K}(\omega)(A^0)^{-1/2}$ and noting that

$$K(\omega)A^0=(A^0)^{1/2}\tilde{K}(\omega)(A^0)^{1/2},$$
$$K(\omega)A(\omega)=(A^0)^{1/2}\tilde{K}(\omega)\tilde{A}(\omega)(A^0)^{1/2},$$

we get a compensating function $K(\omega)$ for the quadruplet $\{A^0, A(\omega), B(\omega), L\}$. Hence we conclude that CONDITION (1) holds for the original system if it holds for the transformed system. The converse assertion can be shown in a similar way. Therefore, CONDITION (1) holds for the original system if and only if it holds for the transformed system.

Next we treat CONDITION (2). We consider the linear homogeneous equation

$$\lambda\tilde{\phi}+\{\tilde{L}+\xi\tilde{A}(\omega)-\xi^2\tilde{B}(\omega)\}\tilde{\phi}=0$$

and look for a nontrivial solution $\tilde{\phi}$, where $\xi\in i\boldsymbol{R}$, $\omega\in S^{n-1}$ and $\lambda\in\boldsymbol{C}$. Namely, $-\lambda$ and $\tilde{\phi}$ are the eigenvalue and the eigenvector of $\tilde{L}+\xi\tilde{A}(\omega)-\xi^2\tilde{B}(\omega)$, respectively. This equation is just (1. 3) with $\{A^0, A(\omega), B(\omega), L\}$ replaced by $\{I, \tilde{A}(\omega), \tilde{B}(\omega), \tilde{L}\}$. Setting $\phi=(A^0)^{-1/2}\tilde{\phi}$ for a given nontrivial solution $\tilde{\phi}$, we obtain a nontivial solution of

$$\lambda A^0\phi+\{L+\xi A(\omega)-\xi^2B(\omega)\}\phi=0,$$

which is (1. 3). Also the converse is true, because $(A^0)^{-1/2}$ is an isomorphism of $\boldsymbol{C}^m$ onto $\boldsymbol{C}^m$. Note that the corresponding value of $\lambda$ coincides with each other. These observations show that CONDITION (2) for the original system implies CONDITION (2) for the transformed system and vice versa. The same holds for CONDITION (4). CONDITION (3) can be dealt with by similar arguments. Therefore, the equivalence of CONDITIONS (1), (2), (3) and (4) for the transformed system implies the same for the original system. This means that we may assume in proving THEOREM 1. 1 that $A^0=I$.

By the above lemma we suppose in the sequel that $A^0$ equals the unit matrix.

LEMMA 3. 2. *CONDITION (2) implies CONDITION (3).*

PROOF. Let $\psi\in\boldsymbol{R}^m\backslash\{0\}$, $\omega\in S^{n-1}$ and $B(\omega)\psi=L\psi=0$. If (3) is false, there exists a number $\mu\in\boldsymbol{R}$ such that $\mu\psi+A(\omega)\psi=0$. Then, for any $\xi\in i\boldsymbol{R}\backslash\{0\}$, we have

$$\{L+\xi A(\omega)-\xi^2B(\omega)\}\psi=-\xi\mu\psi.$$

This shows that (1. 3) has a nontrivial solution $\psi$ for $\lambda=\xi\mu$. But, since $\mathrm{Re}\lambda=\mathrm{Re}(\xi\mu)=0$, we have a contradiction if (2) is assumed. Thus CONDITION (3) follows from CONDITION (2).

LEMMA 3. 3. *CONDITION (1) implies CONDITION (4).*

PROOF. This lemma is proved by an energy method. We refer the reader to PROPOSITION 2.10 of [8] for the proof.

If (4) holds, then (2) holds a fortiori. So we have proved that (1)⇒(4)⇒(2)⇒(3). To show that (3) implies (1), we proceed as follows.

Let $\omega_0\in S^{n-1}$. Let $\lambda_1, \cdots, \lambda_r$ be the distinct eigenvalues of $A(\omega_0)$. We denote by $W(\omega_0)$ a sufficiently small neighborhood of $\omega_0$ and look at the eigenvalues of $A(\omega)$ for $\omega\in W(\omega_0)$. Let $\Gamma_j (j=1, \cdots, r)$ be a small positively-oriented circle centered at $\lambda_j$, respectively. We assume that, for any $j$, $\Gamma_j$ excludes $\Gamma_i$ if $i\neq j$. We assume furthermore that each $\Gamma_j$ contains all the eigenvalues of $A(\omega)$ belonging to the $\lambda_j$-group and excludes other eigenvalues. The $\lambda_j$-group eigenvalues are the totality of the eigenvalues of $A(\omega)$ generated by splitting from the common eigenvalue $\lambda_j$ of $A(\omega_0)$. See Kato [4] for details. We set

$$P_j(\omega)=\frac{1}{2\pi i}\int_{\Gamma_j}(\zeta-A(\omega))^{-1}d\zeta, \tag{3.1}$$

for $\omega\in W(\omega_0)$ and $1\leq j\leq r$. Then $P_j(\omega)$ is the total projection for the $\lambda_j$-group eigenvalues. We define $\hat{A}(\omega)$ for $\omega\in W(\omega_0)$ by

$$\hat{A}(\omega)=\sum_{j=1}^{r}\lambda_jP_j(\omega). \tag{3.2}$$

The following lemma shows that CONDITION (3) holds in a neighborhood of $\omega_0$ for the quadruplet $\{I, \hat{A}(\omega), B(\omega), L\}$.

LEMMA 3.4. *Suppose* CONDITION (3) *holds for the system* $\{I, A(\omega), B(\omega), L\}$. *Let* $\omega_0\in S^{n-1}$. *Then there exists a neighborhood* $V(\omega_0)\subset W(\omega_0)$ *which has the following properties: Let* $\psi\in\boldsymbol{R}^m\setminus\{0\}$ *and let* $B(\omega)\psi=L\psi=0$ *for some* $\omega\in V(\omega_0)$. *Then* $\mu\psi+\hat{A}(\omega)\psi\neq0$ *for any* $\mu\in\boldsymbol{R}$.

PROOF. Suppose the conclusion is false. We choose a sequence of neighborhoods $V_j(\omega_0)$ of $\omega_0$, $j=1, 2, \cdots$, which converges to $\{\omega_0\}$. Then there exist $\omega_j\in V_j(\omega_0)$, $\psi_j\in\boldsymbol{R}^m$ with $\|\psi_j\|=1$ and $\mu_j\in\boldsymbol{R}$ for $j=1, 2, \cdots$ such that

$$B(\omega_j)\psi_j=L\psi_j=0, \quad \mu_j\psi_j+\hat{A}(\omega_j)\psi_j=0. \tag{3.3}$$

By (3.2), $-\mu_j\in\{\lambda_1, \cdots, \lambda_r\}$ for any $j$. Hence we may assume without loss of generality that $\mu_j=-\lambda_k$ for some $k(1\leq k\leq r)$ not depending on $j$ and $j=1, 2, \cdots$. We get also $\psi_j\to\psi_0$ for some $\psi_0\in\boldsymbol{R}^m$ with $\|\psi_0\|=1$ as $j\to\infty$, by taking a subsequence if necessary. It follows then from (3.3) that

$$B(\omega_0)\psi_0=L\psi_0=0, \quad -\lambda_k\psi_0+A(\omega_0)\psi_0=0.$$

Here we used $\hat{A}(\omega_0)=A(\omega_0)$. Since $\|\psi_0\|=1$, this is a contradiction. The proof is complete.

We set $A=\hat{A}(\omega)$ and recall the orthogonal decomposition $\mathscr{H}=\mathscr{C}(A)\oplus\mathscr{G}(A)$ given in (ii) of LEMMA 2.2. We shall use the abbreviation $\hat{\Pi}(\omega)$ for $\Pi_A=\Pi_{\hat{A}(\omega)}$.

LEMMA 3.5. *Suppose* CONDITION (3) *holds. Then there exists a neighbor-*

*hood* $U(\omega_0)\subset V(\omega_0)$ *satisfying the follwoing properties: Let* $\hat{\Pi}(\omega)$ *be the orthogonal projection onto* $\mathscr{C}(\hat{A}(\omega))$. *Let*

(3.4) $$B(\omega)+L=F(\omega)+G(\omega),\ \omega\in U(\omega_0),$$

*where*

$$F(\omega)=\hat{\Pi}(\omega)(B(\omega)+L),$$
$$G(\omega)=(1-\hat{\Pi}(\omega))(B(\omega)+L).$$

*Then* $F(\omega)$ *is real symmetric and positive for* $\omega\in U(\omega_0)$. *Furthermore, there exists a* $C^\infty$*-function* $K(\omega)$ *defined on* $U(\omega_0)$ *and taking values in* $\mathscr{X}$ *such that* $G(\omega)=[A(\omega),\ K(\omega)]$ *for* $\omega\in U(\omega_0)$.

PROOF. It is easily seen that $P_j(\omega)$ defined by (3.1) is a $C^\infty$-function on $W(\omega_0)$ with values in $\mathscr{X}$. Hence, by (2.4), $\hat{\Pi}(\omega)$ is a $C^\infty$-mapping from $W(\omega_0)$ into the space of linear transformations in $\mathscr{X}$. Combining these observations, we conclude that $F(\omega)$ and $G(\omega)$ are $C^\infty$-functions on $W(\omega_0)$ with values in $\mathscr{X}$. That $F(\omega)$ is real symmetric and positive for $\omega\in V(\omega_0)$ is a consequence of THEOREM 2.5 in conjunction with LEMMA 3.4. We write $\Phi(\omega)$ for $\Phi_{A(\omega)}$, which is defined by (2.2), namely,

$$\Phi(\omega)(X)=[A(\omega),\ X],\ X\in\mathscr{X}.$$

Now we consider the following linear inhomogeneous equation

(3.5) $$\Phi(\omega)(X)=G(\omega),$$

and want to find a smooth solution $X=X(\omega)$ defined on a neighborhood of $\omega_0$. Let $\mu_1(\omega),\cdots,\mu_m(\omega)$ be the repeated eigenvalues of $A(\omega)$. Let $\psi_1(\omega),\cdots,\psi_m(\omega)$ be the corresponding eigenvectors, respectively. By LEMMA 2.1, the eigenvalues of $\Phi(\omega)$ are $\mu_j(\omega)-\mu_k(\omega)(1\leq j,\ k\leq m)$ and the corresponding eigenvectors are given by $\psi_j(\omega)\psi_k(\omega)^*$, respectively. We assume that $\{\psi_1(\omega),\cdots,\psi_m(\omega)\}$ forms an orthonormal system in $\boldsymbol{C}^m$.

Let us define an equivalence relation in the set $\{1,\cdots,m\}$ as follows: $j\sim k$ if and only if both $\mu_j(\omega)$ and $\mu_k(\omega)$ belong to one and the same $\lambda_i$-group, viz., there exists an integer $i\in\{1,\cdots,r\}$ such that both $\mu_j(\omega)$ and $\mu_k(\omega)$ are enclosed by $\Gamma_i$ if and only if $j\sim k$. Thus we get a partition of the integers $\{1,\cdots,m\}$. Let

$$\Lambda(\omega)=\{(j,\ k);1\leq j,\ k\leq m,\ j\sim k\}$$

and let

$$\sigma_0(\Phi(\omega))=\{\mu_j(\omega)-\mu_k(\omega);\ (j,\ k)\in\Lambda(\omega)\}.$$

It is easily seen that $\sigma_0(\Phi(\omega_0))=\{0,\cdots,0\}$. More precisely, the number of zeros in the bracket are $m_1^2+\cdots+m_r^2$, where $m_1,\cdots,m_r$ denote the multiplicities of the eigenvalues $\lambda_1,\cdots,\lambda_r$, respectively. Also we see that $\sigma_0(\Phi(\omega))$ gives an enumeration of the eigenvalues of $\Phi(\omega)$ belonging to the 0-group, repeated according to the multiplicities. We define $\mathscr{W}_0(\Phi(\omega))$ to be the direct sum of the eigenspaces corresponding to the eigenvalues contained in $\sigma_0(\Phi(\omega))$. Hence $\mathscr{W}_0(\Phi(\omega))$ is spanned by the orthonomal system $\{\psi_j(\omega)\psi_k(\omega)^*;\ (j,k)\in\Lambda(\omega)\}$. We note that $\sigma_0(\Phi(\omega))$ and $\mathscr{W}_0(\Phi(\omega))$ depend neither on the particular numebering $\{\mu_1(\omega),\cdots,\mu_m(\omega)\}$ of eigenvalues nor on the particular choice of the orthonormal system $\{\psi_1(\omega),\cdots,\psi_m(\omega)\}$ of eigenvectors.

Let us consider now $\hat{A}(\omega)$ instead of $A(\omega)$. We write $\hat{\Phi}(\omega)$ for $\Phi_{\hat{A}(\omega)}$, namely,

$$\hat{\Phi}(\omega)(X)=[\hat{A}(\omega),\ X],\ X\in\mathscr{X}.$$

The totality of the repeated eigenvalues of $\hat{\Phi}(\omega)$ are given by $\{\mu_j(\omega_0)-\mu_k(\omega_0);\ 1\le j,\ k\le m\}$ and the corresponding eigenvectors are $\{\psi_j(\omega)\psi_k(\omega)^*;\ 1\le j,\ k\le m\}$. We define $\sigma_0(\hat{\Phi}(\omega))$ and $\mathscr{W}_0(\hat{\Phi}(\omega))$ in the same way as for $\Phi(\omega)$. Then we have

$$\sigma_0(\hat{\Phi}(\omega))=\sigma_0(\Phi(\omega_0))=\{0,\cdots,0\}$$
$$\mathscr{W}_0(\hat{\Phi}(\omega))=\mathscr{W}_0(\Phi(\omega)),$$

for $\omega\in V(\omega_0)$. It follows that

$$\mathscr{W}_0(\Phi(\omega))=\text{kernel of }\hat{\Phi}(\omega)=\mathscr{C}(\hat{A}(\omega)), \tag{3.6}$$

for $\omega\in V(\omega_0)$. Let $\Gamma$ be a small positively-oriented circle centered at 0. We choose a sufficiently small neighborhood $U(\omega_0)\subset V(\omega_0)$ such that $\Gamma$ encloses all the eigenvalues of $\Phi(\omega)$ contained in $\sigma_0(\Phi(\omega))$, namely, the 0-group eigenvalues of $\Phi(\omega)$ for $\omega\in U(\omega_0)$ but excludes other eigenvalues of $\Phi(\omega)$ for $\omega\in U(\omega_0)$. Then, by (3.6), the orthogonal projection $\hat{\Pi}(\omega)$ onto $\mathscr{C}(\hat{A}(\omega))$ is expressed as

$$\hat{\Pi}(\omega)=\frac{1}{2\pi i}\int_\Gamma(\zeta-\Phi(\omega))^{-1}d\zeta, \tag{3.7}$$

for $\omega\in U(\omega_0)$. Set

$$\Psi(\omega)=-\frac{1}{2\pi i}\int_\Gamma\zeta^{-1}(\zeta-\Phi(\omega))^{-1}d\zeta \tag{3.8}$$

for $\omega\in U(\omega_0)$. We obtain

$$\Phi(\omega)\Psi(\omega)=\Psi(\omega)\Phi(\omega)=1-\hat{\Pi}(\omega).$$

This means that $\Psi(\omega)$ may be regarded as a pseudo-inverse of $\Phi(\omega)$. See Kato [4]. Setting

$$X=X(\omega)=\Psi(\omega)G(\omega),\ \omega\in U(\omega_0), \tag{3.9}$$

we get a solution of (3.5). It is seen from (3.8) that $\Psi(\omega)$ is a $C^\infty$-mapping from $U(\omega_0)$ into the space of linear transformations in $\mathscr{X}$. Hence $X(\omega)$ defined by (3.9) is a $C^\infty$-function on $U(\omega_0)$ with values in $\mathscr{X}$. $X(\omega)$ can be chosen as $K(\omega)$ stated in the lemma. Thus the proof of Lemma 3.5 is completed.

We shall show how the global result is obtained from Lemma 3.5.

Lemma 3.6. *Suppose Condition (3) holds. Then there exist $F(\omega)$ and $K(\omega)$ satisfying the following properties:*

(i) *$F(\omega)$ and $K(\omega)$ are $C^\infty$-functions on $S^{n-1}$ with values in $\mathscr{X}$.*

(ii) *$F(\omega)$ is real symmetric and positive for $\omega\in S^{n-1}$.*

(iii) *We have $[K(\omega), A(\omega)]+B(\omega)+L=F(\omega)$ for $\omega\in S^{n-1}$.*

PROOF. For each $\omega\in S^{n-1}$, we assign an open neighborhood $U(\omega)$ such that LEMMA 3.5 holds. Then $\{U(\omega)\}_{\omega\in S^{n-1}}$ is an open cover of $S^{n-1}$. Since $S^{n-1}$ is compact, there is a finite set of points $\omega_1, \cdots, \omega_N\in S^{n-1}$ such that $\{U(\omega_j)\}_{1\leq j\leq N}$ is a cover of $S^{n-1}$. We have the representation for each $j(1\leq j\leq N)$,

$$[K_j(\omega), A(\omega)]+B(\omega)+L=F_j(\omega), \quad \omega\in U(\omega_j), \tag{3.10}$$

where $K_j(\omega)$ and $F_j(\omega)$ are $C^\infty$-functions on $U(\omega_j)$ with values in $\mathscr{H}$. Moreover, $F_j(\omega)$ is real symmetric and positive for each $j(1\leq j\leq N)$ and $\omega\in U(\omega_j)$. Let $\{\alpha_j(\omega)\}_{1\leq j\leq N}$ be a partition of unity on $S^{n-1}$, subordinate to the cover $\{U(\omega_j)\}_{1\leq j\leq N}$. More precisely, $\alpha_j(\omega)\in C^\infty(S^{n-1})$, supp $\alpha_j(\omega)\subset U(\omega_j)$, $0\leq\alpha_j(\omega)\leq 1$ for $1\leq j\leq N$ and $\sum_{j=1}^{N}\alpha_j(\omega)=1$ for $\omega\in S^{n-1}$. Multiplying (3.10) by $\alpha_j(\omega)$ and summing up with respect to $j$, we obtain

$$[K(\omega), A(\omega)]+B(\omega)+L=F(\omega), \quad \omega\in S^{n-1}. \tag{3.11}$$

Here we set

$$K(\omega)=\sum_{j=1}^{N}\alpha_j(\omega)K_j(\omega),$$

$$F(\omega)=\sum_{j=1}^{N}\alpha_j(\omega)F_j(\omega).$$

Both $K(\omega)$ and $F(\omega)$ can be regarded as $C^\infty$-functions on $S^{n-1}$. We observe that $F(\omega)$ is real symmetric and postive for $\omega\in S^{n-1}$, which completes the proof of LEMMA 3.6.

LEMMA 3.7. *Suppose CONDITION (3) holds. Then there exist $F(\omega)$ and $K(\omega)$ satisfying all the properties enumerated in LEMMA 3.6 and furthermore the following: $K(\omega)$ is real skew-symmetric and $K(-\omega)=-K(\omega)$ for $\omega\in S^{n-1}$.*

PROOF. That $K(\omega)$ is real skew-symmetric is shown in the same way as in the proof of LEMMA 2.4. We shall prove $K(-\omega)=K(\omega)$. Replacing $\omega$ by $-\omega$ in (3.11) and taking into accout of the fact that $A(-\omega)=-A(\omega)$ and $B(-\omega)=B(\omega)$ for $\omega\in S^{n-1}$, we get

$$-[K(-\omega), A(\omega)]+B(\omega)+L=F(-\omega). \tag{3.12}$$

Adding (3.12) to (3.11) and dividing both sides by 2, we have

$$[\frac{1}{2}(K(\omega)-K(-\omega)), A(\omega)]+B(\omega)+L=\frac{1}{2}(F(\omega)+F(-\omega)).$$

The right hand side is again real symmetric and positive and hence we can choose $(K(\omega)-K(-\omega))/2$ and $(F(\omega)+F(-\omega))/2$ in place of $K(\omega)$ and $F(\omega)$, respectively. The proof of LEMMA 3.7 is complete.

Finally we note that, by LEMMA 3.7,

$$\begin{aligned}[K(\omega),\ A(\omega)] &= K(\omega)A(\omega)-A(\omega)K(\omega)\\ &= K(\omega)A(\omega)+{}^t(K(\omega)A(\omega))\\ &= 2[K(\omega)A(\omega)]'.\end{aligned}$$

Therefore, in view of (3.11), we conclude that $2K(\omega)$ satisfies CONDITION (1). Denoting $2K(\omega)$ by $K(\omega)$ again, we obtain the compensating function. Thus CONDITION (1) follows from CONDITION (3). The proof of THEOREM 1.1 is completed.

## § 4. Quasilinear symmetric hyperbolic-parabolic systems

In this section, we give a remark on the system of partial differential equations,

$$(4.1)\qquad \begin{cases} A_1^0(u,v)u_t+\sum_{j=1}^{n}A_{11}^j(u,v)u_{x_j}=f_1(u,v;D_xv),\\ A_2^0(u,v)v_t-\sum_{j,k=1}^{n}B_2^{jk}(u,v)v_{x_jx_k}=f_2(u,v;D_xu,D_xv),\end{cases}$$

which was studied in [6]. Here, $t\geq 0$, $x=(x_1,\cdots,x_n)\in\boldsymbol{R}^n$. The unknowns $u=u(t,x)$ and $v=v(t,x)$ are functions valued in $\boldsymbol{R}^{m'}$ and $\boldsymbol{R}^{m''}$, respectively. We assume that $(u,v)$ takes the values in a open convex set $\mathcal{O}$ contained in $\boldsymbol{R}^m$, where $m=m'+m''$. $A_1^0$ and $A_{11}^j(j=1,\cdots,n)$ are $m'\times m'$ real matrices, while $A_2^0$ and $B_2^{jk}(j,\ k=1,\cdots,n)$ are $m''\times m''$ real matrices. $f_1$ and $f_2$ are functions taking values in $\boldsymbol{R}^{m'}$ and $\boldsymbol{R}^{m''}$, respectively. $D_x$ denotes $\{(\partial/\partial x)^\alpha;\ |\alpha|=1\}$.

We say that the system (4.1) is symmetric hyperbolic-parabolic if the following condition holds.

CONDITION 4.1. $A_1^0(\cdot,\cdot)$, $A_2^0(\cdot,\cdot)$, $A_{11}^j(\cdot,\cdot)(j=1,\cdots,n)$, $B_2^{jk}(\cdot,\cdot)(j,\ k=1,\cdots,n)$ are $C^\infty$-functions defined on $\mathcal{O}\subset\boldsymbol{R}^m$ which satisfy the following properties:

(i) $A_1^0(u,v)$, $A_1^j(u,v)$ are real symmetric matrices for $(u,v)\in\mathcal{O}$ and $j=1,\cdots,n$. $A_1^0(u,v)$ is positive for $(u,v)\in\mathcal{O}$.

(ii) $A_1^0(u,v)$, $B_2^{jk}(u,v)$ are real symmetric matrices for $(u,v)\in\mathcal{O}$ and $j,\ k=1,\cdots,n$. $A_2^0(u,v)$ is positive for $(u,v)\in\mathcal{O}$ $B_2^{jk}(u,v)=B_2^{kj}(u,v)$ for $(u,v)\in\mathcal{O}$ and $j,\ k=1,\cdots,n$. Furthermore, $\sum_{j,k}B_2^{jk}(u,v)\omega_j\omega_k$ is positive for $(u,v)\in\mathcal{O}$ and $\omega=(\omega_1,\cdots,\omega_n)\in S^{n-1}$.

We assume the above condtion and in addition the existence of a constant stationary solution which is as follows.

CONDITION 4.2. $f_1$ is a $C^\infty$-function on $\mathcal{O}\times\boldsymbol{R}^{nm''}$ and $f_2$ is a $C^\infty$-funciton on $\mathcal{O}\times\boldsymbol{R}^{nm}$. Moreover, there exists a constant state $(\bar{u},\ \bar{v})\in\mathcal{O}$ such that $f_1(\bar{u},\ \bar{v};\ 0)=0$ and $f_2(\bar{u},\ \bar{v};\ 0,\ 0)=0$.

It is shown that the initial value problem for (4.1) is well-posed in

Sobolev-spaces if CONDITION 4.1 and 4.2 hold. (See THEOREM 2.9 of [6].) In order to study the global existence of solutions, we linearize (4.1) around the constant stationary solution $(u, v)(t, x)=(\bar{u}, \bar{v})$ whose existence is guaranteed by CONDITION 4.2. Then we examine the decay of solutions to the resulting equation. We shall use $\eta$ and $\zeta$ as variables corresponding to $D_x u=(u_{x_1}, \cdots, u_{x_n})$ and $D_x v=(v_{x_1}, \cdots, v_{x_n})$, respectively. Hence $\eta=(\eta_1, \cdots, \eta_n) \in \boldsymbol{R}^{nm'}$, where $\eta_i \in \boldsymbol{R}^{m'}$ for $1 \leq i \leq n$. Similarly $\zeta=(\zeta_1, \cdots, \zeta_n) \in \boldsymbol{R}^{nm''}$, where $\zeta_i \in \boldsymbol{R}^{m''}$ for $1 \leq i \leq n$. We write therefore $f_1=f_1(u, v; \zeta)$, $f_2=f_2(u, v; \eta, \zeta)$. Note that $u$ and $v$ are used as independent variables here. The linearized equation of (4.1) is given by

$$A^0(\bar{u}, \bar{v}) w_t+\sum_j A^j(\bar{u}, \bar{v}) w_{x_j}-\sum_{j, k} B^{jk}(\bar{u}, \bar{v}) w_{x_j x_k}+L(\bar{u}, \bar{v}) w=0. \tag{4.2}$$

Here

$$A^0(\bar{u}, \bar{v})=\begin{pmatrix} A_1^0(\bar{u}, \bar{v}) & 0 \\ 0 & A_2^0(\bar{u}, \bar{v}) \end{pmatrix},$$

$$A^j(\bar{u}, \bar{v})=\begin{pmatrix} A_{11}^j(\bar{u}, \bar{v}) & -D_{\zeta_j} f_1(\bar{u}, \bar{v}; 0) \\ -D_{\eta_j} f_2(\bar{u}, \bar{v}; 0,0) & -D_{\zeta_j} f_2(\bar{u}, \bar{v}; 0,0) \end{pmatrix},$$

$$B^{jk}(\bar{u}, \bar{v})=\begin{pmatrix} 0 & 0 \\ 0 & B_2^{jk}(\bar{u}, \bar{v}) \end{pmatrix},$$

$$L(\bar{u}, \bar{v})=-\begin{pmatrix} D_u f_1(\bar{u}, \bar{v}; 0) & D_v f_1(\bar{u}, \bar{v}; 0) \\ D_u f_2(\bar{u}, \bar{v}; 0,0) & D_v f_2(\bar{u}, \bar{v}; 0,0) \end{pmatrix}.$$

Besides CONDITION 4.1 and 4.2, CONDITION 1.1 and 1.2 were assumed to hold for (4.2) in [6]. The result obtained is the existence of global solutions near $(\bar{u}, \bar{v})$, which approach $(\bar{u}, \bar{v})$ ultimately as $t \rightarrow \infty$. But we can employ the same arguments without modification even if CONDITION 1.2 is replaced by the existence of compensating function. Hence, by applying THEOREM 1.1, we obtain an improved version of THEOREM 3.6 of [6].

THEOREM 4.1. *(global existence and asymptotic stability for* (4.1)*) Suppose* CONDITION 4.1, 4.2 *for* (4.1) *and* CONDITION 1.1 *for* (4.2)*. Suppose furthermore the existence of compensating function for* (4.2) *or any one of the equivalent conditions given in* THEOREM 1.1. *Let* $n \geq 3$, $s \geq [n/2]+3$, $1 \leq p<2n/(n+1)$ *and let* $(u, v)(0, x)=(u_0, v_0)(x)$. *Suppose* $(u_0-\bar{u}, v_0-\bar{v}) \in H^s(\boldsymbol{R}^n) \cap L^p(\boldsymbol{R}^n)$. *For* $l \leq s$, *we set*

$$\|u_0-\bar{u}, v_0-\bar{v}\|_{l, p}=\|u_0-\bar{u}, v_0-\bar{v}\|_l+\|u_0-\bar{u}, v_0-\bar{v}\|_{L^p},$$

*where* $\|\cdot\|_l$ *denotes the norm in* $H^l(\boldsymbol{R}^n)$. *Then, if* $\|u_0-\bar{u}, v_0-\bar{v}\|_{s, p}$ *is small enough, there exists a unique global solution* $(u, v)(t, x)$ *of the initial value problem for* (4.1) *such that*

$u-\bar{u} \in C^0(0, \infty; H^s(\boldsymbol{R}^n)) \cap C^1(0, \infty; H^{s-1}(\boldsymbol{R}^n)) \cap L^2(0, \infty; H^s(\boldsymbol{R}^n))$,

$v-\bar{v} \in C^0(0, \infty; H^s(\boldsymbol{R}^n)) \cap C^1(0, \infty; H^{s-2}(\boldsymbol{R}^n)) \cap L^2(0, \infty; H^{s+1}(\boldsymbol{R}^n))$,

*Furthermore,* $(u, v)(t, x)$ *satisfies the following estimates*

$$\|(u-\bar{u},\ v-\bar{v})(t)\|_s^2+\int_0^t \|(u-\bar{u})(\tau)\|_s^2+\|(v-\bar{v})(\tau)\|_{s+1}^2 d\tau \tag{4.3}$$
$$\leq C\|u_0-\bar{u},\ v_0-\bar{v}\|_{s,p}^2,$$

$$\|(u-\bar{u},\ v-\bar{v})(t)\|_{s-1}\leq C(1+t)^{-\gamma}\|u_0-\bar{u},\ v_0-\bar{v}\|_{s-1,p} \tag{4.4}$$

*for any* $t\in[0,\infty)$. *Here* $C$ *is a positive constant and* $\gamma=n(1/2p-1/4)$.

As an application of the above theorem, we consider the system of equations describing the motion of three dimensional compressible viscous fluid. Let us denote by $\rho$ the mass density, by $u=(u^1, u^2, u^3)$ the velocity, and by $\theta$ the absolute temperature. These notations are conventional. The system of equations for $(\rho, u, \theta)$ is given by

$$\rho_t+\operatorname{div}(\rho u)=0,$$
$$\rho(u_t+(u\cdot\nabla)u)+\nabla p=\operatorname{div}(2\mu P+\mu' I \operatorname{div} u), \tag{4.5}$$
$$\rho e_\theta(\theta_t+u\cdot\nabla\theta)+\theta p_\theta \operatorname{div} u=\operatorname{div}(\kappa\nabla\theta)+\Psi.$$

Here the pressure $p$ and the internal energy $e$ are the known functions of $(\rho, \theta)$. $p_\theta$ and $e_\theta$ denote $\partial p/\partial\theta$ and $\partial e/\partial\theta$, respectively. $P=(1/2)(u^i_{x_j}+u^j_{x_i})_{1\leq i,j\leq 3}$ is called the deformation tensor and $\Psi=(\mu/2)\sum_{i,j}(u^i_{x_j}+u^j_{x_i})^2+\mu'(\operatorname{div} u)^2$ is the dissipation function. The viscousity coefficients $\mu$, $\mu'$ and the heat conductivity coefficient $\kappa$ are known functions of $(\rho, \theta)$. We set $\mathscr{D}=\{(\rho,\theta);\ \rho>0,\ \theta>0\}$ and assume the foliowing conditions:

(4.6) $p$ and $e$ are $C^\infty$-functions on $\mathscr{D}$. $p_\rho=\partial p/\partial\rho>0$ and $e_\theta=\partial e/\partial\theta>0$ on $\mathscr{D}$.

(4.7) $\mu$, $\mu'$, $\kappa$ are $C^\infty$-functions on $\mathscr{D}$. $\mu>0$, $\nu=2\mu+\mu'>0$ and $\kappa>0$ on $\mathscr{D}$.

We write $w={}^t(\rho, u, \theta)$. Then (4. 5) can be rewritten as follows,

$$A^0(w)w_t+\sum_{j=1}^3 A^j(w)w_{x_j}-\sum_{j,k=1}^3 B^{jk}(w)w_{x_jx_k}=g(w\,;D_xw). \tag{4.8}$$

Here we set

$$A^0(w)=\begin{pmatrix} p_\rho/\rho & 0 & 0\\ 0 & \rho I & 0\\ 0 & 0 & \rho e_\theta/\theta\end{pmatrix},$$

$$\sum_j A^j(w)\xi_j=\begin{pmatrix} (p_\rho/\rho)(u\cdot\xi) & p_\rho\xi & 0\\ p_\rho{}^t\xi & \rho(u\cdot\xi)I & p_\theta{}^t\xi\\ 0 & p_\theta\xi & (\rho e_\theta/\theta)(u\cdot\xi)\end{pmatrix},$$

$$\sum_{j,k} B^{jk}(w)\xi_j\xi_k=\begin{pmatrix} 0 & 0 & 0\\ 0 & \mu|\xi|^2 I+(\mu+\mu')^t\xi\xi & 0\\ 0 & 0 & (\kappa/\theta)|\xi|^2\end{pmatrix},$$

$$g(w\,;D_xw)=\begin{pmatrix} 0\\ 2(\nabla\mu)P+\nabla\mu'\operatorname{div} u\\ (1/\theta)(\Psi+\nabla\kappa\cdot\nabla\theta)\end{pmatrix},$$

with $\xi=(\xi_1, \xi_2, \xi_3)\in \boldsymbol{R}^3$, ${}^t\xi\xi=(\xi_i\xi_j)_{1\le i,j\le 3}$. Let $\mathcal{O}=\{(\rho, u, \theta);\ (\rho, \theta)\in\mathscr{D},\ u\in\boldsymbol{R}^3\}$ and let the constant stationary state be $(\bar\rho, 0, \bar\theta)$ where $(\bar\rho, \bar\theta)\in\mathscr{D}$. Then CONDITION 4.1 and 4.2 are verified for (4.8) by using (4.6), (4.7). Note that, for $\hat w={}^t(\hat\rho, \hat u, \hat\theta)\in\boldsymbol{R}^5$,

$$(4.9)\qquad \sum_{j,k}\langle B^{jk}(w)\omega_j\omega_k\hat w, \hat w\rangle\ge\min\{\mu, \nu\}|\hat u|^2+(\kappa/\theta)|\hat\theta|^2.$$

Also, CONDITION 1.1 is satisfied by $A^0=A^0(\bar w)$, $A^j=A^j(\bar w)$, $B^{jk}=B^{jk}(\bar w)$, $L=0$. Finally we check CONDITION (3) of THEOREM 1.1 which is equivalent to CONDITION (1), i. e., the existence of a compensating function. Let $\psi={}^t(\hat\rho, \hat u, \hat\theta)\in\boldsymbol{R}^5\backslash\{0\}$ and let $\sum_{j,k}B^{jk}(\bar w)\omega_j\omega_k\psi=0$ for some $\omega\in S^2$. Then, by (4.9), $\hat u=\hat\theta=0$. *Hence,* $\psi={}^t(\hat\rho, 0, 0)$, *where* $\hat\rho\neq 0$. *It follows that, for any* $\lambda\in\boldsymbol{R}$,

$$\lambda A^0(\bar w)\psi+\sum_j A^j(\bar w)\omega_j\psi=\bar p_\rho\hat\rho\,{}^t(\lambda/\bar\rho, \omega, 0)\neq 0,$$

where $\bar p_\rho=p_\rho(\bar\rho, \bar\theta)$. This means that CONDITION (3) holds for $A^0=A^0(\bar w)$, $A^j=A^j(\bar w)$, $B^{jk}=B^{jk}(\bar w)$, $L=0$. Hence THEOREM 4.1 can be applied to (4.5).

It should be remarked that CONDITION 1.2 can be checked directly for (4.5). A suitable choice of $K^j (j=1, 2, 3)$ leads to

$$\sum_j K^j\xi_j=\alpha\begin{pmatrix} 0 & \bar p_\rho\xi & 0\\ -\bar p_\rho\xi & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Here $\alpha$ is a sufficiently small constant. We refer the reader to [6] for details including the one-dimensional case.

## §5 Discrete velocity models of the Boltzmann equation

We study the discrete velocity models of the Boltzmann equation in $\boldsymbol{R}^n$ along the lines of Kawashima [5]. In these models, there exists a finite number of velocities (constant vectors in $\boldsymbol{R}^n$), say, $v_1, \cdots, v_m$. It is to be noted that $v_i\neq v_j$ if $i\neq j$. The number $m$ and the velocities $v_1, \cdots, v_m$ depend on the model. The unknown $F_i$, $i=1, \cdots, m$, is a function of time $t\ge 0$ and the space variable $x\in\boldsymbol{R}^n$ and represents the density distribution of particles with the velocity $v_i$. We write $F={}^t(F_1, \cdots, F_m)\in\boldsymbol{R}^m$. The system of equations in the general form is given by

$$(5.1)\qquad (F_i)_t+v_i\cdot\nabla_x F_i=Q_i(F, F),\quad i=1, \cdots, m,$$

where $v_i\cdot\nabla_x F_i$ denotes the Euclidean inner product in $\boldsymbol{R}^n$ of $v_i=(v_i^1, \cdots, v_i^n)$ and $\nabla_x F_i=(\partial F_i/\partial x_1, \cdots, \partial F_i/\partial x_n)$. $Q_i(F, F)$, $i=1, \cdots, m$, is a quadratic form in $\boldsymbol{R}^m$, which comes from the binary collisions of particles. More precisely, the polarization of $Q_i(F, F)$ is expressed as

(5.2) $$Q_i(F, G)=\frac{1}{2\alpha_i}\sum_{jkl}\{A_{kl}^{ij}(F_kG_l+F_lG_k)-A_{ij}^{kl}(F_iG_j+F_jG_i)\}.$$

Here $\alpha_i$ and $A_{kl}^{ij}$ are positive and nonnegative constants, respectively.

We give a formal definition of the collision of two particles. Let $(v_i, v_j)$ be an unordered pair of velocities and let $v_i\neq v_j$. By definition, $(v_i, v_j)=(v_j, v_i)$. A collision is expressed by a couple of such pairs. The first is written before an arrow and the second is written after the same arrow, for example, $(v_i, v_j)\to(v_k, v_l)$. Here, $(v_i, v_j)$ is called the initial state while $(v_k, v_l)$ is refered to as the final state of the collision. It is usual to assume for each collision the following properties: (i) $(v_i, v_j)\neq(v_k, v_l)$ (exclusion of the trivial collision), (ii) $v_i+v_j=v_k+v_l$ (conservation of the momentum), (iii) $|v_i|^2+|v_j|^2=|v_k|^2+|v_l|^2$ (conservation of the energy). Conversely, if (i), (ii) and (iii) are satisfied, then $(v_i, v_j)\to(v_k, v_l)$ represents a collision. Note that for some simple models, the definition of the collision given above should be replaced by less physical one.

CONDITION 5.1. $A_{ij}^{kl}$ satisfies the following properties:

(i) $A_{ij}^{kl}$ is a positive number if $(v_i, v_j)\to(v_k, v_l)$ represents a collision. $A_{ij}^{kl}$ is zero otherwise.

(ii) For any $i, j, k, l$, $A_{kl}^{ij}=A_{lk}^{ij}=A_{kl}^{ji}$ and $A_{kl}^{ij}=A_{ij}^{kl}$.

(iii) For some $i, j, k, l$, $A_{ij}^{kl}$ is a positive number (existence of the collision).

Briefly, $A_{ij}^{kl}$ is related to the rate at which the collision $(v_i, v_j)\to(v_k, v_l)$ takes place. The above condition will be assumed in the sequel.

We denote the $m\times m$ diagonal matrix with diagonal elements $v_1^j, \cdots, v_m^j$ by $V^j$, namely,

(5.3) $$V^j=\mathrm{diag}(v_1^j, \cdots, v_m^j),\ j=1, \cdots, n.$$

We set $Q(F, G)={}^t(Q_1(F, G), \cdots, Q_m(F, G))$ and rewrite (5.1) as

(5.4) $$F_t+\sum_{j=1}^{n} V^jF_{x_j}=Q(F, F).$$

Now we give the definitions of some basic concepts concerning the above equation. One is summational invariant and the other is Maxwellian. Let

(5.5) $$\mathscr{M}=\{\phi={}^t(\phi_1, \cdots, \phi_m)\in\boldsymbol{R}^m;\ A_{kl}^{ij}(\phi_i/\alpha_i+\phi_j/\alpha_j-\phi_k/\alpha_k-\phi_l/\alpha_l)=0 \text{ for any } i, j, k, l\}.$$

By CONDITION 5.1, $\mathscr{M}\neq\boldsymbol{R}^m$. Since ${}^t(\alpha_1, \cdots, \alpha_m)\in\mathscr{M}$, it follows that $0<\dim\mathscr{M}<m$. Any $\phi\in\mathscr{M}$ will be called a summational invariant. It is known that, if CONDITION 5.1 holds, $\phi\in\mathscr{M}$ is equivalent to the follwoing property:

(5.6) $$<\phi,\ Q(F, F)>=0 \text{ for any } F\in\boldsymbol{R}^m,$$

where $\langle\ ,\ \rangle$ denotes the Euclidean inner product in $\boldsymbol{R}^m$. See [3], [5].

Let $F={}^t(F_1, \cdots, F_m)$ and let $F_i>0$ for $1\le i\le m$. Let, furthermore,

(5.7) $A_{kl}^{ij}(F_iF_j - F_kF_l)=0$ for any $i, j, k, l$.
Then $F$ is called a constant Maxwellian. It is known also that, if CONDITION 5.1 holds, (5.7) is equivalent to the following property:
(5.8) $Q(F, F)=0$.

Let $M={}^t(M_1, \cdots, M_m)$ be a constant Maxwellian and set $F=M$. Then $F$, regarded as a function of $t$ and $x$, is a constant stationary solution of (5.4). In order to study the existence of global solutions in time near $F=M$, we introduce a new unknown function $f$ by setting
(5.9) $F=M+\Lambda^{1/2}f$.
Here $\Lambda=\Lambda(M)=\mathrm{diag}(M_1/\alpha_1, \cdots, M_m/\alpha_m)$. Substituting (5.9) into (5.4), we get

$$f_t+\sum_{j=1}^{n} V^j f_{x_j}+L_M f=\Gamma(f, f), \tag{5.10}$$

where

$$\begin{aligned} L_M f &= -2\Lambda^{-1/2}Q(M, \Lambda^{1/2}f), \\ \Gamma(f, f) &= \Lambda^{-1/2}Q(\Lambda^{1/2}f, \Lambda^{1/2}f). \end{aligned} \tag{5.11}$$

The $m\times m$ matrix $L_M$ defined above (the linearized collision operator) satisfies the follwoing properties (see [5]):
(5.12) $L_M$ is real symmetric and nonnegative. Furthermore, the kernel of $L_M$ equals $\Lambda^{1/2}\mathscr{M}$.
It is obvious from (5.3) that $V^j$, $j=1, \cdots, n$, is real symmetric. Then, setting $A^0=I$, $A^j=V^j$, $B^{jk}=0$, $L=L_M$, CONDITDON 1.1 is verified at once. In [5], the existence of global solutions and the asymptotic stability was shown by assuming CONDITION 1.2. But the method used there can be applied without modification even if CONDITION 1.2 is replaced by CONDITION (1) of THEOREM 1.1. We reformulate here CONDITION (3), which is equivalent to CONDITION (1). For $\omega=(\omega_1, \cdots, \omega_n)\in S^{n-1}$, we set

$$V(\omega)=\sum_{j=1}^{n} V^j\omega_j=\mathrm{diag}(v_1\cdot\omega, \cdots, v_m\cdot\omega), \tag{5.13}$$

where $v_i\cdot\omega=\sum_{j=1}^{n} v_i^j\omega_j$ for $i=1, \cdots, m$. CONDITION (3) in the present case reads as follows:

Let $\psi\in\boldsymbol{R}^m\backslash\{0\}$ and let $L_M\psi=0$. Then we have $\mu\psi+V(\omega)\psi\neq 0$ for any $\mu\in\boldsymbol{R}$ and $\omega\in S^{n-1}$.
Observe that $L_M\psi=0$ is equivalent to $\psi=\Lambda^{1/2}\phi$ for some $\phi\in\mathscr{M}$. Since $V(\omega)$ and $\Lambda^{1/2}$ are diagonal matrices, we have $\Lambda^{-1/2}V(\omega)\Lambda^{1/2}=V(\omega)$. Hence, the above mentioned condition is rewritten as

CONDITION (5). Let $\phi\in\mathscr{M}$ and let $\phi\neq 0$. Then $\mu\phi+V(\omega)\phi\neq 0$ for any $\mu\in\boldsymbol{R}$ and $\omega\in S^{n-1}$.

THEOREM 5.2 and 5.3 of [5] are improved as follows.

THEOREM 5.1. *Suppose* CONDITION 5.1. *Let $M$ be a constant Maxwellian and define $L_M$ by* (5.12). *Suppose one of the four condtions in* THEOREM 1.1 *for $A^0=I$, $A^j=V^j$, $B^{jk}=0$, $L=L_M$ or* CONDITION (5) *stated above. Let $F(0, x)=F_0(x)$. Then we have:*

(i) *Let $n\geq 1$, $s\geq[n/2]+1$ and let $F_0-M\in H^s(\mathbf{R}^n)$. If $\|F_0-M\|_s$ is small enough, the initial value problem for* (5.4) *has a unieque global solution $F(t, x)$ such that*

$$F-M\in C^0(0, \infty; H^s(\mathbf{R}^n))\cap C^1(0, \infty; H^{s-1}(\mathbf{R}^n)).$$

*Furthermore,*

$$\|F(t)-M\|_s^2+\int_0^t\|D_xF(\tau)\|_{s-1}^2d\tau\leq C\|F_0-M\|_s^2, \tag{5.14}$$

*for any $t\in[0, \infty)$, where $C$ is a constant not depending on $t$. $F(t, x)$ converges to the Maxwellian $M$ uniformly in $x\in\mathbf{R}^n$ as $t\to\infty$.*

(ii) *Let $n$, $s$ be as in* (i). *Let $p=1$ if $n=1$ and let $p\in[1, 2)$ if $n\geq 2$. Let furthermore $F_0-M\in H^s(\mathbf{R}^n)\cap L^p(\mathbf{R}^n)$. It $\|F_0-M\|_{s,p}$ is small enough (see, for the definition of the norm,* THEOREM 4.1), *the solution obtained in* (i) *satisfies*

$$\|F(t)-M\|_s\leq C(1+t)^{-\gamma}\|F_0-M\|_{s,p}, \tag{5.15}$$

*where $C$ is a constant not depending on $t$, and $\gamma=n(1/2p-1/4)$.*

## § 6. Two-dimensional 8-velocity model

We teat the 8-velocity model introduced in [5]. The velocities $v_1, \cdots, v_8$ are given as follows.

$$v_1=(v, 0),\ v_2=(0, v),\ v_3=-v_1,\ v_4=-v_2,$$
$$v_5=(v, v),\ v_6=(-v, v),\ v_7=-v_5,\ v_8=-v_6.$$

Here $v$ is a positive constant. Note that $|v_i|=v$ for $i=1, \cdots, 4$ while $|v_i|=\sqrt{2}v$ for $i=5, \cdots, 8$ (See Figure 6.1). There exist 12 collisions in this model which are classified into three types.

type 1: $(v_1, v_3)\rightleftarrows(v_2, v_4)$.
type 2: $(v_5, v_7)\rightleftarrows(v_6, v_8)$.
type 3: $(v_1, v_6)\rightleftarrows(v_3, v_5)$, $(v_1, v_7)\rightleftarrows(v_3, v_8)$,
$(v_2, v_7)\rightleftarrows(v_4, v_6)$, $(v_2, v_8)\rightleftarrows(v_4, v_5)$.

Note that two collisions of the same type can be obtained from each other by permutation of the indices corresponding to a transformation of the square (the convex hull of $v_1, \cdots, v_8$) onto itself. This is not the case if the types of two collisions are different. The restitution of a collision, which is obtained by interchanging the initial and the final states with each other, and the original collision are of the same type. Let $1\leq s\leq 3$. We set

$A_{ij}^{kl}=\sigma_s/2$, if $(v_i, v_j)\to(v_k, v_l)$ is a collision of type $s$,

$A_{ij}^{kl}=0$, otherwise.

Here, $\sigma_1, \cdots, \sigma_3$ are positive constants. It is easily seen that CONDITION 5.1 holds for this model. We set $\alpha_i=1$ for $i=1, \cdots, 8$. The equation studied is (5.1) or (5.4) where

$$V^1=v\ \mathrm{diag}(1,0,-1,0,1,-1,-1,1),\\ V^2=v\ \mathrm{diag}(0,1,0,-1,1,1,-1,-1). \tag{6.1}$$

The components of the quadratic term $Q(F, F)$ are, for example,

$$Q_1(F, F)=\sigma_1(F_2F_4-F_1F_3)+\sigma_3\{(F_3F_5-F_1F_6)+(F_3F_8-F_1F_7)\},$$
$$Q_5(F, F)=\sigma_2(F_6F_8-F_5F_7)+\sigma_3\{(F_1F_6-F_3F_5)+(F_2F_8-F_4F_5)\},$$

et cetera. Now we want to verify CONDITION (5). For this purpose, we determine the space of summational invariants $\mathscr{M}$. Let $\phi={}^t(\phi_1, \cdots, \phi_8)\in\mathscr{M}$. Then $\phi$ satisfies the following system of linear homogeneous equations.

$$\phi_2+\phi_4-(\phi_1+\phi_3)=0,\quad \phi_6+\phi_8-(\phi_5+\phi_7)=0,$$
$$\phi_3+\phi_5-(\phi_1+\phi_6)=0,\quad \phi_4+\phi_6-(\phi_2+\phi_7)=0.$$

The converse is true also. It follows that a basis of $\mathscr{M}$ is given by

$$\phi^{(1)}={}^t(1,1,1,1,1,1,1,1),\\ \phi^{(2)}={}^t(1,0,-1,0,1,-1,-1,1),\\ \phi^{(3)}={}^t(0,1,0,-1,1,1,-1,-1),\\ \phi^{(4)}={}^t(1,1,1,1,2,2,2,2). \tag{6.2}$$

In particular, $\dim\mathscr{M}=4$. Note that $\phi^{(1)}, \cdots, \phi^{(4)}$ correspond to the conservation of the number of particles, $x_1$ and $x_2$ components of the momentum and the energy, respectively. Any $\phi\in\mathscr{M}$ is expressed as

$$\phi=\sum_{j=1}^{4}\alpha_j\phi^{(j)},\quad \alpha_j\in\boldsymbol{R}. \tag{6.3}$$

Let $\omega=(\omega_1, \omega_2)\in S^1$. By (6.1),

$$V(\omega)=\mathrm{diag}(v_1\cdot\omega, \cdots, v_8\cdot\omega)\\ =v\ \mathrm{diag}(\omega_1, \omega_2, -\omega_1, -\omega_2, \omega_1+\omega_2, -\omega_1+\omega_2, -\omega_1-\omega_2,\\ \omega_1-\omega_2).$$

Let $e^{(1)}, \cdots, e^{(8)}$ be the standard basis of $\boldsymbol{R}^8$. Let $\phi\neq 0$ and let $\mu\phi+V(\omega)\phi=0$. This implies that $\mu=-v_j\cdot\omega$ for some $j\in\{1, \cdots, 8\}$ and hence

$$J(\mu, \omega)=\{k;\ k\in\{1, \cdots, 8\},\ \mu=-v_k\cdot\omega\}\neq\phi.$$

Furthermore, we have

$$\phi=\sum_{k\in J(\mu, \omega)}\beta_k e^{(k)},\quad \beta_k\in\boldsymbol{R}. \tag{6.5}$$

We set

$$\mathscr{P}=\{J(\mu, \omega);\ \mu\in\boldsymbol{R},\ \omega\in S^1\}.$$

Any $J\in\mathscr{P}$ will be called a $P$-set in the following. The empty set is not a $P$-set by definition. Let $J$ be a $P$-set. If there exists a $P$-set which contains $J$ as a proper subset, $J$ is called a $P$-set of the second category. All other $P$-sets are of the first category. We denote by $\mathscr{Q}$ and $\mathscr{R}$ the totality of the

$P$-sets of the first and the second categories, respectively. Hence $\mathscr{P}=\mathscr{Q}\cup\mathscr{R}$, $\mathscr{Q}\cap\mathscr{R}=\phi$. Let $J\in\mathscr{Q}$. We denote by $|J|$ the cardinality of $J$. We set

$$\mathscr{Q}_i=\{J\,;\,J\in\mathscr{Q},\ |J|=i\},$$

for $1\leq i\leq 8$. It turns out that $\mathscr{Q}_i\neq\phi$ for $i=2$, 3, and $\mathscr{Q}_i=\phi$ for other $i$. More precisely,

1°) $\mathscr{Q}_2$ is classified into four classes:
$\mathscr{Q}_2(A)\ni\{1, 2\}$ and $|\mathscr{Q}_2(A)|=4$,
$\mathscr{Q}_2(B)\ni\{1, 3\}$ and $|\mathscr{Q}_2(B)|=2$,
$\mathscr{Q}_2(C)\ni\{5, 7\}$ and $|\mathscr{Q}_2(C)|=2$,
$\mathscr{Q}_2(D)\ni\{1, 6\}$ and $|\mathscr{Q}_2(D)|=8$,

2°) $\mathscr{Q}_3\ni\{1, 5, 8\}$ and $|\mathscr{Q}_3|=4$.

Let $J_1$ and $J_2$ be two subsets of the integers $\{1,\cdots,8\}$. If $\{v_j\,;j\in J_1\}$ and $\{v_j\,;j\in J_2\}$ are obtained from each other by a two dimensional orthogonal transformation which maps the square (the convex hull of $v_1,\cdots,v_8$) onto itself, we write $J_1\sim J_2$. It is clear that this is an equivalence relation in $\mathscr{P}$ and hence in $\mathscr{Q}$. The classes enumerated above are equivalent classes by this equivalence relation. Only a representative of each class is given there. The $P$-sets of the second category are less important for our purpose. We define $\mathscr{R}_i$ in a similar way. Then, $\mathscr{R}_i=\phi$ for $2\leq i\leq 8$ and $|\mathscr{R}_1|=8$.

Now we turn to check CONDITION (5). It is sufficient to show that, if $\phi$ satisfies (6.3) and (6.5), then $\phi=0$. Observe that

$$\sum_{j=1}^{4}\alpha_j\phi^{(j)}={}^t((\alpha_1+\alpha_4)+\alpha_2,\ (\alpha_1+\alpha_4)+\alpha_3,\ (\alpha_1+\alpha_4)-\alpha_2,\ (\alpha_1+\alpha_4)-\alpha_3,\ (\alpha_1+2\alpha_4)+(\alpha_2+\alpha_3),\ (\alpha_1+2\alpha_4)-(\alpha_2-\alpha_3),\ (\alpha_1+2\alpha_4)-(\alpha_2+\alpha_3),\ (\alpha_1+2\alpha_4)+(\alpha_2-\alpha_3)).$$

It follows from

$$\sum_{j=1}^{4}\alpha_j\phi^{(j)}=\sum_{k=1}^{8}\beta_k e^{(k)}$$

that

(6.6)$_1$ $\quad\alpha_1+\alpha_4=(\beta_1+\beta_3)/2=(\beta_2+\beta_4)/2,$

(6.6)$_2$ $\quad\alpha_1+2\alpha_4=(\beta_5+\beta_7)/2=(\beta_6+\beta_8)/2,$

(6.6)$_3$ $\quad\alpha_2=(\beta_1-\beta_3)/2,\ \alpha_3=(\beta_2-\beta_4)/2,$

(6.6)$_4$ $\quad\alpha_2+\alpha_3=(\beta_5-\beta_7)/2,\ \alpha_2-\alpha_3=(-\beta_6+\beta_8)/2.$

We examine all the possibilities as follows.

Case 1A. Let $\phi=\beta_1e^{(1)}+\beta_2e^{(2)}$. We get (6.6)$_1\sim$(6.4)$_4$ where $\beta_j=0$ for $j\neq 1$, 2. From (6.6)$_4$ follows $\alpha_2=\alpha_3=0$. Hence, by (6.6)$_3$, $\beta_1=\beta_2=0$. Therefore, $\phi=0$.

Case 1B. Let $\phi=\beta_1e^{(1)}+\beta_3e^{(3)}$. We get (6.6)$_1\sim$(6.6)$_4$ where $\beta_j=0$ for $j\neq 1$, 3. It follows from (6.6)$_4$ that $\alpha_2=0$. Hence, by (6.6)$_3$, $\beta_1-\beta_3=0$.

On the other hand, $\beta_1+\beta_3=0$ by $(6.6)_1$. Combining these, we obtain $\beta_1=\beta_3=0$. Therefore, $\phi=0$.

Case 1C. Let $\phi=\beta_5 e^{(5)}+\beta_7 e^{(7)}$. We get $(6.6)_1\sim(6.6)_4$ where $\beta_j=0$ for $j\neq 5.7$. By $(6.6)_3$, $\alpha_2=\alpha_3=0$. Substituting this into $(6.6)_4$, we obtain $\beta_5-\beta_7=0$. On the other hand, $\beta_5+\beta_7=0$ by $(6.6)_4$. Hence, $\beta_5=\beta_7=0$. Therefore, $\phi=0$.

Case 1D. Let $\phi=\beta_1 e^{(1)}+\beta_6 e^{(6)}$. We get $(6.6)_1\sim(6.6)_4$ where $\beta_j=0$ for $j\neq 1, 6$. By $(6.6)_4$, $\beta_1=0$. Also, by $(6.6)_2$, $\beta_6=0$. Hence, $\phi=0$.

Case 2. Let $\phi=\beta_1 e^{(1)}+\beta_5 e^{(5)}+\beta_8 e^{(8)}$. We get $(6.6)_1\sim(6.6)_4$ where $\beta_j=0$ for $j\neq 1, 5, 8$. From $(6.6)_1$ follows $\beta_1=0$. Then, by $(6.6)_3$, $\alpha_2=\alpha_3=0$. Substituting this into $(6.6)_4$, we obtain $\beta_5=\beta_8=0$. Hence, $\phi=0$.

We conclude therefore that CONDITION (5) is satisfied for this model. Hence THEOREM 5.1 holds for the two-dimensional 8-velocity model. We remark finally that in [5] CONDITION 1.2 was verified by constructing $K^1$ and $K^2$ explicitly for some Maxwellian $M$.

## §7 Three-dimensional 14-velocity model

We investigate the 14-velocity model introduced by Cabannes [1]. The velocities are given by

$$v_1=(v,0,0),\ v_2=(0,v,0),\ v_3=(0,0,v),\ v_{j+3}=-v_j(j=1,2,3),$$
$$v_7=(v,v,v),\ v_8=(-v,v,v),\ v_9=(-v,-v,v),\ v_{10}=(v,-v,v),$$
$$v_{k+4}=-v_k(k=7,8,9.10).$$

Here $v$ is a positive constant. Note that $|v_i|=v$ for $i=1,\cdots,6$ while $|v_i|=\sqrt{3}v$ for $i=7,\cdots,14$. There exist 54 nontrivial collisions which preserve the momentum and the energy. These collisions are classified into four types.

type 1: $(v_1, v_4)\rightleftarrows(v_2, v_5)$, $(v_1, v_4)\rightleftarrows(v_3, v_6)$
$(v_2, v_5)\rightleftarrows(v_3, v_6)$

type 2: $(v_7, v_{11})\rightleftarrows(v_8, v_{12})$, $(v_7, v_{11})\rightleftarrows(v_9, v_{13})$,
$(v_7, v_{11})\rightleftarrows(v_{10}, v_{14})$, $(v_8, v_{12})\rightleftarrows(v_9, v_{13})$,
$(v_8, v_{12})\rightleftarrows(v_{10}, v_{14})$, $(v_9, v_{13})\rightleftarrows(v_{10}, v_{14})$.

type 3: $(v_7, v_9)\rightleftarrows(v_8, v_{10})$, $(v_7, v_{12})\rightleftarrows(v_{10}, v_{13})$,
$(v_7, v_{14})\rightleftarrows(v_8, v_{13})$, $(v_8, v_{11})\rightleftarrows(v_9, v_{14})$,
$(v_9, v_{12})\rightleftarrows(v_{10}, v_{11})$, $(v_{11}, v_{13})\rightleftarrows(v_{12}, v_{14})$.

type 4: $(v_1, v_8)\rightleftarrows(v_4, v_7)$, $(v_1, v_9)\rightleftarrows(v_4, v_{10})$,
$(v_1, v_{11})\rightleftarrows(v_4, v_{12})$, $(v_1, v_{14})\rightleftarrows(v_4, v_{13})$,
$(v_2, v_9)\rightleftarrows(v_5, v_8)$, $(v_2, v_{11})\rightleftarrows(v_5, v_{14})$,
$(v_2, v_{12})\rightleftarrows(v_5, v_{13})$, $(v_2, v_{10})\rightleftarrows(v_5, v_7)$,
$(v_3, v_{13})\rightleftarrows(v_6, v_7)$, $(v_3, v_{14})\rightleftarrows(v_6, v_8)$,
$(v_3, v_{11})\rightleftarrows(v_6, v_9)$, $(v_3, v_{12})\rightleftarrows(v_6, v_{10})$.

The principle for the classification is analoguous to that for the 8-velocity model. A modification needed is to replace the square by the cube (the convex hull of $v_1, \cdots, v_{14}$). Let $1\leq s\leq 4$. We define

$A_{ij}^{kl}=\sigma_s/2$, if $(v_i, v_j)\rightarrow(v_k, v_l)$ represents a collision of type s,

$A_{ij}^{kl}=0$, otherwise.

Here $\sigma_1, \cdots, \sigma_4$ are positive constants. It is clear that CONDITION 5.1 is satisfied for this model. We set $\alpha_i=1$ for $1\leq i\leq 14$. The equation to be considered is (5.1) or (5.4) where the explicit forms of $V^1$, $V^2$ and $V^3$ are given by

$$\begin{aligned}&V^1=v\ \mathrm{diag}(1,0,0,-1,0,0,1,-1,-1,1,-1,1,1,-1),\\ (7.1)\qquad &V^2=v\ \mathrm{diag}(0,1,0,0,-1,0,1,1,-1,-1,-1,-1,1,1),\\ &V^3=v\ \mathrm{diag}(0,0,1,0,0,-1,1,1,1,1,-1,-1,-1,-1).\end{aligned}$$

The components of $Q(F, F)$ are, for example,

$$\begin{aligned}Q_1(F, F)=&\sigma_1\{(F_2F_5-F_1F_4)+(F_3F_6-F_1F_4)\}+\sigma_4\{(F_4F_7-F_1F_8)\\ &+(F_4F_{10}-F_1F_9)+(F_4F_{12}-F_1F_{11})+(F_4F_{13}-F_1F_{14})\},\\ Q_7(F, F)=&\sigma_2\{F_8F_{12}-F_7F_{11})+(F_9F_{13}-F_7F_{11})+(F_{10}F_{14}-F_7F_{11})\}\\ &+\sigma_3\{F_8F_{10}-F_7F_9)+(F_{10}F_{13}-F_7F_{12})+(F_8F_{13}-F_7F_{14})\}\\ &+\sigma_4\{(F_1F_8-F_4F_7)+(F_2F_{10}-F_5F_7)+(F_3F_{13}-F_6F_7)\},\end{aligned}$$

et cetera.

First we determine the subspace $\mathscr{M}$. By elementary computations, we see that $\phi={}^t(\phi_1, \cdots, \phi_{14})\in\mathscr{M}$ if and only if $\phi$ satisfies the following system of linear homogeneous equations.

$$\begin{aligned}&\phi_1+\phi_4-(\phi_2+\phi_5)=0,\ \phi_1+\phi_4-(\phi_3+\phi_6)=0,\\ &\begin{cases}\phi_7+\phi_{11}-(\phi_8+\phi_{12})=0,\ \phi_7+\phi_{11}-(\phi_9+\phi_{13})=0,\\ \phi_7+\phi_{11}-(\phi_{10}+\phi_{14})=0,\end{cases}\\ &\phi_7+\phi_9-(\phi_8+\phi_{10})=0,\\ &\begin{cases}\phi_1+\phi_8-(\phi_4+\phi_7)=0,\ \phi_2+\phi_9-(\phi_5+\phi_8)=0,\\ \phi_3+\phi_{13}-(\phi_6+\phi_7)=0.\end{cases}\end{aligned}$$

It is shown that $\dim\mathscr{M}=5$. A basis of $\mathscr{M}$ is given by

$$\begin{aligned}&\phi^{(1)}={}^t(1,1,1,1,1,1,1,1,1,1,1,1,1,1),\\ &\phi^{(2)}={}^t(1,0,0,-1,0,0,1,-1,-1,1,-1,1,1,-1),\\ (7.2)\qquad &\phi^{(3)}={}^t(0,1,0,0,-1,0,1,1,-1,-1,-1,-1,1,1),\\ &\phi^{(4)}={}^t(0,0,1,0,0,-1,1,1,1,1,-1,-1,-1,-1),\\ &\phi^{(5)}={}^t(1,1,1,1,1,1,3,3,3,3,3,3,3,3).\end{aligned}$$

Note that $\phi^{(1)}$ and $\phi^{(5)}$ correspond to the conservation of the number of particles and the energy, respectively, while $\phi^{(2)}$, $\phi^{(3)}$ and $\phi^{(4)}$ are related to the conservation of the momentum in $x_1$, $x_2$ and $x_3$ directions, respectively. Any $\phi\in\mathscr{M}$ can be expressed as

$$(7.3)\qquad \phi=\sum_{j=1}^{5}\alpha_j\phi^{(j)},\ \alpha_j\in\boldsymbol{R}.$$

Let $\omega=(\omega_1, \omega_2, \omega_3)\in S^2$. By (7.1), we have

$$\begin{aligned} V(\omega)&=\operatorname{diag}(v_1\cdot\omega, \cdots, v_{14}\cdot\omega) \\ &=v\ \operatorname{diag}(\omega_1, \omega_2, \omega_3, -\omega_1, -\omega_2, -\omega_3, \omega_1+\omega_2+\omega_3, \\ &\quad -(\omega_1-\omega_2-\omega_3), -(\omega_1+\omega_2-\omega_3), \omega_1-\omega_2+\omega_3, \\ &\quad -(\omega_1+\omega_2+\omega_3), \omega_1-\omega_2-\omega_3, \omega_1+\omega_2-\omega_3, -(\omega_1-\omega_2+\omega_3)). \end{aligned} \tag{7.4}$$

Let $e^{(1)}, \cdots, e^{(14)}$ be the standard basis of $\boldsymbol{R}^{14}$. Let $\phi\neq 0$ and let $\mu\phi+V(\omega)\phi=0$ for some $\mu\in\boldsymbol{R}$ and $\omega\in S^2$. Then,

$$J(\mu, \omega)=\{k;\ k\in\{1, \cdots, 14\},\ \mu=-v_k\cdot\omega\}\neq\phi.$$

We obtain

$$\phi=\sum_{k\in J(\mu,\omega)}\beta_k e^{(k)}, \quad \beta_k\in\boldsymbol{R}. \tag{7.5}$$

Let us set

$$\mathscr{P}=\{J(\mu, \omega);\ \mu\in\boldsymbol{R},\ \omega\in S^2\}.$$

Any $J\in\mathscr{P}$ is called a $P$-set. Namely, $J$ is a $P$-set if $J=J(\mu, \omega)$ for some $\mu\in\boldsymbol{R}$ and $\omega\in S^{n-1}$ and in addition $J\neq\phi$. The empty set is excluded from $\mathscr{P}$ by definition. We define also the first and the second categories for $P$-sets as in the preceeding section. We denote by $\mathscr{Q}$ the totality of the $P$-sets of the first category and set

$$\mathscr{Q}_i=\{J;J\in\mathscr{Q},\ |J|=i\},$$

for $1\leq i\leq 14$. It is shown that $\mathscr{Q}_i\neq\phi$ only for $3\leq i\leq 6$. We use the equivalence relation introduced in the preceeding section with slight modification: Two dimensional orthogonal transformation which maps the square onto itself is replaced by three dimensional orthogonal transformation mapping the cube (the convex hull of $v_1, \cdots, v_{14}$) onto itself. Thus we obtain the following result.

1°) $\mathscr{Q}_3$ consists of two classes:
$\mathscr{Q}_3(A)\ni\{1, 2, 9\}$ and $|\mathscr{Q}_3(A)|=24$,
$\mathscr{Q}_3(B)\ni\{1, 8, 9\}$ and $|\mathscr{Q}_3(B)|=24$.

2°) $\mathscr{Q}_4\ni\{1, 2, 4, 5\}$ and $|\mathscr{Q}_4|=3$.

3°) $\mathscr{Q}_5\ni\{1, 7, 10, 12, 13\}$ and $|\mathscr{Q}_5|=6$.

4°) $\mathscr{Q}_6$ consists of two classes:
$\mathscr{Q}_6(A)\ni\{1, 2, 3, 8, 10, 13\}$ and $|\mathscr{Q}_6(A)|=8$,
$\mathscr{Q}_6(B)\ni\{1, 4, 7, 8, 11, 12\}$ and $|\mathscr{Q}_6(B)|=6$.

We denote by $\mathscr{R}$ the totality of the $P$-sets of the second category and define $\mathscr{R}_i$ by

$$\mathscr{R}_i=\{J;J\in\mathscr{R},\ |J|=i\}$$

for $1\leq i\leq 14$. Then $\mathscr{R}_i\neq\phi$ only for $i=1, 2, 3$. We note that $|\mathscr{R}_1|=14$, $|\mathscr{R}_2|=55$ and $|\mathscr{R}_3|=12$.

We shall show that $\phi=0$ if $\phi$ satisfies (7.3) and (7.5). We note that

$$\sum_{j=1}^{5}\alpha_j\phi^{(j)}={}^t((\alpha_1+\alpha_5)+\alpha_2,\ (\alpha_1+\alpha_5)+\alpha_3,\ (\alpha_1+\alpha_5)+\alpha_4,\ (\alpha_1+\alpha_5)-\alpha_2,$$

$$(\alpha_1+\alpha_5)-\alpha_3,\ (\alpha_1+\alpha_5)-\alpha_4,\ (\alpha_1+3\alpha_5)+(\alpha_2+\alpha_3+\alpha_4),$$
$$(\alpha_1+3\alpha_5)-(\alpha_2-\alpha_3-\alpha_4),\ (\alpha_1+3\alpha_5)-(\alpha_2+\alpha_3-\alpha_4),$$
$$(\alpha_1+3\alpha_5)+(\alpha_2-\alpha_3+\alpha_4),\ (\alpha_1+3\alpha_5)-(\alpha_2+\alpha_3+\alpha_4),$$
$$(\alpha_1+3\alpha_5)+(\alpha_2-\alpha_3-\alpha_4),\ (\alpha_1+3\alpha_5)+(\alpha_2+\alpha_3-\alpha_4),$$
$$(\alpha_1+3\alpha_5)-(\alpha_2-\alpha_3+\alpha_4)).$$

Let

$$\sum_{j=1}^{5}\alpha_j\phi^{(j)}=\sum_{k=1}^{14}\beta_k e^{(k)}.$$

Then we get the following system of linear equations.

$(7.6)_1$ $\quad \alpha_1+\alpha_5=(\beta_1+\beta_4)/2=(\beta_2+\beta_5)/2=(\beta_3+\beta_6)/2,$

$(7.6)_2$ $\quad \alpha_1+3\alpha_5=(\beta_7+\beta_{11})/2=(\beta_8+\beta_{12})/2=(\beta_9+\beta_{13})/2=(\beta_{10}+\beta_{14})/2,$

$(7.6)_3$ $\quad \alpha_2=(\beta_1-\beta_4)/2,\ \alpha_3=(\beta_2-\beta_5)/2,\ \alpha_4=(\beta_3-\beta_6)/2,$

$(7.6)_4$ $\quad \alpha_2+\alpha_3+\alpha_4=(\beta_7-\beta_{11})/2,\ \alpha_2-\alpha_3-\alpha_4=(-\beta_8+\beta_{12})/2,$
$\alpha_2+\alpha_3-\alpha_4=(-\beta_9+\beta_{13})/2,\ \alpha_2-\alpha_3+\alpha_4=(\beta_{10}-\beta_{14})/2.$

Case 1A. Let $\phi=\beta_1e^{(1)}+\beta_2e^{(2)}+\beta_9e^{(9)}$. We obtain $(7.6)_1\sim(7.6)_4$ where $\beta_j=0$ for $j\neq1,2,9$. From $(7.6)_1$ follows $\beta_1=\beta_2=0$. Hence, by $(7.6)_3$, $\alpha_2=\alpha_3=\alpha_4=0$. Substituting this into $(7.6)_4$, we get $\beta_9=0$. Hence $\phi=0$.

Case 1B. Let $\phi=\beta_1e^{(1)}+\beta_8e^{(8)}+\beta_9e^{(9)}$. We obtain $(7.6)_1\sim(7.6)_4$ where $\beta_j=0$ for $j\neq1,8,9$. From $(7.6)_1$ follows $\beta_1=0$. Also, by $(7.6)_2$, $\beta_8=\beta_9=0$. Hence $\phi=0$.

Case 2. Let $\phi=\beta_1e^{(1)}+\beta_2e^{(2)}+\beta_4e^{(4)}+\beta_5e^{(5)}$. We obtain $(7.6)_1\sim(7.6)_4$ where $\beta_j=0$ for $j\neq1,2,4,5$. It follows from $(7.6)_4$ that $\alpha_2=\alpha_3=\alpha_4=0$. Substituting this into $(7.6)_3$, we get $\beta_1-\beta_4=\beta_2-\beta_5=0$. On the other hand, $\beta_1+\beta_4=\beta_2+\beta_5=0$ follows from $(7.6)_1$. Hence $\beta_1=\beta_2=\beta_4=\beta_5=0$, which implies $\phi=0$.

Case 3. Let $\phi=\beta_1e^{(1)}+\beta_7e^{(7)}+\beta_{10}e^{(10)}+\beta_{12}e^{(12)}+\beta_{13}e^{(13)}$. We obtain $(7.6)_1\sim(7.6)_4$ where $\beta_j=0$ for $j\neq1,7,10,12,13$. From $(7.6)_1$ follows $\beta_1=0$. Hence, by $(7.6)_3$, $\alpha_2=\alpha_3=\alpha_4=0$. Substituting this into $(7.6)_4$, we get $\beta_7=\beta_{10}=\beta_{12}=\beta_{13}=0$. Hence $\phi=0$.

Case 4A. Let $\phi=\beta_1e^{(1)}+\beta_2e^{(2)}+\beta_3e^{(3)}+\beta_8e^{(8)}+\beta_{10}e^{(10)}+\beta_{13}e^{(13)}$. We obtain $(7.6)_1\sim(7.6)_4$ where $\beta_j=0$ for $j\neq1,2,3,8,10,13$. From $(7.6)_2$ follows $\beta_8=\beta_{10}=\beta_{13}=0$. Substituting this into $(7.6)_4$, we get $\alpha_2=\alpha_3=\alpha_4=0$. Hence, by $(7.6)_3$, $\beta_1=\beta_2=\beta_3=0$. Therefore, $\phi=0$.

Case 4B. Let $\phi=\beta_1e^{(1)}+\beta_4e^{(4)}+\beta_7e^{(7)}+\beta_8e^{(8)}+\beta_{11}e^{(11)}+\beta_{12}e^{(12)}$. We obtain $(7.6)_1\sim(7.6)_4$ where $\beta_j=0$ for $j\neq1,4,7,8,11,12$. From $(7.6)_3$ follows $\alpha_3=\alpha_4=0$. Combining this with the third and the fourth equations of $(7.6)_4$, we get $\alpha_2=0$. Substituting this into the first and the second equations of $(7.6)_4$ gives $\beta_7-\beta_{11}=\beta_8-\beta_{12}=0$. On the other hand, $\beta_7+\beta_{11}=\beta_8+\beta_{12}=0$ follows from $(7.6)_2$. Hence $\beta_7=\beta_8=\beta_{11}=\beta_{12}=0$. Since $\alpha_2=0$, $\beta_1-\beta_4=0$ by $(7.6)_3$. Also, $\beta_1+\beta_4=0$ follows from $(7.6)_1$. Hence $\beta_1=\beta_4=0$. We conclude that

$\phi=0$.

Thus CONDITION (5) is verified. This means that THEOREM 5.1 can be applied to the three-dimensional 14-velocity model by Cabannes.

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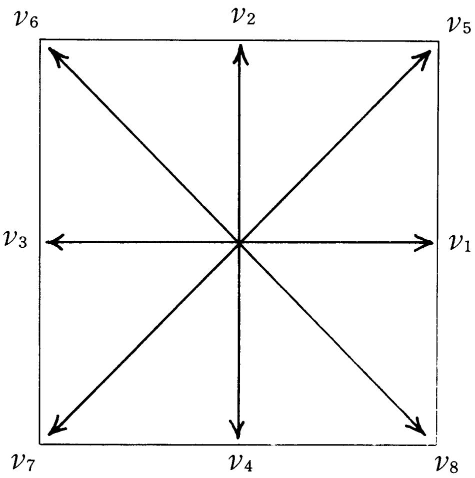


**Figure 6.1**

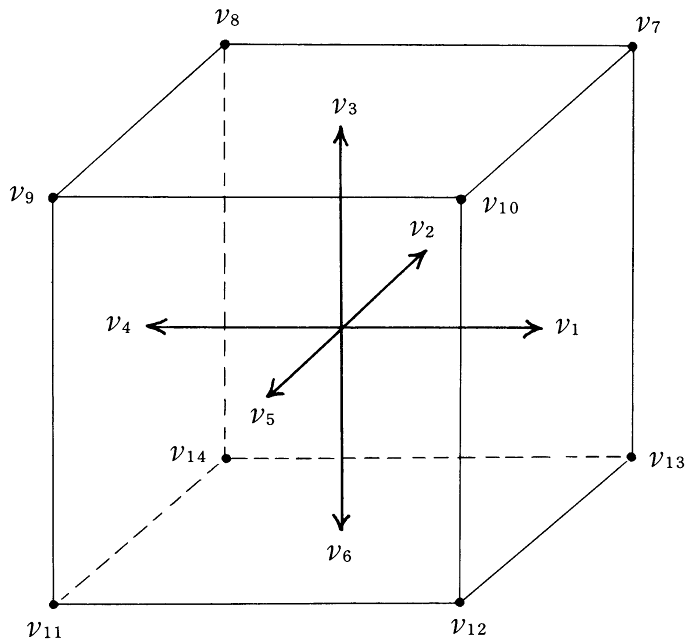


**Figure 7.1**