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Characterizations of subspaces, quotients and subspaces of quotients of L_p

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§ 1. Introduction

In his paper [5] Kwapien has shown that the following: Let E be a Banach space, $1<p<\infty$ and $1/p+1/p'=1$. Then E is isomorphic with a subspace of L_p if and only if every dual p-absolutely summing operator from $l_{p'}$ into E is p-absolutely summing; E is isomorphic with a quotient of L_p if and only if every dual p-integral operator from $l_{p'}$ into E is p-integral; and E is isomorphic with a subspace of a quotient of L_p if and only if every dual p-integral operator from $l_{p'}$ into E is p-absolutely summing. In this paper, we shall extend his results. Let X be a Banach space whose dual X' is a closed subspace of l_p. Let f_n be the projection of X' onto the n-th coordinate. Then we have $f_n \in X = X''$. The sequence $\{f_n\}$ is called that of unit vectors in X. In particular, we denote by $\{e_n\}$ the sequence of unit vectors in $l_{p'}$. Then the main results of this paper are the following:

(1) E is isomorphic with a subspace of L_p if and only if for each Banach space X with $X' \subset l_p$ and each operator $u: X \to E$, $\sum \| u(f_n)\|^p < \infty$ implies u is p-absolutely summing.

(2) E is isomorphic with a quotient of L_p if and only if for each operator $u: l_{p'} \to E$, $\sum \| u(e_n)\|^p < \infty$ implies u is p-integral.

(3) E is isomorphic with a subspace of a quotient of L_p if and only if for each operator $u: l_{p'} \to E$, $\sum \| u(e_n)\|^p < \infty$ implies u is p-absolutely summing.

REMARK. Similar characterizations of subspaces of L_p were given by Holub [2], Kalton and Ruckle [3], and Lindenstrauss and Pelczyński [6] (see also Cohen [1] and Kwapien [4] for the case $p=2$.) Let us recall that an operator $u: l_{p'} \to E$ is called dual p-decomposed if $\sum \| u(e_n)\|^p < \infty$. From (1) it follows that E is isomorphic with a subspace of L_p if and only if every dual sub-p-nuclear operator from $l_{p'}$ into E is dual p-decomposed. This extends the results of [2] and [5]. Note that sub-p-nuclear operators are always p-absolutely summing, but in general, the converse is not true (see Persson [7]). On the other hand, (2) and (3) extend the results of

Kwapien [5] since every dual p-decomposed operator from $l_{p'}$ into E is dual p-integral. Finally, we note that E is isomorphic with a subspace of a quotient of L_p if and only if every dual p-integral (or p-nuclear) operator from $l_{p'}$ into E is dual p-decomposed.

§ 2. Definitions and notations

Let E be a Banach space with the dual E', p be a real number such that $1 \leqq p < \infty$ and $1/p + 1/p' = 1$. Denote by $L(E, F)$ the set of all continuous linear operators from E into a Banach space F. For an operator u in $L(E, F)$, the adjoint of u will be denoted by u'.

DEFINITION 2.1. An operator u in $L(E, F)$ is called p-nuclear if it has a factorization of the form

$$E \to l_\infty \xrightarrow{(\alpha)} l_p \to F$$

where (α) is multiplication by an element α in l_p.

The set of all p-nuclear operators from E into F will be denoted by $N_p(E, F)$; $N_p(E, F)$ is clearly a linear space. For $p=1$ the class $N_1(E, F)$ coincides with the space of all nuclear operators from E into F.

Similarly, we can define "sub-p-nuclear operator" by introducing a sub-factorization in the obvious way.

DEFINITION 2.2. An operator u in $L(E, F)$ is called p-integral if it has a factorization of the form

$$E \to L_\infty(\Omega, \mu) \xrightarrow{i} L_p(\Omega, \mu) \to F$$

where (Ω, μ) is a probability space and i is the natural injection.

The set of all p-integral operators from E into F will be denoted by $I_p(E, F)$; $I_{p'}(E, F)$ is clearly a linear space. Of course the inclusion $N_p(E, F) \subset I_p(E, F)$ always holds, but in general, the converse inclusion does not holds. It is known that if E is reflexive or E' is separable, then for each Banach space F the inclusion $I_p(E, F) \subset N_p(E, F)$ holds (see Persson [7], Corollary 1 and Theorem 5).

DEFINITION 2.3. An operator u in $L(E, F)$ is called p-absolutely summing if there exists a constant $C>0$ such that for each $x_1, x_2, \dots, x_n$ in E the inequality

$$\left(\sum_{i=1}^n \|u(x_i)\|^p\right)^{1/p} \leqq C \sup\left\{\left(\sum_{i=1}^n |<x_i, x'>|^p\right)^{1/p}; x' \in E', \|x'\| \leqq 1\right\}$$

holds.

The set of all p-absolutely summing operators from E into F will be denoted by $\Pi_p(E, F)$. It is known that $\Pi_p(E, F)$ is a linear space and in fact a Banach space when equipped with the norm $\Pi_p(u)=\inf C$.

Let us recall that a sequence $\{x_n\}$ in E is called weakly p-summable if $\sum|<x_n, x'>|^p<\infty$ for all $x'\in E'$, and it is also called absolutely p-summable if $\sum\|x_n\|^p<\infty$. It is easy to see that an operator u in $L(E, F)$ is p-absolutely summing if and only if it takes each weakly p-summable sequence $\{x_n\}$ in E into an absolutely p-summable sequence $\{u(x_n)\}$ in F. (For the details of p-absolutely summing operators; see Pietsch [8] and [9].)

§ 3. Main results

We shall say that a Banach space E is of S_p type, resp. Q_p type, resp. SQ_p type, resp. QS_p type if it is isomorphic with a subspace of L_p, resp. with a quotient of L_p, resp. with a subspace of a quotient of L_p, resp. with a quotient of a subspace of L_p. Let us mention that every Banach space is of Q_1 type, of SQ_1 type and of QS_1 type. As mentioned in Section 1, if X is a Banach space whose dual X' is a closed subspace of l_p, $1<p<\infty$, then we denote by $\{f_n\}$ the sequence of unit vectors in X. In prticular, we also denote by $\{e_n\}$ the sequence of unit vectors in $l_{p'}$, where $1/p+1/p'=1$.

We shall first give characterizations of Banach spaces of SQ_p type and QS_p type, which extend the results of Kwapien [5]. From now on let us assume that $1<p<\infty$ and $1/p+1/p'=1$.

THEOREM 3.1. *For a Banach space E, the following statements are equivalent.*

(1) *E is of QS_p type.*

(2) *E is of SQ_p type.*

(3) *For an operator u in $L(l_{p'}, E)$, $u\in\Pi_p(l_{p'}, E)$ if and only if $\sum\|u(e_n)\|^p<\infty$.*

(4) *For an operator u in $L(l_{p'}, E)$, $u'\in N_p(E', l_p)$ if and only if $\sum\|u(e_n)\|^p<\infty$.*

PROOF. We shall first show the equivalence of (1) and (2). Suppose (1) holds. Then there are a Banach space F of S_p type and a quotient map ϕ from F onto E. For each Banach space G if $u\in I_{p'}(E, G)$, then $u\phi\in I_{p'}(F, G)$, so that $\phi'u'\in\Pi_{p'}(G', F')$ by Kwapien [5], Corollary 8. Hence we have $u'\in\Pi_{p'}(G', E')$ since $\phi': E'\to F'$ is isomorphism. Using Kwapien [5], Corollary 8, it follows that E is of SQ_p type. On the other hand, suppose (2) holds. Since E' is of $QS_{p'}$ type, by the first proof (1)$\Longrightarrow$(2) it is also of

$SQ_{p'}$ type, so that E is of QS_p type.

We shall next show the implication (1)$\Longrightarrow$(4). Suppose (1) holds. In order to prove (4), it is enough to show that $u' \in N_p(E', l_p)$ implies $\sum \|u(e_n)\|^p < \infty$. However this follows from Kwapien [5], Corollary 8 and the fact that E' is of $SQ_{p'}$ type and every p-nuclear operator is also p-integral.

Finally, we shall show the implications (4)$\Longrightarrow$(3)$\Longrightarrow$(2).

(4)$\Longrightarrow$(3) Of course we only have to prove one implication. Suppose (4) is satisfied and let $\sum \|u(e_n)\|^p < \infty$, where $u \in L(l_{p'}, E)$. Then the operator $u' : E' \to l_p$ is clearly p-nuclear. Let $\{x_n\}$ be an weakly p-summable sequence in $l_{p'}$. If we define the operator v in $L(l_{p'}, l_{p'})$ by $v(e_n) = x_n$ for $n = 1, 2, \dots$, then $v'u' \in N_p(E', l_p)$, so that by the assumption (4) we get

$$\sum \|u(x_n)\|^p = \sum \|uv(e_n)\|^p < \infty,$$

which means $u \in \Pi_p(l_{p'}, E)$.

(3)$\Longrightarrow$(2) If we put

$$\Lambda_p(l_{p'}, E) = \{u \in L(l_{p'}, E) ; \quad |||u||| = (\sum \|u(e_n)\|^p)^{1/p} < \infty\},$$

then it is easy to see that $\Lambda_p(l_{p'}, E)$ is a Banach space with the norm $|||\cdot|||$. Suppose now (3) is satisfied. Since the identity map: $\Pi_p(l_{p'}, E) \to \Lambda_p(l_{p'}, E)$ is clearly continuous, by the closed graph theorem there is a constant $C > 0$ such that for each $u \in \Lambda_p(l_{p'}, E)$ there holds

$$(*) \qquad \Pi_p(u) \leqq C \ |||u||| .$$

Let $\{x_i\}$ be an absolutely p-summable sequence in E, and let (a_{ij}) be a matrix defining an operator v in $L(l_p, l_p)$. Define the operator w in $L(l_{p'}, E)$ by $w(e_i) = x_i$ for $i = 1, 2, \dots$. Since $w \in \Lambda_p(l_{p'}, E)$, by the assumption (3) $w \in \Pi_p(l_{p'}, E)$, and so we get $wv' \in \Pi_p(l_{p'}, E)$. It follows from $(*)$ that the estimations

$$\begin{aligned}(\sum \|wv'(e_i)\|^p)^{1/p} &\leqq \Pi_p(wv') \leqq \|v'\| \Pi_p(w) \\ &\leqq C\|v\| \cdot |||w||| = C\|v\| (\sum \|x_i\|^p)^{1/p}\end{aligned}$$

hold. Since $v'(e_i) = \sum_j a_{ij} e_j$ for $i = 1, 2, \dots$, we get

$$\sum_i \|wv'(e_i)\|^p = \sum_i \|\sum_j a_{ij} x_j\|^p.$$

Thus using Kwapien [5], Theorem 2′, it follows that E is of SQ_p type.

This completes the proof.

COROLLARY 3.2. *For a Banach space E, the following statements are* *equivalent.*

(1) *E is of SQ_p type.*

(2) *For each absolutely p-summable sequence $\{x_n\}$ in E, there exists a Banach subspace G of E which is of SQ_p type such that the sequence $\{x_n\}$ is contained in G and it is absolutely p-summable in G. (Here " Banach subspace " means that G is a linear subspace of E and itself is a Banach space such that the inclusion map : $G \to E$ is continuous.)*

(3) *Every separable subspace of E is of SQ_p type.*

Proof. Of course we only have to prove the implication (2)$\Longrightarrow$(1). Suppose (2) is satisfied and let $\sum \|u(e_n)\|^p < \infty$, where $u \in L(l_{p'}, E)$. If we can show $u \in \Pi_p(l_{p'}, E)$, then the assertion follows from Theorem 3. 1. In fact, let $x_n = u(e_n)$. Since the sequence $\{x_n\}$ in E is absolutely p-summable, by the assumption there exists a Banach subspace G of E as in (2). Since G is of SQ_p type and u may be regarded as a continuous linear operator from $l_{p'}$ into G, using Theorem 3. 1 it follows that $u \in \Pi_p(l_{p'}, G)$, and so we get $u \in \Pi_p(l_{p'}, E)$.

This completes the proof.

Next we shall give characterizations of Banach spaces of Q_p type, which extend a result of Kwapien [5].

Theorem 3. 3. *For a Banach space E, the following statements are equivalent.*

(1) *E is of Q_p type.*

(2) *For an operator u in $L(l_{p'}, E)$, $u \in I_p(l_{p'}, E)$ if and only if $\sum \|u(e_n)\|^p < \infty$.*

(3) *For an operator u in $L(l_{p'}, E)$, $u \in I_p(l_{p'}, E)$ if and only if $u' \in N_p(E', l_p)$.*

Proof. First we shall show (1)$\Longrightarrow$(3). Suppose (1) holds. Then E' is of $S_{p'}$ type. Hence the assertion follows from Kwapien [5], Corollary 4, since $N_p(E', l_p) \subset I_p(E', l_p)$.

(3)$\Longrightarrow$(2) is clear.

Finally, we shall show (2)$\Longrightarrow$(1). Suppose (2) holds. Since $I_p(l_{p'}, E) \subset \Pi_p(l_{p'}, E)$, by Theorem 3. 1 E is of SQ_p type, and hence it is reflexive. To prove (1), it is enough to see that E' is of $S_{p'}$ type. Let G be a Banach space and $u \in I_p(E', G)$. Since E is reflexive, by Persson [7], Corollary 1, $u \in N_p(E', G)$ so that it has a factorization of the form

$$E' \xrightarrow{v} l_\infty \xrightarrow{(\alpha)} l_p \xrightarrow{w} G$$

where $u=w(\alpha)v$ and (α) is multiplication by an element $\alpha\in l_p$. It is easy to see that $v'(\alpha)'\in L(l_{p'}, E)$ and $\sum\|v'(\alpha)'(e_n)\|^p<\infty$. Hence from the assumption (2) it follows that $v'(\alpha)'\in I_p(l_{p'}, E)$, and so we get $u'=v'(\alpha)'w'\in I_p(G', E)$. Thus using Kwapien [5], Corollary 4, E' is of $S_{p'}$ type.

This completes the proof.

COROLLARY 3.4. *For a Banach space E, the following statements are equivalent.*

(1) *E is of Q_p type.*

(2) *For each absolutely p-summable sequence $\{x_n\}$ in E, there exists a Banach subspace G of E which is of Q_p type such that the sequence $\{x_n\}$ is contained in G and it is absolutely p-summable in G.*

(3) *Every separable subspace of E is of Q_p type.*

Using Theorem 3.3, the proof can be done by the same way as in that of Corollary 3.2, and so we omit it.

Finally, we shall give characterizations of Banach spaces of S_p type, which extend the results of Holub [2], Kalton and Ruckle [3] and Kwapien [5]. In particular, taking $p=2$, we get characterizations of Banach spaces isomorphic with Hilbert spaces, which extend the results of Cohen [1] and Kwapien [4].

THEOREM 3.5. *For a Banach space E, the following statements are equivalent.*

(1) *E is of S_p type.*

(2) *For each Banach space X with $X'\subset l_p$ and each operator u in $L(X, E)$, $u\in\Pi_p(X, E)$ if and only if $\sum\|u(f_n)\|^p<\infty$.*

(3) *For an operator u in $L(l_{p'}, E)$, $\sum\|u(e_n)\|^p<\infty$ if and only if $u': E'\to l_p$ is sub-p-nuclear.*

(4) *For a sequence $\{x_i\}$ in E, if*

$$\sum_i\sum_j|<x_i, x_j'>|^p<\infty$$

for every weakly p-summable sequence $\{x_j'\}$ in E', then $\sum\|x_i\|^p<\infty$.

(5) *For a sequence $\{x_i\}$ in E, if $\sum\|u(x_i)\|^p<\infty$ for every u in $L(E, l_p)$, then $\sum\|x_i\|^p<\infty$.*

PROOF. (1)$\Longrightarrow$(5) Let $\{x_i\}$ be a sequence in E satisfying that

$$\sum\|u(x_i)\|^p<\infty \text{ for every } u\in L(E, l_p).$$

Then for each u in $L(E, l_p)$ and each sequence $\{\alpha_i\}$ of complex numbers such that $\sum|\alpha_i|^{p'}<\infty$, there holds

$$\sum\|u(\alpha_i x_i)\|\leqq(\sum\|u(x_i)\|^{p})^{1/p}(\sum|\alpha_i|^{p'})^{1/p'}<\infty.$$

Since E is of S_p type, using Kalton and Ruckle [3], Theorem, it follows that $\sum\|\alpha_i x_i\|<\infty$, and so we get $\sum\|x_i\|^p<\infty$ since the sequence $\{\alpha_i\}$ is arbitrary.

(5)$\Longrightarrow$(4) Let $\{x_i\}$ be a sequence in E satisfying that

$$\sum_i\sum_j|<x_i, x'_j>|^p<\infty$$

for every weakly p-summable sequence $\{x'_j\}$ in E'. Then for each u in $L(E, l_p)$, we get

$$\sum_i\|u(x_i)\|^p=\sum_i\sum_j|<u(x_i), e_j>|^p$$
$$=\sum_i\sum_j|<x_i, u'(e_j)>|^p<\infty$$

since the sequence $\{u'(e_j)\}$ in E' is weakly p-summable.
Hence by the assumption (5) we get $\sum\|x_i\|^p<\infty$.

(4)$\Longrightarrow$(3) Of course it is enough to show that for an operator u in $L(l_{p'}, E)$, if $u': E'\to l_p$ is sub-p-nuclear, then $\sum\|u(e_n)\|^p<\infty$. Suppose that u' is sub-p-nuclear. Then u' is clearly p-absolutely summing. Hence for each weakly p-summable sequence $\{x'_j\}$ in E', there holds

$$\sum_i\sum_j|<u(e_i), x'_j>|^p=\sum_i\sum_j|<e_i, u'(x'_j)>|^p$$
$$=\sum_j\|u'(x'_j)\|^p<\infty.$$

Thus by the assumption (4) we get $\sum\|u(e_i)\|^p<\infty$.

(3)$\Longrightarrow$(2) Let X be a Banach space whose dual X' is a closed subspace of l_p. Suppose that $\sum\|u(f_n)\|^p<\infty$, where $u\in L(X, E)$. Then it is easy to see that $u': E'\to X'$ is sub-p-nuclear. Now we shall prove that $u: X\to E$ is p-absolutely summing. Let $\{x_n\}$ be an weakly p-summable sequence in X. If we define the operator $v: l_p\to X$ by $v(e_n)=x_n$ for $n=1, 2, \ldots$, then $v'u': E'\to l_p$ is sub-p-nuclear since u' is sub-p-nuclear. Hence by the assumption (3) we get

$$\sum\|uv(e_n)\|^p=\sum\|u(x_n)\|^p<\infty,$$

which shows $u: X\to E$ is p-absolutely summing.

(2)$\Longrightarrow$(1) For the proof, we use the Lindenstrauss-Pelczyński criterion [6] embedding of a Banach space into L_p. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in E satisfying the following conditions

$$\sum|<y_n, x'>|^p\leqq\sum|<x_n, x'>|^p \text{ for all } x'\in E',$$

and

$$\sum \|x_n\|^p < \infty.$$

Consider the operator $u: E' \to l_p$ defined by

$$u(x') = (<x_n, x'>) \text{ for } x' \in E'.$$

If we put $G = u(E')$, then G is a normed space when equipped with the induced topology by l_p. Denote by X the dual of G. Evidently, X is a reflexive Banach space whose dual X' is a closed subspace of l_p, and u may be regarded as a continuous linear operator from E' into X'. Note that u has a dense range in X'. Hence we can define the operator $v: X' \to l_p$ by

$$v: (<x_n, x'>) \to (<y_n, x'>) \text{ for } x' \in E'.$$

Here we may assume that E is reflexive and in fact by the assumption (2) and Theorem 3, 1 it is of SQ_p type. Thus $u' \in L(X, E)$. Since

$$\sum \|u'(f_n)\|^p = \sum \|x_n\|^p < \infty,$$

from the assumption (2) it follows that $u' \in \Pi_p(X, E)$, and so $u'v' \in \Pi_p(l_{p'}, E)$. Consequently, we get

$$\sum \|y_n\|^p = \sum \|u'v'(e_n)\|^p < \infty.$$

Using Lindenstrauss-Pelczyński criterion [6] it follows that E is S_p type.

This completes the proof.

REMARK. In the statement (3) of Theorem 3. 5, "sub-p-nuclear" can be replaced by "p-absolutely summing", but it can not be replaced by "p-nuclear" or "p-integral" (see, Theorem 3. 1.) For the case $p=1$, we consider the following statements (2') and (3') instead of (2) and (3):

(2') For each Banach space X isomorphic with a quotient of c_0 and each operator u in $L(X, E)$, $u \in \Pi_1(X, E)$ if and only if $\sum \|u(f_n)\| < \infty$. (In this case, the sequence $\{f_n\}$ is defined by $f_n = \phi(e_n)$ for $n = 1, 2, \dots$, where $\{e_n\}$ is the sequence of unit vectors in c_0 and ϕ is a quotient map from c_0 onto X.)

(3') For an operator u in $L(c_0, E)$, $\sum \|u(e_n)\| < \infty$ if and only if $u': E' \to l_1$ is sub-1-nuclear.

Then for $1 \leqq p < \infty$, Theorem 3. 5 is also true.

By Theorem 3. 5 and the Remark we get

COROLLARY 3. 6. *Let E be a Banach space and $1 \leqq p < \infty$. Then the following statements are equivalent.*

(1) *E is of S_p type.*

(2) *For each absolutely p-summable sequence $\{x_n\}$ in E, there exists a Banach subspace G of E which is of S_p type such that the sequence $\{x_n\}$ is contained in G and it is absolutely p-summable in G.*

(3) *Every separable subspace of E is of S_p type.*

In particular, taking $p=2$, we get

COROLLARY 3.7 *For a Banach space E, the following statements are equivalent.*

(1) *E is isomorphic with a Hilbert space.*

(2) *For an operator u in $L(l_2, E)$, $u \in \Pi_2(l_2, E)$ if and only if $\sum \|u(e_n)\|^2 < \infty$.*

(3) *For an operator u in $L(l_2, E)$, $u' \in N_2(E', l_2)$ if and only if $\sum \|u(e_n)\|^2 < \infty$.*

(4) *For a sequence $\{x_n\}$ in E, if $\sum \|u(x_n)\|^2 < \infty$ for every u in $L(E, l_2)$, then $\sum \|x_n\|^2 < \infty$.*

(5) *For each absolutely 2-summable sequence $\{x_n\}$ in E, there exists a Hilbert subspace H such that the sequence $\{x_n\}$ is contained in H and it is absolutely 2-summable in H.*

(6) *Every separable subspace of E is isomorphic with l_2.*

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