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Path integral for diffusion equations

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Introduction

The purpose of this paper is to consider the path integral for the diffusion equation defined on a Riemannian manifold, which is compared to Feynman's path integral for the Schrödinger equations.

For a certain Lagrangian function of the form $L(x, v)=2^{-1}|v|^{2}-V(x)$ on the Euclidean $(d-)$ space $\boldsymbol{R}^{d}$, Ito [10-11] defined a generalized uniform measure on the Hilbert space of paths on $\boldsymbol{R}^{d}$. By using this measure, he proposed the concept of the path integral for the Schrödinger equation which corresponds to this Lagrangian function. It seems to be natural to extend his idea to the general Lagrangian function $L(x, v)$ on the Riemannian manifold M. Though, by following [10-11], we can define the Hilbert space $\Omega(t, x, M)$ of paths on M (cf. § 1, (1. 1)), there may be a slight difficulty to give a "uniform measure" on $\Omega(t, x, M)$ rigorously.

Our main aim is to give a meaning of the path integral for diffusion epuations on the Riemannian manifold by using the Lebesgue measure on the space $\Omega^{\Delta}(t, x, M)$ of the polygon paths on M with the mesh $|\Delta|$ (See § 1, (1. 2)). This idea is similarly discussed by Elworthy-Truman [3] for a heat equation on a Riemannian manifold. We generalize this idea to non--degenerate diffusion equations on $\boldsymbol{R}^{d}$ (or on a compact manifold). Namely, using the Lebesgue measure on $\Omega^{\Delta}(t, x, M)$, we consider the (approximate) functional integration u_{Δ} which corresponds to the given Lagrangian function. Then, we obtain the convergence of u_{Δ} by tending the limit $|\Delta| \rightarrow 0$ and show that it gives the solution of a diffusion equation. As a result, it can be defined the path integral for the diffusion equation and also, the rate of their convergence is given explicitly.

Lastly, we note that these analogies of Feynman's path integral on curved space are based on the stochastic development which was studied by Gangoli [6], Eells-Elworthy [1] and so on (See also [2], p. 157). Also, we refer that other than these probabilistic approach, there are analytic ones by Inoue-Maeda [7] and Fujiwara [4-5].

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§ 1. Statement of the result

Let (M, g) be a d-dimensional Riemannian manifold, x a point of M and t a positive number. We denote by $C_b^r(\boldsymbol{R}^d)$ the set of $C^r(\boldsymbol{R}^d)$ class functions whose i-th derivatives $(i=0, \ldots, r)$ are all bounded. We consider the path space on M as follows:

$$(1.1) \qquad \Omega(t, x, M)= \{c: [0, t] \to M \text{ ; absolutely continuous, } c(0)=x \text{ and } \int_0^t g_{c(\tau)}(\dot{c}(\tau), \dot{c}(\tau))d\tau < \infty\}.$$

In § 2, we shall introduce a Hilbert space structure into $\Omega(t, x, M)$ (See Theorem 2. 4). Next let $T>0$ and

$$\Delta: 0=t<t_1<\ldots<t_L=T$$

be an arbitrary subdivision of the interval $[0, T]$. We put

$$|\Delta| = \max_{1 \leqq k \leqq L} |t_k - t_{k-1}|.$$

We also put, for any t $(0 \leqq t \leqq T)$,

$$(1.2) \qquad \Omega^\Delta(t, x, M)$$

$= \{c \in \Omega(t, x, M)$; For eack $k=1, \ldots, t(\Delta)$, $c|[s_{k-1}, s_k]$ is smooth and satisfies $(D_{\dot{c}(\tau)}\dot{c})(\tau)=0(\tau \in (s_{k-1}, s_k)).\}$.

Here D is the covariant derivative. As for the definitions of s_k and $t(\Delta)$, see (2. 2) and (2. 3).

It will be shown in Theorem 2.5 that $\Omega^\Delta(t, x, M)$ is a $dt(\Delta)$-dimensional linear subspace of $\Omega(t, x, M)$. Using the inner product of $\Omega(t, x, M)$, we will give a uniform measure $F_{t,x}^\Delta(dc)$ to $\Omega^\Delta(t, x, M)$ at the end of § 2.

To show our result, we need some preliminaries. Throughout this section we shall assume that (M, g) is of the type (A) or (B):

(A) M is compact

(B) $M=\boldsymbol{R}^d$ and if we write g using the global coordinates as

$$g = \sum_{i,j=1}^d g_{ij}(x)dx_i \otimes dx_j,$$

then we have

(1) $g_{ij}(x) \in C_b^3(\boldsymbol{R}^d)(i. j=1, \ldots, d)$

(2) there exists a positive constant K_1 such that

$$\sum_{i,j=1}^{d} g_{ij}(x)\xi^i\xi^i \geqq K_1|\xi|^2 \quad (\xi\in\boldsymbol{R}^d).$$

We note that in both cases (M, g) is complete.

Now let b be a C^2 vector field on M and V a C^2 function on M with compact support. In case of (B), we further assume that $b^i(x) \in C_b^3(\boldsymbol{R}^d)$ $(i=1, \dots, d)$, where

$$b(x)=\sum_{i=1}^{d} b^i(x)\frac{\partial}{\partial x^i}.$$

We consider the diffusion equation on M :

$$(1.3) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t, x)=(\dfrac{1}{2}\Delta_g+b+V)u(t, x) \\ u(+0, x)=\phi(x), \end{cases}$$

were $\phi\in C_0^\infty(M)$ (C^∞ function with compact support) and Δ_g is the Laplace--Beltrami operator of (M, g). We note that general non-degenrate diffusion equation of second order on $\boldsymbol{R}^d$ is rewritten as (1. 3) (See Ikeda-Watanabe [9], p. 274). It is known that bounded $C^{1,2}([0. \infty)\times M)$ class solution of (1. 1) uniquely exists. We denote it as $u(t, x)$. We put, for $t\in[0, T]$,

$$(1.4) \qquad u_\Delta(t, x)=\int_{\Omega^\Delta(t,x,M)} \exp\{\int_0^t L(c(\tau), \dot{c}(\tau))d\tau\}\phi(c(t))F_{t,x}^\Delta(dc),$$

where $L: TM\to\boldsymbol{R}$ is defined by

$$(1.5) \qquad L(x, v)=-\frac{1}{2}|v-b(x)|^2{}_x-\frac{1}{2}divb(x)+V(x),\ |\cdot|_x=g_x(\cdot,\cdot)^{1/2}$$
$$((x, v)\in TM).$$

Now let us show our main theorem in the present paper.

THEOREM 1. 1. *Assume (A) or (B) and that $\phi\in C_0^\infty(\boldsymbol{R}^d)$. Then there exists a positive constant $K_2=K_2(T)$ such that, for any $t\in[0, T]$, $x\in M$ and Δ, we have*

$$(1.6) \qquad |u(t, x)-u_\Delta(t, x)|\leqq K_2|\Delta|^{1/2}.$$

Here, the constant K_2 is independent of t, x and Δ. In particular, $u_\Delta(t, x)$ converges to $u(t, x)$ uniformly in $(t, x)\in[0, T]\times\boldsymbol{R}^d$ as $|\Delta|\to 0$.

The proof of Theorem 1. 1 will be found in § 4. In § 2, we will study path space on a Riemannian manifold. § 3 is devoted to prove some facts which will be used in the proof of Theorem 1. 1.

§ 2. A path space on a Riemannian manifold

Let (M, g) be a d-dimensional C^3-Riemannian manifold, TM_x the tangent space of M at $x\in M$ and TM the tangent bundle of M. We often regard TM_x as an affine space. The metric $g_x(\ ,\ )$ of TM_x is often written simply as $(\ ,\ )_x$. Let $O(M)$ be the set of $(d+1)$-tuples $(x, e_1, \dots, e_d)$, where $x\in M$ and $\{e_1, \dots, e_d\}$ is an orthonormal basis of TM_x. Let π: $O(M)\to M$ be given by $\pi(x, e_1, \dots, e_d)=x$. Now we have the bundle of orthonormal flames $(O(M), \pi, M)$ with the strucure group $O(d)$. We will denote the bundle by $O(M)$ alone. If we take local coordinates $(x^1, \dots, x^d)$ in a coordinate neighborhood U of M, every orthonormal frame $r\in\pi^{-1}(U)$ may be expressed in the form

$$r=(x, e_1, \dots, e_d) \text{ and } \qquad e_i=\sum_{k=1}^d e_i^k\frac{\partial}{\partial x^k} \quad (i=1, \dots, d),$$

where we have

$$\sum_{k,l=1}^d e_i^k e_j^l g_{kl}=\delta_{ij} \quad (i, j=1, \dots, d)$$

and

$$g_x=\sum_{i,j=1}^d g_{ij}(x)\,dx^i\otimes dx^j.$$

Let Γ_{pq}^i be the coefficients of the Riemannian connection associated with the Riemannian metric g:

$$\Gamma_{pq}^i=\frac{1}{2}\sum_{k=1}^d\left(\frac{\partial}{\partial x^p}g_{kq}+\frac{\partial}{\partial x^q}g_{pk}-\frac{\partial}{\partial x^k}g_{pq}\right)g^{ki} \quad (i, p, q=1, \dots, d),$$

where

$$(g^{ij})=(g_{ij})^{-1}.$$

We introduce a path space on M.

Definition 2. 1. *For $x\in M$ and $t>0$, $\Omega(t, x, M)$ is defined by*

$$\begin{aligned}(2.1)\quad &\Omega(t, x, M)\\ &=\{c:[0, t]\to M\ ;\ \textit{absolutely continuous},\ c(0)=x\\ &\qquad \textit{and}\ \int_0^t (\dot{c}(\tau), \dot{c}(\tau))_{c(\tau)}d\tau<\infty\}.\end{aligned}$$

Let $T>0$ and

$$\Delta\ :\ 0=t_0<t_1<\dots<t_L=T$$

be an arbitrary subdivision of the interval $[0, T]$.
We put

(2.2) $\quad [\tau]^{+}(\Delta)=t_k,\ [\tau]^{-}(\Delta)=t_{k-1}$ and $\tau(\Delta)=k$ if $t_{k-1}\leq\tau<t_k$.

Also put, for any $t\in[0, T]$,

(2.3) $\quad s_0=t_0,\ s_1=t_1, \dots, s_{t(\Delta)-1}=t_{t(\Delta)-1}$ and $s_{t(\Delta)}=t$.

DEFINITION 2.2. *For $x\in M$, $t\in[0, T]$ and a subdivision Δ of $[0, T]$, $\Omega^{\Delta}(t, x, M)$ is defined by*

(2.4)
$$\Omega^{\Delta}(t, x, M)$$
$$=\{c\in\Omega(t, x, M)\,;\ \textit{For each } k=1, \dots, t(\Delta),\ c|[s_{k-1}, s_k] \textit{ is smooth and satisfies } (D_{\dot{c}(\tau)}\dot{c})(\tau)=0(\tau\in(s_{k-1}, s_k)).\}.$$

We want to regard $\Omega(t, x, M)$ as a Hilbert space and $\Omega^{\Delta}(t, x, M)$ as its finite dimensional linear subspace. For that purpose, some notions which are usually defined for smooth curves need to be generalized to the elements of $\Omega(t, x, M)$. Let c be an element of $\Omega(t, x, M)$ and $v(\tau)(0\leq\tau\leq t)$ an adsolutely continuous vector field along c. Then v is said to be parallely transported along c if the equality

(2.5) $\quad (D_{\dot{c}(\tau)}v)(\tau)=0\ \ (a.e.\ \tau\in[0, t])$

is satisfied, where the left hand side of (2.5) is expressed in local coordinates as

(2.6)
$$(D_{\dot{c}(\tau)}v)(\tau)=\sum_{\alpha=1}^{d}\{\frac{d}{d\tau}v^{\alpha}(\tau)+\sum_{p,q=1}^{d}\Gamma_{pq}^{\alpha}(c(\tau))\frac{dc^{p}}{d\tau}(\tau)v^{q}(\tau)\}\frac{\partial}{\partial x^{\alpha}}.$$

For any $c\in\Omega(t, x, M)$ and $v\in TM_x$, there exists a unique absolutely continuous curve $(c(\tau),v(\tau))(0\leqq\tau\leqq t)$ in TM which satisfies equation (2.5) with the initial condition $v(0)=v$. In fact, if there exists a local coordinate neighborhood U such that $c(\tau)\in U$ for $\tau\in[0, t]$, then equation (2.5) is written as

$$\frac{d}{d\tau}v^{\alpha}(\tau)=-\sum_{p,q=1}^{d}\Gamma_{pq}^{\alpha}(c(\tau))\frac{dc^{p}}{d\tau}(\tau)v^{p}(\tau)\ \ (\alpha=1, \dots, d,\ a.e.\ \tau\in[0, t])$$

and the solution $(v^1(\tau), \dots, v^d(\tau))$ is expressed as

$$^{t}(v^{1}(\tau), \dots, v^{d}(\tau))$$
$$=\{I+\sum_{i=1}^{\infty}\int_{0\leqq\lambda_1\leqq\dots\leqq\lambda_i\leqq\tau}A(\lambda_i)\dots A(\lambda_1)d\lambda_1\dots d\lambda_i\}$$
$$\times{}^{t}(v^{1}(0), \dots, v^{d}(0)),$$

where $A(\lambda)$ is a $d\times d$ matrix defined by

$$A(\lambda)_{\alpha q}=-\sum_{p=1}^{d}\Gamma_{pq}^{\alpha}(c(\lambda))\frac{dc^{p}}{d\tau}(\lambda)\quad(\alpha=1,\dots,d,\ q=1,\dots,d).$$

We note that the above series is convergent since $\dot{c}(\tau)$ is square integrable. Even if c is not contained in a single coordinate neighborhood, we can reduce it to the above case by deviding the interval $[0, t]$ as usual. Thus given a curve $c\in\Omega(t, x, M)$, we obtain a unique vector at $c(s')(0\leqq s'\leqq t)$ by parallely transporting any given vector from $c(s)(0\leqq s\leqq t)$ along c. This parallel transfer from $c(s)$ to $c(s')$ is a linear isomorphism from $TM_{c(s)}$ to $TM_{c(s')}$ which preserves all scalar products. This linear isomorphism is denoted by $c_{s'}^{s}$.

If we rtansport an orthonormal basis of TM_x along a given curve $c\in\Omega(t, x, M)$ parallely, then we obtain an absolutely continuous curve $\tilde{c}=(c(\tau), e(\tau))$ $(0\leqq\tau\leqq t)$ in $O(M)$. We call it the horizontal lift of c. Namely, $\tilde{c}(\tau)=(c(\tau), e(\tau))$ is the horizontal lift of $c\in\Omega(t, x, M)$, iff

$$(2.7)\qquad (D_{\dot{c}(\tau)}e_{\alpha})(\tau)=0\ \ (a.e.\ \ \tau\in[0, t],\ \alpha=1,\dots,d).$$

Next we shall prove the existence and uniqueness theorem for solutions of ordinary differential equations in the form needed here.

Let D_0 be a domain in $\boldsymbol{R}^n$, a a point in D_0 and $f_j^i(y)$ $(i=1,\dots,n,\ j=1,\dots,m)$ continuous functions on D_0. Furthermore let $\gamma(\tau)$ $(-\delta\leqq\tau\leqq\delta)$ $(\delta>0)$ be an absolutely continuous curve in $\boldsymbol{R}^m$ such that

$$\gamma(0)=0\ \ \textit{and}\ \ \int_{-\delta}^{\delta}|\dot{\gamma}(\tau)|^2 d\tau<\infty.$$

Now we consider the equation

$$(2.8)\qquad \begin{cases}\dfrac{d}{d\tau}x^i(\tau)=\displaystyle\sum_{j=1}^{m}f_j^i(x(\tau))\frac{d\gamma^j}{d\tau}(\tau)\quad(i=1,\dots,n)\\ x(0)=(x^1(0),\dots,x^n(0))=a.\end{cases}$$

THEOREM 2.1. *Suppose that* $f_j^i(y)$ $(i=1,\dots,n,\ j=1,\dots,m)$ *belongs to* $C^1(D_0)$. *Then, for any point* a *in* D_0, *there exists a unique family of* n *functions* $x(\tau)=(x^1(\tau),\dots,x^n(\tau))$ *defined on* $[-\delta', \delta']$ $(0<\delta'<\delta)$ such that

(1) $x(\tau)$ *is absolutely continuous*

and

(2) $x(\tau)$ *satisfies equation* (2.8) *for a.e.* $\tau\in[-\delta', \delta']$.

PROOF. Let c_1 be a positive constant such that the *set* $\{y\in\boldsymbol{R}^n; |y-a|\leqq c_1\}$ is contained in D_0. For a positive number δ', we put $F=\{x\in C([-\delta', \delta']\to\boldsymbol{R}^n); |x(\tau)-a|\leqq c_1(-\delta'\leqq\tau\leqq\delta')\}$. Then F becomes a Banach space with the norm $|x|_\infty=\sup_{-\delta'\leqq\tau\leqq\delta'}|x(\tau)|$.

Now we put

$$(Tx)(\tau)=a+\sum_{j=1}^{m}\int_0^\tau f_j(x(s))\dot{\gamma}^j(s)ds \quad (-\delta'\leqq\tau\leqq\delta',\ x\in F),$$

where $f_j=(f_j^1, \dots, f_j^n)(j=1, \dots, m)$. For any $x\in F$, it holds that

$$|Tx-a|_\infty\leqq c_2(\int_{-\delta'}^{\delta'}|\dot{\gamma}(s)|^2ds)^{1/2},$$

where c_2 is a positive constant which does not depend on x nor δ' $(<\delta)$. Therefore, by choosing δ' small enough, we may assume that T maps F into F. Furthermore, for any x any $y\in F$, it holds that

$$|Tx-Ty|_\infty\leqq c_3(\int_{-\delta'}^{\delta'}|\dot{\gamma}(s)|^2ds)^{1/2}|x-y|_\infty,$$

where c_3 is a positive constant which depends on neither x, y nor δ'. Thus, again, by choosing δ' small enough, we may assume that T is a contraction map from F to F. Then the theorem follows from the usual iteration technique. This completes the proof.

Let $c_0^\tau: TM_{c(\tau)}\to TM_x$ be the parallel displacement along $c\in\Omega(t, x, M)$ from $c(\tau)$ to $c(0)=x$. Since it holds that

$$\int_0^t(c_0^\tau(\dot{c}(\tau)),c_0^\tau(\dot{c}(\tau)))_x d\tau=\int_0^t(\dot{c}(\tau),\dot{c}(\tau))_{c(\tau)}d\tau<\infty,$$

we have the next definition

DEFINITION 2.3. *We define a map* Φ *from* $\Omega(t, x, M)$ *to* $\Omega(t, 0, TM_x)$ *by*

$$\Phi(c)(\tau)=\int_0^\tau c_0^s(\dot{c}(s))ds \quad (0\leqq\tau\leqq t,\ c\in\Omega(t, x, M)) \tag{2.9}$$

and call it the development of c *into the affine tangent space* TM_x.

Next we shall construct the inverse map of Φ when M is complete. This is carried out by "rolling" M along a curve $\bar{\gamma}\in\Omega(t, 0, \boldsymbol{R}^d)$. To be precise, let $\bar{\gamma}\in\Omega(t, 0, \boldsymbol{R}^d)$ and $(x, e)\in O(M)$ and define an absolutely continuous curve $\tilde{c}(\tau)=(c(\tau),e(\tau))$ in $O(M)$ by

$$(2.10)\quad \begin{cases} \dfrac{d}{d\tau}c(\tau)=\sum_{\alpha=1}^{d}\dfrac{d}{d\tau}\gamma^{\alpha}(\tau)e_{\alpha}(\tau) \\ (D_{\dot{c}(\tau)}e)(\tau)=0 \\ c(0)=x \\ e(0)=e. \end{cases} \qquad (a.e.\ \tau\in[0, t])$$

Or in local coordinates,

$$(2.11)\quad \begin{cases} \dfrac{d}{d\tau}c^{i}(\tau)=\sum_{\alpha=1}^{d}\dfrac{d}{d\tau}\gamma^{\alpha}(\tau)e_{\alpha}^{i}(\tau) \\ e_{\alpha}^{i}(\tau)=-\sum_{m,\,l=1}^{d}\Gamma_{ml}^{i}(c(\tau))e_{\alpha}^{l}(\tau)\dfrac{d}{d\tau}c^{m}(\tau) \\ c^{i}(0)=x^{i} \\ e_{\alpha}^{i}(0)=e_{\alpha}^{i}\ \ (i, \alpha=1,\dots, d,\ a.e.\ \tau\in[0, t]). \end{cases}$$

By the next theorem we know that $\tilde{c}$ is well defined if M is complete.

THEOREM 2.2. *Suppose that M is complete. Then, for any $\gamma\in\Omega(t, 0, \boldsymbol{R}^{d})$ and $(x, e)\in O(M)$, there exists a unique absolutely continuous curve $\tilde{c}(\tau)=(c(\tau), e(\tau))(0\leqq\tau\leqq t)$ in $O(M)$ which satisfies equation* (2.10).

PROOF. The proof is in two steps.
(1) Suppose that there are two such curves $\tilde{c}(\tau)=(c(\tau), e(\tau))$ and $\tilde{c}'(\tau)=(c'(\tau), e'(\tau))$. We put

$$t_1=\inf\{\tau\in[0, t]\ ;\ \tilde{c}(\tau)=\tilde{c}'(\tau)\}.$$

Then $\tilde{c}(\tau)=\tilde{c}'(\tau)$ for any $\tau\in[0, t_1)$. While, since both $\tilde{c}$ and $\tilde{c}'$ are continuous, we have $\tilde{c}(t_1)=\tilde{c}'(t_1)$. If $t_1<t$, in view of Theorem 2.1 there exists a positive number δ such that $\tilde{c}(\tau)=\tilde{c}'(\tau)(t_1\leqq\tau\leqq t_1+\delta)$. This is a contradiction. Therefore, t_1 must coincide with t.
(2) In view of Theorem 2.1, we have an absolutely continuous curve $\tilde{c}(\tau)$ which is defined on $[0, \delta]$ for some δ (>0) and satisfies equation (2.10). Let t_2 be the supremum of those δ' s. Then, with the aid of (1), we have a curve $\tilde{c}(\tau)(0\leqq\tau<t_2)$ which is absolutely continuous on each $[0, s]$ $(0\leqq s<t_2)$ and satisfies equation (2.10). Now we shall show that $\tilde{c}$ can be extended up to t_2 as an absolutely continuous curve. For a sequence $\{\tau_k\}_{k=1}^{\infty}$ such that $\tau_k\uparrow t_2$, $\{c(\tau_k)\}_{k=1}^{\infty}$ forms a Cauchy sequence in M. In fact, in view of equation (2.10) it holds that

$$\begin{aligned} &\mathrm{dis}(c(\tau_k), c(\tau_l)) \\ &\leqq \int_{\tau_l}^{\tau_k} g_{c(\tau)}(\dot{c}(\tau),\dot{c}(\tau))^{1/2}d\tau \end{aligned}$$

$$= \int_{\tau_l}^{\tau_k} (\dot{\gamma}(\tau), \dot{\gamma}(\tau))^{1/2} d\tau \to 0 \ (k, l \to \infty).$$

Therefore, since M is complete, $\{c(\tau_k)\}_{k=1}^{\infty}$ converges to some point $p \in M$. As $\{c(\tau_k)\}_{k=1}^{\infty}$ is arbitrary, we know that $c(\tau)$ convereges to p as $\tau \uparrow t_2$. Now we put $c(t_2)=p$. We wish to show that $c(\tau)(0 \leqq \tau \leqq t_2)$ is absolutely continuous. Take a local coordinate neighborhood U in M such that the closure $\bar{U}$ of U is compact and $c(t_2)$ is in U. Then $c(\tau)(s \leqq \tau \leqq t_2)$ is contained in U for some $s(<t_2)$. Since $\pi^{-1}(\bar{U})$ is compact, $e_{\alpha}^{i}(\tau)(\tau \in [s, t_2),$ $i, \alpha=1, \ldots, d)$ are all bounded. Therefore, it follows that

$$\int_0^t \left| \sum_{\alpha=1}^{d} \frac{d\bar{\gamma}^{\alpha}}{d\tau}(\tau) e_{\alpha}^{i}(\tau) \right| d\tau < \infty \ (i=1, \ldots, d).$$

Hence, by the dominated convergence theorem, if we let $\tau \uparrow t_2$ in

$$c^i(\tau) = c^i(s) + \int_0^{\tau} \sum_{\alpha=1}^{d} \frac{d\bar{\gamma}^{\alpha}}{d\tau}(\tau) d\tau \ (i=1, \ldots, d),$$

we obtain

$$c^i(t_2) = c^i(s) + \int_0^{t} \sum_{\alpha=1}^{d} \frac{d\bar{\gamma}^{\alpha}}{d\tau}(\tau) e_{\alpha}^{i}(\tau) d\tau \ (i=1, \ldots, d).$$

This shows that $c(\tau)$ $(0 \leqq \tau \leqq t_2)$ is absolutely continuous. Now let $\tilde{c}'(\tau) = (c(\tau), e'(\tau))$ be the horizontal lift of $c(\tau)$ $(0 \leqq \tau \leqq t_2)$. Then, by the definition, we have the equality

$$(D_{\dot{c}(\tau)} e')(\tau) = 0 \ (a.e.\ \tau \in [0, t_2]).$$

Since the solution of equation (2.5) is unique, it follows that $e(\tau)=e'(\tau)$ $(0 \leqq \tau < t_2)$. This shows that $\tilde{c}'$ is the absolutely continuous extension of $\tilde{c}$ $(\tau)(0 \leqq \tau < t_2)$ to $[0, t_2]$ as desired. We write this $\tilde{c}'$ as $\tilde{c}$ again. Now suppose that $t_2 < t$. Then, in view of Theorem 2.1, $\tilde{c}$ is extended up to $t_2 + \delta$ for some $\delta > 0$. But this contradicts the definition of t_2. Hence t_2 must coincide with t. This completes the proof of the theorem.

Let M be a complete Riemannian manifold and $(x, e) \in O(M)$. We denote by Ψ the following map from $\Omega(t, 0, TM_x)$ to $\Omega(t, x, M)$

(2.12) $\quad \Psi(\gamma)(\tau) = \pi(\tilde{c}(\tau)) \ (\gamma \in \Omega(t, 0, TM_x),\ 0 \leqq \tau \leqq t),$

where $\tilde{c}(\tau) = (c(\tau), e(\tau))$ is the solution of equation (2.10) with

(2.13) $\quad \bar{\gamma}(\tau) = ((\gamma(\tau), e_1)_x, \ldots, (\gamma(\tau), e_d)_x).$

By the next theorem, we know that Ψ does not depend on the choice of orthonormal basis of TM_x. Furthermore, this theorem will be used to regard $\Omega(t, x, M)$ as a Hilbert space.

THEORM 2.3. *If M is complete, Φ is bijective and Φ^{-1} is given by Ψ.*

REMARK 2.1. We can also show the converse: if Φ is bijective, M is complete.

PROOF. In view of Theorem 2.2, we have a map Ψ defined by equation (2.12) when M is complete, where the orthonormal basis e of TX_x is chosen arbitrarily. Take an element γ of $\Omega(t, 0, TM_x)$ and let $\tilde{c}(\tau)=(c(\tau), e(\tau))$ be the solution of equation (2.10) with (2.13). Then, by the definition, $c=\Psi(\gamma)$. In view of (2.10) and (2.13), we derive

$$\begin{aligned} c_0^\tau(\dot{c}(\tau)) &= c_0^\tau\Big(\sum_{\alpha=1}^d \dot{\gamma}^\alpha(\tau)e_\alpha(\tau)\Big) \\ &= \sum_{\alpha=1}^d \dot{\gamma}^\alpha(\tau)c_0^\tau(e_\alpha(\tau)) \\ &= \sum_{\alpha=1}^d (\dot{\gamma}(\tau), e_\alpha)_x e_\alpha \\ &= \dot{\gamma}(\tau). \end{aligned}$$

Therefore, we obtain

$$\int_0^\tau c_0^s(\dot{c}(s))ds=\gamma(\tau) \quad (0\leqq\tau\leqq t),$$

which shows that $\Phi\circ\Psi=$the identity.

Conversely, let c be an arbitrary element of $\Omega(t, x, M)$ and $\tilde{c}(\tau)=(c(\tau), e(\tau))$ be its horizontal lift with $\tilde{c}(0)=(x, e)$. We put $\gamma=\Phi(c)$. Then, in view of (2.9), we obtain

$$\begin{aligned} (\dot{\gamma}(\tau), e_\alpha)_x &= (c_0^\tau(\dot{c}(\tau)), c_0^\tau(e_\alpha(\tau)))_x \\ &= (\dot{c}(\tau), e_\alpha(\tau))_{c(\tau)} \end{aligned}$$

and from this and (2.5) we see that $\tilde{c}$ is the solution of equation (2.10) with (2.13). This implies

$$c=\Phi(\gamma)=\Psi\circ\Phi(c)$$

and therefore $\Psi\circ\Phi=$the identity. Now the proof of the theorem is complete.

Now let us introduce a Hilbert space structure into $\Omega(t, x, M)$. At first, we note that $\Omega(t, 0, TM_x)$ is regarded as a Hilbert space in a natural way, because TM_x is a finite dimensional vector space with an inner product. In particular, the inner product $<\,,\,>$ of $\Omega(t, 0, TM_x)$ is defined by

$$<\gamma_1, \gamma_2>=\int_0^t (\dot{\gamma}_1(\tau), \dot{\gamma}_2(\tau))_x d\tau \quad (\gamma_1, \gamma_2\in\Omega(t, 0, TM_x)).$$

In view of Theorem 2.3, we have a bijection Φ from $\Omega(t, x, M)$ to

$\Omega(t, 0, TM_x)$ when M is a complete Riemannian manifold. Therefore, we can regard $\Omega(t, x, M)$ as a Hilbert space through Φ. To be precise, we define linear combination and inner product of $\Omega(t, x, M)$ as

(1) *linear combination*

$$(2.14) \quad \alpha_1 c + \alpha_2 c' = \Phi^{-1}(\alpha_1 \Phi(c) + \alpha_2 \Phi(c'))$$

(2) *inner product*

$$(2.15) \quad \langle c, c' \rangle = \langle \Phi(c), \Phi(c') \rangle. \ (\alpha_1, \alpha_2 \in \boldsymbol{R}, \ c, \ c' \in \Omega(t, x, M)).$$

It is easy to see that the Hilbert space structure induced from $\Omega(t, 0, TM_x)$ into $\Omega(t, x, M)$ through Φ is characterized as follows.

THEOREM 2.4. *Suppose that M is complete and let $x \in M$. Then,*

(1) *for any $\alpha_1, \alpha_2 \in \boldsymbol{R}$ and any $c, c' \in \Omega(t, x, M)$, there exists a unique element $c'' \in \Omega(t, x, M)$ such that*

(2.16) $c''^{\tau}_0(\dot{c}''(\tau)) = \alpha_1 c^{\tau}_0(\dot{c}(\tau)) + \alpha_2 c'^{\tau}_0(\dot{c}'(\tau))$ (*a.e.* $\tau \in [0, t]$) *and if we write c'' as $\alpha_1 c + \alpha_2 c'$, then $\Omega(t, x, M)$ becomes a vector space.*
Furthermore,

(2) *if we define an inner product of $\Omega(t, x, M)$ as*

$$(2.17) \quad \langle c, c' \rangle = \int_0^t (c^{\tau}_0(\dot{c}(\tau)), c'^{\tau}_0(\dot{c}'(\tau)))_x d\tau,$$

then $\Omega(t, x, M)$ becomes a Hilbert space.

REMARK 2.2. As mentioned before, c^{τ}_0 denotes the parallel displacement along c from $c(\tau)$ to $c(0)$. Therefore, the both sides of (2.16) should be regarded as elements of TM_x.

Now let Δ be the subdivision of the interval $[0, T]$ as before. Since geodesics in TM_x are nothing but line segments, $\Omega^{\Delta}(t, 0, TM_x)$ introduced in (2.4) consists of piecewise linear curves with respect to Δ. If we put $(x, e) \in O(M)$ and

$$(2.18) \quad \gamma^{\alpha}_{\Delta k}(\tau) = \begin{cases} \dfrac{\tau - s_{k-1}}{(s_k - s_{k-1})^{1/2}} e_\alpha & (\tau \in [s_{k-1}, s_k]) \\ 0 \ (\tau \in [0, t] - [s_{k-1}, s_k]) \end{cases}$$
$$(\alpha = 1, \dots, d, \ k = 1, \dots, t(\Delta)).$$

then $\{\gamma^{\alpha}_{\Delta k}; \alpha = 1, \dots, d, \ k = 1, \dots, t(\Delta)\}$ forms a basis of $\Omega^{\Delta}(t, 0, TM_x)$. Here we have used the notations in (2.2) and (2.3). In particular, $\Omega^{\Delta}(t, 0, TM_x)$ is a $dt(\Delta)$-dimensional linear subspace of $\Omega(t, 0, TM_x)$.

THEOREM 2.5. *Suppose that M is complete. Then $\Phi(\Omega^{\Delta}(t, x, M)) =$*

$\Omega^{\Delta}(t, 0, TM_x)$. *In particular,* $\Omega^{\Delta}(t, x, M)$ *is a* $dt(\Delta)$*-dimensional linear subspace of* $\Omega(t, x, M)$.

PROOF. Let $\gamma\in\Omega^{\Delta}(t, 0, TM_x)$. Then, $(\dot{\gamma}(\tau), e_{\alpha})_x=$constant for any $\tau \in(s_{k-1}, s_k)(k=1, \ldots, t(\Delta),\ \alpha=1, \ldots, d)$. Now we put $c=\Psi(\gamma)$. In view of equation (2.10) with (2.13), we have

$$\begin{aligned}(D_{\dot{c}(\tau)}\dot{c})(\tau)&=(D_{\dot{c}(\tau)}(\sum_{\alpha=1}^{d}\dot{\gamma}^{\alpha}e_{\alpha}))(\tau)\\&=\sum_{\alpha=1}^{d}\dot{\gamma}^{\alpha}(\tau)(D_{\dot{c}(\tau)}e_{\alpha})(\tau)\\&=0\quad(s_{k-1}<\tau<s_k,\ k=1, \ldots, t(\Delta)).\end{aligned}$$

Thus we have proved that

$$\Phi^{-1}(\Omega^{\Delta}(t, 0, TM_x))\subset\Omega_{\Delta}(t, x, M).$$

Since Φ is bijective, this shows that

$$\Omega^{\Delta}(t, 0, TM_x)\subset\Phi(\Omega^{\Delta}(t, x, M)).$$

Conversely, if $c\in\Omega^{\Delta}(t, x, M)$, then $c_0^{\tau}(\dot{c}(\tau))$ is the same element of TM_x for any $\tau\in(s_{k-1}, s_k)$. Hence $\Phi(c)\in\Omega^{\Delta}(t, 0, TM_x)$ and this shows that

$$\Phi(\Omega^{\Delta}(t, x, M))\subset\Omega^{\Delta}(t, 0, TM_x).$$

This completes the proof of the theorem.

Now let us introduce a *uniform measure* into $\Omega^{\Delta}(t, x, M)$. Let $\{\gamma_j\}_{j=1}^{dt(\Delta)}$ be an arbitrary orthonormal basis of $\Omega^{\Delta}(t, x, M,)$ and define a linear isomorphism $T: \Omega^{\Delta}(t, x, M)\to\boldsymbol{R}^{dt(\Delta)}$ by

(2.19) $\quad T(c)=(<c, c_1>, \ldots, <c, c_{dt(\Delta)}>)\ (c\in\Omega^{\Delta}(t, x, M))$.

DEFINITION 2.4. *We define the uiform measure* $F_{t,x}^{\Delta}(dc)$ *of* $\Omega^{\Delta}(t, x, M)$ *by*

(2.20) $\quad F_{t,x}^{\Delta}=F\circ T$,

where F *is defined by*

(2.21) $\quad F=(2\pi)^{-dt(\Delta)/2}\cdot$(*Lebesgue measure of* $\boldsymbol{R}^{dt(\Delta)}$).

We note that $F_{t,x}^{\Delta}$ does not depend on the choice of orthonormal basis of $\Omega^{\Delta}(t, x, M)$.

§ 3. Approximation theorems and some related estimates.

In this section, we shall prove two approximation theorems and some estimates related to stochastic integrals and stochastic differential equations. They will be used in § 4.

We first introduce some notations. Fix an arbitrary positive number T. Let $(W_0^{r,T}, \mathscr{F}, P)$ be the r-dimensional Wiener space with the usual reference family $(\mathscr{F}_t)_{0\leqq t\leqq T}$. That is, $W_0^{r,T}=\{w\in C([0, T]\to\boldsymbol{R}^r); w(0)=0\}$, $\mathscr{F}_t$ and $\mathscr{F}$ denote the smallest σ-field with respect to which $w(\tau)$ are measurable for $0\leqq\tau\leqq t$ and for $0\leqq\tau\leqq T$ respectively and P is the Wiener measure on $(W_0^{r,T}, \mathscr{F})$.

Let

$$\Delta : 0=t_0<t_1< \dots <t_L=T$$

be an arbitrary subdivision of the interval $[0, T]$. We put as in § 2 that

$$[\tau]^+(\Delta)=t_k,\ [\tau]^-(\Delta)=t_{k-1}\ \textit{and}\ \tau(\Delta)=k \quad \textit{if}\ t_{k-1}\leqq\tau<t_k.$$

DEFINITION 3.1. *By piecewise linear approximation of Wiener process, we mean a family* $\{w_\Delta(\tau)=(w_\Delta^1(\tau), \dots, w_\Delta^r(\tau))\}$ *of r-dimensional continuous processes defined over the Wiener space such that*

$$(3.1)\qquad w_\Delta^i(\tau)=w^i(t_{k-1})+\frac{\tau-t_{k-1}}{t_k-t_{k-1}}\{w^i(t_k)-w^i(t_{k-1})\}, \quad (t_{k-1}\leqq\tau<t_k,\ k=1, \dots, L,\ i=1, \dots, r).$$

Let σ_j^i $(i=1, \dots, d,\ j=1, \dots, r)$ be real valued functions on $\boldsymbol{R}^d$ such that $\sigma_j^i\in C_b^2(\boldsymbol{R}^d)$. Consider the system of ordinary differential equations

$$(3.2)\qquad \frac{d}{d\tau}X_\Delta^i(\tau, w)=\sum_{j=1}^r\sigma_j^i(X_\Delta(\tau, w))\frac{d}{d\tau}w_\Delta^i(\tau),\ X_\Delta^i(0, w)=x^i \quad (i=1, \dots, d).$$

We also cosider the system of stochastic differential equations

$$(3.3)\qquad dX^i(\tau, w)=\sum_{j=1}^r\sigma_j^i(X(\tau, w))\circ dw^j(\tau),\ X^i(0, w)=x^i(i=1, \dots, d).$$

Here $x=(x^1, \dots, x^d)\in\boldsymbol{R}^d$. For any $x\in\boldsymbol{R}^d$, the solutions of equations (3.2) and (3.3) exist uniquely, which we shall denote by $X_\Delta(\tau, x, w)=(X_\Delta^1(\tau, x, w), \dots, X_\Delta^d(\tau, x, w))$ and $X(\tau, x, w)=(X^1(\tau, x, w), \dots, X^d(\tau, x, w))$ respectively. For simplicity, we shall often suppress w.

First, we shall prove two approximation theorems in the form needed in § 4.

THEOREM 3.1. *Let* $T>0$ *be fixed. Then, there exists a positive* *constant* $K_3=K_3(T)$ *such that*

$$(3.4)\qquad E[|X(t,x,w)-X_\Delta(t,x,w)|]^2]\leqq K_3|\Delta|$$

for any $t\in[0,T]$ *and* $x\in\boldsymbol{R}^d$. *The constant* K_3 *is independent of* Δ, t *and* x.

THEOREM 3.2. *Let* $T>0$ *be fixed and assume that* $u\in C_b^2(\boldsymbol{R}^d)$. *Then, there exists a positive constant* $K_4=K_4(T)$ *such that*

$$(3.5)\qquad E[|\int_0^t u(X_\Delta(\tau,x,w))\dot{w}_\Delta^j(\tau)d\tau-\int_0^t u(X(\tau,x,w))\circ dw^j(\tau)|^2]$$
$$\leqq K_4|\Delta|$$

for any $t\in[0,T]$, $x\in\boldsymbol{R}^d$ and $j=1,\dots,d$. The constant K_4 is independent of Δ, t and x.

These are slight modifications of approximation theorems in Ikeda-Nakao-Yamato [8] and in the following proof we will follow their idea.

First, we shall prepare some lemmas.

LEMMA 3.1.

$$(3.6)\qquad E[\{\int_{t_{k-1}}^{t_k}\dot{w}_\Delta^j(\tau)(w_\Delta^i(t_k)-w_\Delta^i(\tau))d\tau\}^2|\mathscr{F}_{t_{k-1}}]$$
$$=(\frac{1}{4}+\frac{1}{2}\delta_{ij})(t_k-t_{k-1})^2$$

and

$$(3.7)\qquad E[\int_{t_{k-1}}^{t_k}\dot{w}_\Delta^i(\tau)(w_\Delta^j(t_{k-1})-w_\Delta^j(\tau))d\tau|\mathscr{F}_{t_{k-1}}]=\frac{1}{2}\delta_{ij}(t_k-t_{k-1}).$$

PROOF. Both are proved by direct calculations.

LEMMA 3.2. *Let* $Z_1(\tau,w)$ *be a bounded* $(\mathscr{F}_\tau)$*-adapted process defined on* $(W_0^{r,T},\mathscr{F},P)$ *with piecewise continuous sample paths. Then,*

$$(3.8)\qquad E[\{\int_0^{[t]^-(\Delta)}Z_1([\tau]^-(\Delta))[\dot{w}_\Delta^i(\tau)(w_\Delta^i([\tau]^+(\Delta))-w_\Delta^i(\tau))-\frac{1}{2}\delta_{ij}]$$
$$d\tau\}^2]$$
$$\leqq\frac{1}{2}(K_5)^2T|\Delta|\ (i,j=1,\dots,r),$$

where $K_5=\sup_{0\leqq\tau\leqq T,\ w}|Z_1(\tau,w)|$.

PROOF. In view of (3.7), we have

$$E[\int_{t_{k-1}}^{t_k}\{\dot{w}_\Delta^i(\tau)(w_\Delta^j(t_k)-w_\Delta^j(\tau))-\frac{1}{2}\delta_{ij}\}d\tau\,|\,\mathscr{F}_{t_{k-1}}]=0$$

and therefore it follows that

$$(3.9)\quad E[\{\int_0^{[t]^-(\Delta)} Z_1([\tau]^-(\Delta))[\dot{w}_\Delta^i(\tau)(w_\Delta^j([\tau]^+(\Delta))-w_\Delta^j(\tau))-\frac{1}{2}\delta_{ij}]\, d\tau\}^2]$$
$$=E[\sum_{k=1}^{t(\Delta)-1} Z_1(t_{k-1})^2\{\int_{t_{k-1}}^{t_k}[\dot{w}_\Delta^i(\tau)(w_\Delta^j(t_{k-1})-w_\Delta^j(\tau))-\frac{1}{2}\delta_{ij}]d\tau\}^2].$$

While, using Lemma 3. 1, we have

$$(3.10)\quad E[\{\int_{t_{k-1}}^{t_k}[\dot{w}_\Delta^i(\tau)(w_\Delta^j(t_k)-w_\Delta^j(\tau))-\frac{1}{2}\delta_{ij}]d\tau\}^2|\mathscr{F}_{t_{k-1}}]$$
$$=E[\{\int_{t_{k-1}}^{t_k}\dot{w}_\Delta^i(\tau)(w_\Delta^j(t_k)-w_\Delta^j(\tau))d\tau\}^2|\mathscr{F}_{t_{k-1}}]-(\frac{1}{2}\delta_{ij})^2(t_k-t_{k-1})^2$$
$$\leqq\frac{1}{2}(t_k-t_{k-1})^2.$$

Combining (3. 9) and (3. 10), we derive

$$E[\{\int_0^{[t]^-(\Delta)} Z_1([\tau]^-(\Delta))[\dot{w}_\Delta^i(\tau)(w_\Delta^j([\tau]^+(\Delta))-w_\Delta^j(\tau)) -\frac{1}{2}\delta_{ij}]d\tau\}^2]$$
$$\leqq\frac{1}{2}(K_5)^2\sum_{k=1}^{L}(t_k-t_{k-1})^2$$
$$\leqq\frac{1}{2}(K_5)^2|\Delta|\,T.$$

This completes the proof of the lemma.

LEMMA 3. 3. *Let K_6 be a positive constant and $Z_2(\tau, w)$ a stochastic process defined on $(W_0^{r,T}, \mathscr{F}, P)$ with piecewise continuous sample paths satisfying the condition*

$$(3.11)\quad |Z_2(t)|\leqq K_6\sum_{m=1}^{r}\int_{[t]^-(\Delta)}^{[t]^+(\Delta)}|\dot{w}_\Delta^m(\tau)|d\tau\quad(0\leqq t\leqq T).$$

Then, there exists a positive constant $K_7=K_7(T)$ such that, for any $t\in[0, T]$, we have

$$(3.12)\quad E[\{\int_0^{[t]^-(\Delta)} Z_2(\tau)\dot{w}_\Delta^i(\tau)(w_\Delta^j([\tau]^+(\Delta))-w_\Delta^j(\tau))d\tau\}^2]$$
$$\leqq K_7|\Delta|\quad(i, j=1, \dots, r).$$

Here, the constant K_7 is independent of t and Δ.

PROOF. Using (3. 11), we obtain

$$
\begin{aligned}
&|\int_0^{[t]^-(\Delta)} Z_2(\tau)\dot{w}_\Delta^i(\tau)\{w_\Delta^j([\tau]^+(\Delta))-w_\Delta^j(\tau)\}d\tau|^2\\
&\leqq T\int_0^{[t]^-(\Delta)} |Z_2(\tau)|^2|\dot{w}_\Delta^i(\tau)|^2\{w_\Delta^j([\tau]^+(\Delta))-w_\Delta^j(\tau)\}^2d\tau\\
&\leqq T(K_6)^2\sum_{k=1}^{L}\{\sum_{m=1}^{r}\int_{t_{k-1}}^{t_k}|\dot{w}_\Delta^m(\tau)|d\tau\}^2\{\int_{t_{k-1}}^{t_k}|\dot{w}_\Delta^i(\tau)|^2d\tau\}\\
&\times\{\int_{t_{k-1}}^{t_k}|\dot{w}_\Delta^j(\tau)|d\tau\}^2\\
&=T(K_6)^2\sum_{k=1}^{L}(t_k-t_{k-1})^{-1}\{\sum_{m,\ l=1}^{r}|w^m(t_k)-w^m(t_{k-1})|\\
&\times|w^l(t_k)-w^l(t_{k-1})|\,|w^i(t_k)-w^i(t_{k-1})|^2|w^j(t_k)-w^j(t_{k-1})|^2\}.
\end{aligned}
$$

Hence, the left hand side of (3. 12) is bounded by

$$
\begin{aligned}
&T(K_6)^2\sum_{k=1}^{L}(t_k-t_{k-1})^{-1}\{\sum_{m=1}^{r}\sum_{l=1}^{r}E[|w^m(t_k)-w^m(t_{k-1})|\\
&\times|w_\Delta^l(t_k)-w_\Delta^l(t_{k-1})|\,|w_\Delta^i(t_k)-w_\Delta^i(t_{k-1})|^2|w_\Delta^j(t_k)-w_\Delta^j(t_{k-1})|^2]\}\\
&\leqq T(K_6)^2\sum_{k=1}^{L}(t_k-t_{k-1})^{-1}\{\sum_{m=1}^{r}\sum_{l=1}^{r}E[|w^m(t_k)-w^m(t_{k-1})|^4]^{1/4}\\
&\times E[|w^l(t_k)-w^l(t_{k-1})|^4]^{1/4}E[|w^i(t_k)-w^i(t_{k-1})|^8]^{1/4}\\
&\times E[|w^j(t_k)-w^j(t_{k-1})|^8]^{1/4}\}\\
&=T(K_6)^2 315^{1/2}r^2\sum_{k=1}^{r}(t_k-t_{k-1})^2\\
&\leqq 315^{1/2}(K_6Tr)^2|\Delta|.
\end{aligned}
$$

This completes the proof.

PROPOSITION 3. 1. *Assume that $u\in C_b^2(\boldsymbol{R}^d)$. Then, there exists a positive constant $K_8=K_8(T)$ such that, for any $t\in[0, T]$ and $j=1, \dots, r$, we have*

$$
\begin{aligned}
(3.13)\quad &E[|\int_0^t u(X_\Delta(\tau, x, w))\dot{w}_\Delta^j(\tau)d\tau-\int_0^t u(X(\tau, x, w))\circ dw^j(\tau)|^2]\\
&\leqq K_8\{|\Delta|+\int_0^t E[|X_\Delta(\tau, x, w))-X(\tau, x, w))|^2]d\tau\}.
\end{aligned}
$$

Here, the constant K_8 is independent of t, x, j and Δ.

PROOF To begin with, we note that, for every Δ and $\tau\in[0, T]$, it holds that

$$
(3.14)\quad |X_\Delta(\tau, x)-X_\Delta([\tau]^-(\Delta), x)|\leqq c_1\sum_{m=1}^{r}\int_{[\tau]^-(\Delta)}^{[\tau]^+(\Delta)}|\dot{w}_\Delta^m(s)|ds,
$$

where c_1 is a positive constant determined by σ_j^i $(i=1, \dots, d,\ j=1, \dots, r)$.

Integration by parts yields

$$(3.15)\qquad \int_{t_{k-1}}^{t_k} u(X_\Delta(\tau, x))\dot{w}_\Delta^j(\tau)d\tau$$
$$= -\int_{t_{k-1}}^{t_k} u(X_\Delta(\tau, x))\frac{d}{d\tau}\{w_\Delta^j(t_k) - w_\Delta^j(\tau)\}d\tau$$
$$= u(X_\Delta(t_{k-1}, x))\{w^j(t_k) - w^j(t_{k-1})\}$$
$$+ \sum_{m=1}^{d}\sum_{i=1}^{r}\int_{t_{k-1}}^{t_k}\left(\frac{\partial u}{\partial x^m}\sigma_i^m\right)(X_\Delta(\tau, x))\dot{w}_\Delta^i(\tau)\{w_\Delta^j(t_k) - w_\Delta^j(\tau)\}d\tau.$$

In view of (3. 3), *we have*

$$(3.16)\qquad \int_0^t u(X(\tau, x))\circ dw^j(\tau)$$
$$= \int_0^t u(X(\tau, x))dw^j(\tau) + \frac{1}{2}\sum_{m=1}^{d}\int_0^t\left(\frac{\partial u}{\partial x^m}\sigma_j^m\right)(X(\tau, x))d\tau.$$

It follows from (3. 15) and (3. 16) that

$$\int_0^t u(X_\Delta(\tau, x))\dot{w}_\Delta^j(\tau)d\tau - \int_0^t u(X(\tau, x))\circ dw^j(\tau) = \sum_{i=1}^{5} I_i(\Delta, t, x),$$

where

$$I_1(\Delta, t, x) = \int_{[t]^-(\Delta)}^{t} u(X(\tau, x))\dot{w}_\Delta^j(\tau)d\tau - \int_{[t]^-(\Delta)}^{t} u(X(\tau, x))dw^j(\tau)$$
$$-\frac{1}{2}\sum_{m=1}^{d}\int_{[t]^-(\Delta)}^{t}\left(\frac{\partial u}{\partial x^m}\sigma_j^m\right)(X(\tau, x))d\tau.$$

$$I_2(\Delta, t, x) = \int_0^{[t]^-(\Delta)}\{u(X_\Delta([\tau]^-(\Delta), x) - u(X(\tau, x))\}dw^j(\tau),$$

$$I_3(\Delta, t, x) = \sum_{m=1}^{d}\sum_{i=1}^{r}\int_0^{[t]^-(\Delta)}\left(\frac{\partial u}{\partial x^m}\sigma_i^m\right)(X_\Delta([\tau]^-(\Delta), x))$$
$$\times[\dot{w}_\Delta^i(\tau)\{w_\Delta^j([\tau]^+(\Delta)) - w_\Delta^j(\tau)\} - \frac{1}{2}\delta_{ij}]d\tau.$$

$$I_4(\Delta, t, x) = \sum_{m=1}^{d}\sum_{i=1}^{d}\int_0^{[t]^-(\Delta)}\{\left(\frac{\partial u}{\partial x^m}\sigma_i^m\right)(X_\Delta(\tau, x))$$
$$-\left(\frac{\partial u}{\partial x^m}\sigma_i^m\right)(X_\Delta([\tau]^-(\Delta), x))\}$$
$$\times\dot{w}_\Delta^i(\tau)\{w_\Delta^j([\tau]^+(\Delta)) - w_\Delta^j(\tau)\}d\tau.$$

$$I_5(\Delta, t, x) = \frac{1}{2}\sum_{m=1}^{d}\int_0^{[t]^-(\Delta)}\{\left(\frac{\partial u}{\partial x^m}\sigma_j^m\right)(X_\Delta([\tau]^-(\Delta), x))$$
$$-\left(\frac{\partial u}{\partial x^m}\sigma_j^m\right)(X(\tau, x))\}d\tau.$$

Now, we easily obtain

$$
\begin{aligned}
&E[|I_1(\Delta, t, x)|^2] \\
&\leqq c_2\{E[|w^j([t]^+(\Delta)) - w^j([t]^-(\Delta))|^2] \\
&+ \int_{[t]^-(\Delta)}^{t} E[u^2(X(\tau, x))]d\tau + |\Delta|^2\} \leqq c_3|\Delta|.
\end{aligned}
$$

Next, by (3.14), we have

$$
\begin{aligned}
&E[|I_2(\Delta, t, x)|^2] \\
&= \int_0^{[t]^-(\Delta)} E[\{u(X_\Delta([\tau]^-(\Delta), x)) - u(X_\Delta(\tau, x))\}^2]d\tau \\
&\leqq c_4 \int_0^{[t]^-(\Delta)} E[|X_\Delta([\tau]^-(\Delta), x) - X(\tau, x)|^2]d\tau \\
&\leqq 2c_4\{\int_0^t E[|X_\Delta([\tau]^-(\Delta), x) - X_\Delta(\tau, x)|^2]d\tau \\
&+ \int_0^t E[|X_\Delta(\tau, x) - X(\tau, x)|^2]d\tau\} \\
&\leqq c_5\{\sum_{k=1}^{L}(t_k - t_{k-1})E[\{\sum_{m=1}^{r}\int_{t_{k-1}}^{t_k} |\dot{w}_\Delta^m(\tau)|d\tau\}^2] \\
&+ \quad E[|X_\Delta(\tau, x) - X(\tau, x)|^2]d\tau\} \\
&\leqq c_6\{|\Delta| + \int_0^t E[|X_\Delta(\tau, x) - X(\tau, x)|^2]\}d\tau.
\end{aligned}
$$

If we fix j and put

$$
Z_1(\tau, x, w) = \sum_{m=1}^{d}(\frac{\partial u}{\partial x^m}\sigma_j^m)(X_\Delta([\tau]^-(\Delta), x)),
$$

then $Z_1(\tau, x, w)$ satisfies the conditions of Lemma 3.2. Therefore, by (3.8), we have

$$
E[|I_3(\Delta, t, x)|^2] \leqq c_7|\Delta|.
$$

Next, we put

$$
Z_2(\tau, x, w) = \sum_{m=1}^{d}\{(\frac{\partial u}{\partial x^m}\sigma_i^m)(X_\Delta(\tau, x)) - (\frac{\partial u}{\partial x^m}\sigma_i^m)(X_\Delta([\tau]^-(\Delta), x))\}.
$$

Then (3.14) shows that $Z_2(\tau, x, w)$ satisfies (3.11) in Lemma 3.3 for each $x \in \boldsymbol{R}^d$. Therefore (3.12) yields

$$
E[|I_4(\Delta, x)|^2] \leqq c_8|\Delta|.
$$

Finally, as in the case of $I_2(\Delta, t, x)$, we have

$$
E[|I_4(\Delta, t, x)|^2]
$$

$$\leqq c_9 \int_0^t E[|X_\Delta([\tau]^-(\Delta), x) - X(\tau, x)|^2] d\tau$$
$$\leqq c_{10}\{|\Delta| + \int_0^t E[|X_\Delta(\tau, x) - X(\tau, x)|^2] d\tau\}.$$

This completes the proof of the proposition.

PROOF OF THEOREM 3.1. Using ~~Proposition~~ 3. ~~1~~, we have

$$E[|X(t, x, w) - X_\Delta(t, x, w)|^2]$$
$$\leqq c_1\{\sum_{i=1}^d \sum_{j=1}^r E[|\int_0^t \sigma_j^i(X(\tau, x, w)) \circ dw^j(\tau) - \int_0^t \sigma_j^i(X_\Delta(\tau, x, w))$$
$$\times w_\Delta^{\dot{j}}(\tau) d\tau|^2\}$$
$$\leqq c_2\{|\Delta| + \int_0^t E[|X_\Delta(\tau, x, w) - X(\tau, x, w)|^2] d\tau\}.$$

Then, by Gronwall's inequality, we obtain

$$E[|X(t, x, w) - X(t, x, w)|^2] \leqq c_2 |\Delta| e^{tc_2},$$

which completes the proof.

PROOF OF THEOREM 3.2. This follows easily from Theorem 3.1 and Proposition 3.1.

The following two propositions will be used in § 4 to prove Theorem 1. 1, too.

PROPOSITION 3.2. *Let $u \in C_b^1(\boldsymbol{R}^d)$ and $T>0$. Then, there exists a positive constant $K_9 = K_9(T)$ such that, for any $t \in [0, T]$, we have*

$$E[\exp\{|\int_0^t u(X(\tau, x, w)) \circ dw^j(\tau)|\}] < K_9 \quad (j=1, \ldots, r). \tag{3.17}$$

Here, the constant K_9 is independent of t, x and j.

PROOF. In view of the property of exponential martingales, we have

$$E[\exp\{\varepsilon \int_0^t u(X(\tau, x)) dw^j(\tau) - \frac{1}{2} \int_0^t u(X(\tau, x))^2 d\tau\}] = 1 \quad (\varepsilon = 1, -1). \tag{3.18}$$

Since u is bounded, there exists a positive constant c_1 such that

$$\exp\{-\frac{1}{2} \int_0^t u(X(\tau, x))^2 d\tau\} > c_1. \tag{3.19}$$

(3.18) and (3.19) show that

(3.20) $$E[\exp\{\varepsilon\int_0^t u(X(\tau, x))dw^j(\tau)\}]<c_1^{-1}.$$

From (3.16) and (3.20), we derive

$$\begin{aligned}&E[\exp|\int_0^t u(X(\tau, x))\circ dw^j(\tau)|]\\ &\leqq c_2E[\exp|\int_0^t u(X(\tau, x))dw^j(\tau)|]\\ &\leqq c_2\{E[\exp(\int_0^t u(X(\tau, x))dw^j(\tau))+\exp(-\int_0^t u(X(\tau, x))\\ &dw^j(\tau))]\}\\ &\leqq 2c_2c_1^{-1}.\end{aligned}$$

This completes the proof.

PROPOSITION 3.3. *Let $u\in C_b^1(\boldsymbol{R}^d)$ and $T>0$. Then, there exists a positive constant $K_{10}=K_{10}(T)$ such that, for any $t\in[0, T]$ and $x\in\boldsymbol{R}^d$, we have*

(3.21) $$E[\exp|\int_0^t u(X_\Delta(\tau, x, w))\dot{w}_\Delta^j(\tau)d\tau|]<K_{10},$$

where the constant K_{10} is independent of Δ, t, x and j.

PROOF. In this proof, c_i $(i=1, \dots, 8)$ denotes positive constant which is independent of Δ, t, j and x. The proof is in six steps.

(1) First, we shall introduce some notations. We put

$$f_i(x)=\sum_{m=1}^d\left(\frac{\partial u}{\partial x^m}\sigma_i^m\right)(x)\quad(i=1, \dots, r)$$

and

$$c_1=\max_{1\leqq i\leqq r}|f_i|_\infty.$$

We shall prove the proposition in the case of $c_1>0$ only. If $c_1=0$, the inequality (3.21) is proved in the same way and more easily.

Let Δ be a subdivision of $[0, T]$. If we devide the set $A=\{1, \dots, t(\Delta)-1\}$ into two subsets

$$A_1=\{k\in A\,;\ |t_k-t_{k-1}|>6^{-1}(r+1)^{-1}c_1^{-1}\}$$

and

$$A_2=\{k\in A\,;\ |t_k-t_{k-1}|\leqq 6^{-1}(r+1)^{-1}c_1^{-1}\},$$

then we have

$$(3.22)\quad E[\exp|\int_0^t u(X_\Delta(\tau,x))\dot{w}^j_\Delta(\tau)d\tau|]$$
$$\leqq\{E[\exp|\sum_{k\in A_1}\int_{t_{k-1}}^{t_k}3u(X_\Delta(\tau,x))\dot{w}^j_\Delta(\tau)d\tau|]\}^{1/3}$$
$$\times\{E[\exp|\sum_{k\in A_2}\int_{t_{k-1}}^{t_k}3u(X_\Delta(\tau,x))\dot{w}^j_\Delta(\tau)d\tau|]\}^{1/3}$$
$$\times\{E[\exp|\int_{[t]-(\Delta)}^{t}3u(X_\Delta(\tau,x))\dot{w}^j_\Delta(\tau)d\tau|]\}^{1/3}.$$

(2) We note that $\sharp A_1<6(r+1)tc_1$. Then, putting

$$x_k=w^j(t_k)-w^j(t_{k-1})\quad(k\in A_1),$$

we obtain

$$(3.23)\quad E[\exp|\sum_{k\in A_1}\int_{t_{k-1}}^{t_k}3u(X_\Delta(\tau,x))\dot{w}^j_\Delta(\tau)d\tau|]$$
$$\leqq E[\exp\{c_2\sum_{k\in A_1}|w^j(t_k)-w^j(t_{k-1})|\}]$$
$$=\prod_{k\in A_1}\int_{\boldsymbol{R}}\{2\pi(t_k-t_{k-1})\}^{-1/2}\exp\{c_2|x_k|-\frac{1}{2(t_k-t_{k-1})}|x_k|^2\}dx_k$$
$$\leqq\prod_{k\in A_1}2\exp\{2^{-1}(c_2)^2(t_k-t_{k-1})\}\leqq 2^{6(r+1)tc_1}\exp\{2^{-1}(c_2)^2t\}.$$

Similarly we have

$$(3.24)\quad E[\exp|\int_{[t]-(\Delta)}^{t}3u(X_\Delta(\tau,x))\dot{w}^j_\Delta(\tau)d\tau|]\leqq c_3.$$

(3) Integration by parts and (3.3) yield

$$\sum_{k\in A_2}\int_{t_{k-1}}^{t_k}u(X_\Delta(\tau,x))\dot{w}^j_\Delta(\tau)d\tau$$
$$=\sum_{k\in A_2}u(X_\Delta(t_{k-1},x))\{w^j(t_k)-w^j(t_{k-1})\}$$
$$+\sum_{i=1}^{r}\sum_{k\in A_2}\int_{t_{k-1}}^{t_k}f_i(X_\Delta(\tau,x))\dot{w}^i_\Delta(\tau)\{w^j_\Delta(t_k)-w^j_\Delta(\tau)\}d\tau$$
$$=M(\Delta,x)+\sum_{i=1}^{r}I_i(\Delta,x).$$

Hence we have

$$(3.25)\quad E[\exp\{|\sum_{k\in A_2}\int_{t_{k-1}}^{t_k}3u(X_\Delta(\tau,x))\dot{w}^j_\Delta(\tau)d\tau|\}]$$
$$\leqq\{E[\exp|3(r+1)M(\Delta,x)|]\}^{1/r+1}$$
$$\times\prod_{i=1}^{r}\{E[\exp|3(r+1)I_i(\Delta,x)|]\}^{1/r+1}.$$

(4) Since $M(\Delta, x)$ is a stochastic integral with

$$<M(\Delta, x)> = \sum_{k\in A_2} u(X_\Delta(t_{k-1}))^2(t_k - t_{k-1}) < c_4,$$

as in the proof of Proposition 3.2 we have

(3.26) $$E[\exp|3(r+1)M(\Delta, x)|] \leqq c_5.$$

(5) If we put

$$x_k^i = w^i(t_k) - w^i(t_{k-1}) \quad (i=1, \dots, r,\ k\in A_2),$$

then it follows that, for each $i=1, \dots, r$,

(3.27)
$$\begin{aligned}
&E[\exp|3(r+1)I_i(\Delta, x)|]\\
&\leqq E[\exp\{3c_1(r+1)\sum_{k\in A_2}|w^i(t_k)-w^i(t_{k-1})||w^j(t_k)-w^j(t_{k-1})|\}]\\
&= \prod_{k\in A_2}\int_{R^2}\exp\{3c_1(r+1)|x_k^i||x_k^i| - \frac{(x_k^i)^2+(x_k^i)^2}{2(t_k-t_{k-1})}\}\\
&\times\{2\pi(t_k-t_{k-1})\}dx_k^i dx_k^i\\
&\leqq [\prod_{k\in A_2}\int_R \exp\{\frac{3}{2}(r+1)c_1(x_k)^2 - \frac{(x_k)^2}{2(t_k-t_{k-1})}\}\\
&\times\{2\pi(t_k-t_{k-1})\}^{-1/2}dx_k]^2\\
&= \prod_{k\in A_2}\{1-3c_1(r+1)(t_k-t_{k-1})\}^{-1}.
\end{aligned}$$

Consider the function $\log(1-x)^{-1}$ $(0<x\leqq\frac{1}{2})$. By the mean value theorem, we have

$$\log(1-x)^{-1} = x + \frac{1}{2(1-\theta)^2}x^2 \quad (0<\theta<x).$$

Hence, noting $0<3c_1(r+1)(t_k-t_{k-1})\leqq\frac{1}{2}$, we obtain the estimate

$$\begin{aligned}
&\log[\prod_{k\in A_2}\{1-3c_1(r+1)(t_k-t_{k-1})\}^{-1}]\\
&\leqq 3c_1(r-1)\sum_{k\in A_2}(t_k-t_{k-1}) + 18(c_1)^2(r+1)^2\sum_{k\in A_2}(t_k-t_{k-1})^2\\
&\leqq 3c_1(r+1)t + 18(c_1)^2(r+1)^2t^2.
\end{aligned}$$

This and (3.27) show that

(3.28) $$E[\exp|3(r+1)I_i(\Delta, x)|] < c_6 \quad (i=1, \dots, r).$$

(6) Now we are ready to prove the proposition. (3.25), (3.26) and

(3.28) imply the inequality

$$(3.29)\qquad E[\exp|\sum_{k\in A_2}\int_{t_{k-1}}^{t_k} 3u(X_\Delta(\tau,x))\dot{w}_\Delta^j(\tau)d\tau|]<c_7.$$

Then, by (3.22), (3.23), (3.24) and (3.29) we have

$$E[\exp|\int_0^t u(X_\Delta(\tau,x))\dot{w}_\Delta^j(\tau)d\tau|]\leqq c_8.$$

This completes the proof of the proposition.

§ 4. Proof of Theoren 1.1.

First we shall assume that M satisfies the assumption (B). The case of the assumption (A) will be proved at the end of this section. Thus $M=\boldsymbol{R}^d$ and if we write g and b as

$$g(x)=\sum_{i,j=1}^d g_{ij}(x)dx^i\otimes dx^j,\quad b(x)=\sum_{i=1}^d b^i(x)\frac{\partial}{\partial x_i},$$

then it holds that $g_{ij}(\mathrm{x})\in C_b^3(\boldsymbol{R}^d)(i,j=1,\dots,d)$ and $b^i(x)\in C_b^2(\boldsymbol{R}^d)(i=1,\dots,d)$. We also assume that V is a compact support C^2 function and ϕ is a compact support C^∞ function. Now we put

$$(4.1)\qquad \bar{b}_\alpha(r)=\sum_{j,k=1}^d g_{jk}(x)b^k(X)e_\alpha^j$$
$$(\alpha=1,\dots,d,\ r=(x,e)\in O(M)).$$

We note that $\bar{b}_\alpha$ $(\alpha=1,\dots,d)$ can be extended to $\boldsymbol{R}^{d(d+1)}$ as a C_b^2 class function. Next we put

$$(4.2)\qquad f(x)=-\frac{1}{2}div(b)(x)-\frac{1}{2}|b|^2(x)+V(x),$$

where

$$|b|^2(x)=\sum_{i,j=1} g_{ij}(x)b^i(x)b^j(x).$$

The horizontal Brownian motion $\tilde{c}(\tau,r,w)=(c(\tau,r,w),\ e(\tau,r,w))$ is a diffusion on $O(M)$ governed by the following stochastic differential equation

$$(4.3)\qquad \begin{cases} dc^i(\tau, r, w) = \sum_{\alpha=1}^{d} e_\alpha^i(\tau, r, w) \circ dw^\alpha(\tau) \quad (i=1, \dots, d) \\ de_\alpha^i(\tau, r, w) = -\sum_{k,\, m=1}^{d} \Gamma_{mk}^i(c(\tau, r, w)) e_\alpha^k(\tau, r, w) \circ dc^m(\tau, r, w) \\ c^i(0, r, w) = x^i \quad (i,\ \alpha=1, \dots, d) \\ e_\alpha^i(0, r, w) = e_\alpha^i . \end{cases}$$

where the initial point $(x, e)=(x^i, e_\alpha^i)$ is in $O(M)$. If we put

$$(4.4)\qquad u(x, t) = E[\exp\{\sum_{\alpha=1}^{d} \int_0^t \bar{b}_\alpha(\tilde{c}(\tau, r)) \circ dw^\alpha(\tau) + \int_0^t f(c(\tau, r))\, d\tau\} \phi(c(t, r))]$$

for $\phi \in C_0^\infty(M)$, then we have the next theorem (cf. [10]).

THEOREM 4.1. *$u(t, x)$ is the unique bounded $C^{1,2}$ $([0, \infty) \times M)$ class solution of equation* (1.1).

Now consider the following equation

$$(4.5)\qquad \begin{cases} \dfrac{d}{d\tau} c_\Delta^i(\tau, r, w) = \sum_{\alpha=1}^{d} e_{\Delta\alpha}^i(\tau, r, w) \dfrac{d}{d\tau} w_\Delta^\alpha(\tau) \quad (i=1, \dots, d) \\ \dfrac{d}{d\tau} e_{\Delta\alpha}^i(\tau, r, w) = -\sum_{k,\, m=1}^{d} \Gamma_{mk}^i(c_\Delta(\tau, r, w)) e_{\Delta\alpha}^k(\tau, r, w) \dfrac{d}{d\tau} c_\Delta^m(\tau, r, w) \\ \qquad\qquad (i,\ \alpha=1, \dots, d) \\ c_\Delta(0, r, w) = x \\ e_\Delta(0, r, w) = e, \end{cases}$$

where $r=(x, e) \in O(M)$. We denote the solution of equation (4.5) as $\tilde{c}_\Delta(\tau, r, w) = (c_\Delta(\tau, r, w), e_\Delta(\tau, r, w))$. We note that $\tilde{c}_\Delta$ is in $O(M)$.

PROPOSITION 4.1. *For every $r=(x, e) \in O(M)$ and $t \in [0, T]$, we have*

$$(4.6)\qquad \int_{\Omega^\Delta(t,x,M)} \exp\{\int_0^t L(c(\tau), \dot{c}(\tau))\, d\tau\} \phi(c(t)) F_{t,\,x}^\Delta(dc)$$
$$= E[\exp\{\sum_{\alpha=1}^{d} \int_0^t \bar{b}_\alpha(\tilde{c}_\Delta(\tau, r)) \dot{w}_\Delta^\alpha(\tau)\, d\tau + \int_0^t f(c_\Delta(\tau, r))\, d\tau\} \phi(c_\Delta(t, r))].$$

PROOF. Put

$$c_\Delta = \Phi^{-1}(\gamma_\Delta) \quad (\gamma_\Delta \in \Omega^\Delta(t, 0, TM_x)).$$

Let $\tilde{c}_\Delta(\tau, r) = (c_\Delta(\tau, r), e_\Delta(\tau, r))$ be the horizontal lift of $c_\Delta(\tau, r)$. Then it holds that

$$(4.7)\quad \begin{cases} \dfrac{d}{d\tau}c_\Delta(\tau, r)=\sum_{\alpha=1}^{d}(\dot{\gamma}_\Delta(\tau), e_\alpha)_x e_\alpha(\tau, r) \\ (D_{\dot{c}\ \Delta(\tau, r)}e)(\tau, r)=0\ (a.e.\ \tau\in[0, t]) \\ \tilde{c}_\Delta(0, r)=r=(x, e), \end{cases}$$

where $r=(x, e)\in O(M)$. Hence we have

$$(4..8)\quad \int_0^t L(c_\Delta(\tau, r), \dot{c}_\Delta(\tau, r))d\tau$$
$$=-\frac{1}{2}\int_0^t(\dot{\gamma}_\Delta(\tau), \dot{\gamma}_\Delta(\tau))_x d\tau+\sum_{\alpha=1}^{d}\int_0^t \bar{b}_\alpha(\tilde{c}_\Delta(\tau, r))(\dot{\gamma}_\Delta(\tau), e_\alpha)_x d\tau$$
$$+\int_0^t f(c_\Delta(\tau, r))d\tau).$$

Let $\{\gamma^\alpha_{\Delta k}(\tau)\ ;\ \alpha=1,\dots, d,\ k=1,\dots, t(\Delta)\}$ be the orthonormal basis of $\Omega^\Delta(t, 0, TM_x)$ defined by (2.18). We put, as in § 2,

$$s_0=t_0,\ s_1=t_1,\ \dots,\ s_{t(\Delta)-1}=t_{t(\Delta)-1} \text{ and } s_{t(\Delta)}=t.$$

Then we have

$$<\gamma_\Delta, \gamma^\alpha_{\Delta k}>=\frac{(\gamma_\Delta(s_k), e_\alpha)_x-(\gamma_\Delta(s_{k-1}), e_\alpha)_x}{(s_k-s_{k-1})^{1/2}}$$
$$(k=1,\dots, t(\Delta),\ \alpha=1,\dots, d).$$

Now we put

$$x^\alpha_k=(\gamma_\Delta(s_k), e_\alpha)_x\ (k=1,\dots,\ t(\Delta),\ \alpha=1,\dots,\ d).$$

Then, by this coordinate transformation of $\Omega^\Delta(t, 0, TM_x)$, $F^\Delta_{t,0}(d\gamma_\Delta)$ changes into

$$(4.9)\quad \prod_{k=1}^{t(\Delta)}\frac{dx^1_k\dots dx^d_k}{\{2\pi(s_k-s_{k-1})\}^{d/2}}.$$

We note that

$$(4.10)\quad \frac{1}{2}\int_0^t(\dot{\gamma}_\Delta(\tau), \dot{\gamma}_\Delta(\tau))_x d\tau=\frac{1}{2}\sum_{k=1}^{t(\Delta)}\frac{|x_k-x_{k-1}|^2}{s_k-s_{k-1}}$$

and

$$(4.11)\quad (\bar{\gamma}_\Delta(\tau),\ e_\alpha)_x=\frac{x^\alpha_k-x^\alpha_{k-1}}{s_k-s_{k-1}}$$
$$(x_0=0,\ s_{k-1}\leqq\tau<s_k,\ \alpha=1,\ \dots,\ d,\ k=1,\ \dots,\ t(\Delta)).$$

We define a curve $\lambda_\Delta(\tau)=\lambda_\Delta(\tau\,|\,x_1,\ \dots,\ x_{t(\Delta)})$ in $\boldsymbol{R}^d$ by

$$\lambda_{\Delta}(\tau)=x_{k-1}+\frac{\tau-s_{k-1}}{s_k-s_{k-1}}(x_k-x_{k-1})$$
$$(s_{k-1}\leqq\tau<s_k,\ k=1,\dots,t(\Delta)),$$

where we put $x_k=(x_k^1,\dots,x_k^d)$ $(k=1,\dots,t(\Delta))$ and $x_0=0$. Let $\tilde{\theta}_{\Delta}(\tau,r)=\tilde{\theta}(\tau,r\,|\,x_1,\dots,x_{t(\Delta)})=(\theta_{\Delta}(\tau,r),\ \eta_{\Delta}(\tau,r))$ be the solution of the following equation on $O(M)$

$$(4.12)\qquad \begin{cases} \dfrac{d}{d\tau}\theta_{\Delta}(\tau,r)=\displaystyle\sum_{\alpha=1}^{d}\frac{d}{d\tau}\lambda_{\Delta}^{\alpha}(\tau)\eta_{\Delta\alpha}(\tau,r) \\ (D_{\dot{\theta}_{\Delta}(\tau,r)}\eta_{\Delta})(\tau,r)=0\ \ (a.\ e.\ \tau\in[0,t]) \\ \tilde{\theta}_{\Delta}(0,r)=r=(x,e). \end{cases}$$

Then, by (4.7)~(4.12), we have

$$\begin{aligned}
&\int_{\Omega^{\Delta}(t,x,M)}\exp\{\int_0^t L(c(\tau),\dot{c}(\tau))d\tau\}\phi(c(t))F_{t,x}^{\Delta}(dc)\\
&=\int_{\Omega^{\Delta}(t,0,TM_x)}\exp\{\int_0^t L(c_{\Delta}(\tau),\dot{c}_{\Delta}(\tau))d\tau\}\phi(c_{\Delta}(\tau))F_{t,0}^{\Delta}(d\gamma_{\Delta})\\
&=\int_{R^{dt(\Delta)}}\exp\{\sum_{\alpha=1}^{d}\int_0^t\bar{b}_{\alpha}(\tilde{\theta}_{\Delta}(\tau))\dot{\lambda}_{\Delta}^{\alpha}(\tau)d\tau+\int_0^t f(\theta_{\Delta}(\tau))d\tau\}\\
&\times\phi(\theta_{\Delta}(t))\\
&\times\prod_{k=1}^{t(\Delta)}\frac{1}{\{2\pi(s_k-s_{k-1})\}^{d/2}}\exp\{-\sum_{k=1}^{t(\Delta)}\frac{|x_k-x_{k-1}|^2}{2(s_k-s_{k-1})}\}\prod_{k=1}^{t(\Delta)}\prod_{\alpha=1}^{d}dx_k^{\alpha}\\
&=E[\exp\{\sum_{\alpha=1}^{d}\int_0^t\bar{b}_{\alpha}(\tilde{c}_{\Delta}(\tau,r,w))\dot{w}_{\Delta}^{\alpha}(\tau)d\tau\\
&+\int_0^t f(c_{\Delta}(\tau,r,w))d\tau\}\phi(c_{\Delta}(t,r,w))].
\end{aligned}$$

This completes the proof of the proposition.

Now let us prove Theorem 1.1 in the case of (B). We put

$$\begin{aligned}
u_{\Delta}(t,x)=E[&\exp\{\sum_{i=1}^{d}\int_0^t\bar{b}_{\alpha}(\tilde{c}_{\Delta}(\tau,r,w))\dot{w}_{\Delta}^{\alpha}(\tau)d\tau\\
&+\int_0^t f(c_{\Delta}(\tau,r,w))d\tau\}\phi(c_{\Delta}(t,r,w))].
\end{aligned}$$

We note that

$$\sum_{i=1}^{d}e_{\alpha}^{i}(\tau,r,w)^2\leqq\frac{1}{K^1}g(e_{\alpha}(\tau,r,w),\ e_{\alpha}(\tau,r,w))=\frac{1}{K^1}\ \ (\alpha=1,\dots,d)$$

and

$$\sum_{i=1}^{d}e_{\Delta\alpha}^{i}(\tau,r,w)^2\leqq\frac{1}{K^1}g(e_{\Delta\alpha}(\tau,r,w),\ e_{\Delta\alpha}(\tau,r,w))=\frac{1}{K^1}$$

$(\alpha=1, \ldots, d)$.

Hence we may regard the two families of coefficients of equations (4.3) and (4.5) as the same bounded C^2 class functions. We may also regard $\bar{b}_\alpha$ as bounded C^2 class functions. Therefore, we can apply the results of § 3 to them. Now we have

$$\begin{aligned}
&|u(t, x)-u_\Delta(t, x)| \\
&\leqq E[|I_1(t, x)-I_1^\Delta(t, x)||I_2(t, x)||I_3(x)|] \\
&+E[|I_1^\Delta(t, x)||I_2(t, x)-I_2^\Delta(t, x)||I_3(t, x)|] \\
&+E[|I_1^\Delta(t, x)||I_2^\Delta(t, x)||I_3(t, x)-I_3^\Delta(t, x)|] \\
&=J_1+J_2+J_3,
\end{aligned}$$

where

$$\begin{aligned}
&I_1(t, x)=\exp\{\sum_{\alpha=1}^d \int_0^t \bar{b}_\alpha(\tilde{c}(\tau, r, w))\circ dw^\alpha(\tau)\} \\
&I_2(t, x)=\exp\{\int_0^t f(c(\tau, r, w))\,d\tau\} \\
&I_3(t, x)=\phi(c(t, r, w))
\end{aligned}$$

and

$$\begin{aligned}
&I_1^\Delta(t, x)=\exp\{\sum_{\alpha=1}^d \int_0^t \bar{b}_\alpha(\tilde{c}_\Delta(\tau, r, w))\dot{w}_\Delta^\alpha(\tau)\,d\tau\} \\
&I_2^\Delta(t, x)=\exp\{\int_0^t f(c_\Delta(\tau, r, w))\,d\tau\} \\
&I_3^\Delta(t, x)=\phi(c_\Delta(t, r, w)).
\end{aligned}$$

We note that

$$(4.13)\qquad |e^x-e^y|\leqq|x-y|\exp(|x|+|y|) \quad (x, y\in \boldsymbol{R}^d).$$

Then, from (3.5), (3.17), (3.21) and (4.13), we see that

$$\begin{aligned}
J_1 &\leqq c_1 E[|\sum_{\alpha=1}^d \int_0^t \bar{b}_\alpha(\tilde{c}(\tau, r))\circ dw^\alpha(\tau)-\sum_{i=1}^d \int_0^t \bar{b}_\alpha(\tilde{c}_\Delta(\tau, r))\dot{w}_\Delta^\alpha(\tau)\,d\tau| \\
&\times\exp\{|\sum_{\alpha=1}^d \int_0^t \bar{b}_\alpha(\tilde{c}(\tau, r))\circ dw^\alpha(\tau)| \\
&+|\sum_{\alpha=1}^d \int_0^t \bar{b}_\alpha(\tilde{c}_\Delta(\tau, r))\dot{w}_\Delta^\alpha(\tau)\,d\tau|\}] \\
&\leqq c_2\{E[|\sum_{\alpha=1}^d \int_0^t \bar{b}^\alpha(\tilde{c}(\tau, r))\circ dw^\alpha(\tau) \\
&-\sum_{\alpha=1}^d \int_0^t \bar{b}^\alpha(\tilde{c}_\Delta(\tau, r))\dot{w}_\Delta^\alpha(\tau)\,d\tau|^2]\}^{1/2} \\
&\leqq c_3|\Delta|^{1/2}.
\end{aligned}$$

Next, in view of (3. 5), (3, 21) and (4. 13), we have

$$J_2 \leqq c_4\{E[|\int_0^t f(c(\tau, r, w))d\tau - \int_0^t f(c_\Delta(\tau, r, w))d\tau|^2]\}^{1/2}$$
$$\leqq c_5\{\int_0^t E[|c(\tau, r, w) - c_\Delta(\tau, r, w)|^2]d\tau\}^{1/2}$$
$$\leqq c_6|\Delta|^{1/2}.$$

Finally, (3. 4) and (3. 21) show that

$$J_3 \leqq c_7\{E[|c(t, r, w) - c_\Delta(t, r, w)|^2]\}^{1/2}$$
$$\leqq c_8|\Delta|^{-/2}.$$

This completes the proof.

Finally, we shall prove Theorem 1. 1 with the assumption (A). We note that Proposition 4. 1 also holds for compact manifold. Take an embedding i: $O(M)\to \boldsymbol{R}^n$ for some n. Since $O(M)$ is compact, we can extend the vector fields on $O(M)$ which define holizontal Brownian motion as well as $\bar{b}$, f, and ϕ smoothly to $\boldsymbol{R}^n$. Furthermore, we may assume that they all have compact supports. Then, using Proposition 4. 1, the theorem is proved in the same way with (B) (cf. [2], Theorem 4). This completes the proof.

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