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On j-algebras and homogeneous Kähler manifolds

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Introduction.

The notion of j-algebras introduced by Pyatetskii-Shapiro played an important role in the theory of realization of homogeneous bounded domains as homogeneous Siegel domains. Vinberg, Gindikin and Pyatetskii-Shapiro [16] stated that the Lie algebra of a transitive holomorphic transformation group of a homogeneous bounded domain admits a structure of an effective proper j-algebra and that every effective proper j-algebra can be regarded as the Lie algebra of a transitive holomorphic transformation group of a homogeneous Siegel domain of the second kind. In this paper, we remove the properness and study the structure of homogeneous complex manifolds corresponding to effective j-algebras.

By an effective j-algebra $(\mathfrak{g}, \mathfrak{k}, j, \omega)$ we mean a system of a Lie algebra $\mathfrak{g}$, a subalgebra $\mathfrak{k}$, an endomorphism j, and a linear form ω satisfying certain conditions. (For a precise definition, see § 3.) Let G be a connected Lie group with $\mathfrak{g}$ as its Lie algebra and let K be the connected subgroup corresponding to $\mathfrak{k}$. Then K is closed and G/K admits a G-invariant Kähler structure. The homogeneous space G/K is said to be the homogeneous complex manifold associated with the effective j-algebra $(\mathfrak{g}, \mathfrak{k}, j, \omega)$. We shall prove the following theorems.

THEOREM A. *Let G/K be the homogeneous complex manifold associated with an effective j-algebra $(\mathfrak{g}, \mathfrak{k}, j, \omega)$. Then G/K is biholomorphic to a product of a homogeneous bounded domain M_1 and a compact simply connected homogeneous complex manifold M_2.*

THEOREM B. *Conversely, let M_1 be a homogeneous bounded domain and let M_2 be a compact simply connected homogeneous complex manifold. Let G be a connected Lie group acting on $M_1 \times M_2$ transitively, effectively and holomorphically. Assume further that $M_1 \times M_2$ admits a G-invariant Kähler metric. Then the Lie algebra of G admits a structure of an effective j-algebra so that the associated homogeneous complex manifold coincides with $M_1 \times M_2$.*

Gindikin, Pyatetskii-Shapiro and Vinberg [17] stated that Theorem A was essentially proved in [16]. But it seems to the author that there is no description in [16] which enables us to obtain our Proposition 8.

We can prove Theorem B by a simple consideration of the canonical hermitian form due to Koszul [8]. In order to prove Theorem A, we use the decomposition theorem of an effective j-algebra which was obtained in [16] by using an algebraic technique, i. e., by considering the algebraic hull of an effective j-algebra. In view of [11], we can show this decomposition theorem without taking the algebraic hull. This enables us to obtain a structure theorem similar to Theorem A for a certain class of homogeneous Kähler manifolds. Vinberg and Gindikin [15] conjectured: *Every homogeneous Kähler manifold admits a holomorphic fibering whose base space is a homogeneous bounded domain and whose fiber is, equipped with the induced metric, a product of a locally flat homogeneous Kähler manifold and a compact simply connected homogeneous Kähler manifold.* By a result of Borel [1], this conjecture is true for a homogeneous Kähler manifold of a semi-simple Lie group. We say that a homogeneous Kähler manifold satisfies the condition (C) if it does not contain any locally flat homogeneous Kähler submanifold. Recently, Shima [13] showed that under an "algebraic" assumption on G, if a homogeneous Kähler manifold G/K satisfies the condition (C), then G/K admits a fibering whose base space M_1 is diffeomorphic to a homogeneous bounded domain and whose fiber M_2 is a compact simply connected homogeneous Kähler submanifold. But it has not been proved in [13] that the projection: $G/K \rightarrow M_1$ is holomorphic. In this paper, we also prove the following

THEOREM C. *Let G/K be a homogeneous Kähler manifold satisfying the condition (C). Then G/K is, as a complex manifold, biholomorphic to a product of a homogeneous bounded domain and a compact simply connected homogeneous Kähler manifold.*

In the special case where G is solvable, Theorem C is obtained by Kodama and Shima [7] by using the main result of [16]. To prove Theorems A and C, we shall find a connected closed subgroup U containing K so that G/U is a homogeneous bounded domain and U/K is a compact simply connected homogeneous Kähler submanifold of G/K (Theorem 11). We can see that the fibering: $G/K \rightarrow G/U$ is holomorphic. Therefore the conjecture of Vinberg and Gindikin is true for a homogeneous Kähler manifold satisfying (C) and for a homogeneous Kähler manifold associated with an effective j-algebra.

From our Theorem 11, we can reprove the main result of [16]. We can also show by our method the following unpublished result of Koszul [10]: *Every homogeneous complex manifold with a positive definite canonical* *hermitian form is biholomorphic to a homogeneous bounded domain.*

§ 1. Preliminaries.

Let G/K be a homogeneous manifold of a connected Lie group G by a closed subgroup K. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K respectively. Assume that G/K admits a G-invariant complex structure J. Then there corresponds an endomorphism j of $\mathfrak{g}$ satisfying the followings:

(1.1) $$j\,\mathfrak{k}\subset\mathfrak{k},\ j^2x\equiv -x(\mathrm{mod}\ \mathfrak{k}),$$

(1.2) $$\mathrm{Ad}\ k\circ jx\equiv j\circ\mathrm{Ad}\ k\ x\ (\mathrm{mod}\ \mathfrak{k}),$$

(1.3) $$[jx, jy]\equiv[x, y]+j[jx, y]+j[x, jy]\ (\mathrm{mod}\ \mathfrak{k}),$$

where $x, y\in\mathfrak{g}$ and $k\in K$. The operator j induces an endomorphism $\tilde{j}$ of $\mathfrak{g}/\mathfrak{k}$ in a natural manner. Then under the identification of $\mathfrak{g}/\mathfrak{k}$ with the tangent space $T_o(G/K)$ at the origin o of G/K,

(1.4) $$\tilde{j}v=J_ov \text{ for any } v\in T_o(G/K).$$

Conversely, if an endomorphism j satisfies (1.1)$\sim$(1.3), then there corresponds a unique G-invariant complex structure J satisfying (1.4). Another endomorphism j' is called equivalent to j if it satisfies $j'x\equiv jx$ (mod $\mathfrak{k}$) for any $x\in\mathfrak{g}$. Then j' also satisfies (1.1)$\sim$(1.3) and determines the same G-invariant complex structure. If the group K is connected, then (1.2) is equivalent to

(1.5) $$[h, jx]\equiv j[h, x]\ (\mathrm{mod}\ \mathfrak{k}) \text{ for } h\in\mathfrak{k},\ x\in\mathfrak{g}.$$

We remark that the following conditions for a homogeneous complex manifold G/K are mutually equivalent:

(a) *G/K is biholomorphic to a homogeneous Siegel domain of the second kind.*

(b) *G/K is biholomorphic to a homogeneous bounded domain in $\boldsymbol{C}^n$.*

(c) *The universal covering space of G/K is biholomorphic to a homogeneous bounded domain in $\boldsymbol{C}^n$.*

This fact is proved in [11] by using the Kobayashi hyperbolicity. Of course, if we admit the results of [16], then it is obtained from [5], [6] and [16].

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{k}$ be a subalgebra equipped with an endomorphism j satisfying (1.1), (1.3) and (1.5). For every $x\in\mathfrak{g}$, ad $jx-j\circ\mathrm{ad}\ x$ induces an endomorphism of $\mathfrak{g}/\mathfrak{k}$. Assume that $\mathrm{Tr}_{\mathfrak{g}/\mathfrak{k}}$ ad $h=0$ for any $h\in\mathfrak{k}$. According to Koszul [8], we define *the Koszul form* ψ of $(\mathfrak{g}, \mathfrak{k}, j)$ by

$$(1.6) \qquad \psi(x)=\mathrm{Tr}_{\mathfrak{g}/\mathfrak{k}}(\mathrm{ad}\, jx-j\circ \mathrm{ad}\, x) \text{ for } x\in\mathfrak{g}.$$

We then have ([8])

$$(1.7) \qquad \psi([jx, jy])=\psi([x, y]), \ \psi([h, x])=0 \text{ for } x, y\in\mathfrak{g}, \ h\in\mathfrak{k}.$$

Let G/K be a homogeneous complex manifold with a G-invariant volume element V. If V is expressed as

$$V=V^* \, dz^1\wedge \ldots \wedge dz^n \wedge d\bar{z}^1 \wedge \ldots \wedge d\bar{z}^n$$

in terms of local coordinate system $(z^1, \ldots, z^n)$, then we get a G-invariant hermitian form η on G/K, called *the canonical hermitian form,* by

$$\eta=\sum_{i,j}\frac{\partial^2 \log V^*}{\partial z^i \partial\bar{z}^j} dz^i\otimes d\bar{z}^j.$$

Let us denote by π the projection of G onto G/K. Then $\pi^*\eta$ is a left invariant symmetric bilinear form on G. Therefore it is regarded as a symmetric bilinear form on $\mathfrak{g}$. Under this identification, Koszul [8] showed

$$\pi^*\eta(x, y)=\frac{1}{2}\psi([jx, y]) \text{ for any } x, y\in\mathfrak{g}.$$

The following facts are well known:

(i) *If G/K is a homogeneous bounded domain, then η coincides with the Bergman metric. Therefore $\psi([jx, x])\geqq 0$ for any $x\in\mathfrak{g}$ and $\psi([jx, x])=0$ if and only if $x\in\mathfrak{k}$.*

(ii) *Let G/K be a compact simply connected homogeneous complex manifold with a G-invariant Kähle metric. Then the group of all holomorphic isometries of G/K is a compact semi-simple group. Therefore by* [8], *η is negative definite. Consequently, $\psi([jx, x])\leqq 0$ for any $x\in\mathfrak{g}$ and $\psi([jx, x])=0$ if and only if $x\in\mathfrak{k}$.*

(iii) *Let G/K be a homogeneous Kähler manifold of a semi-simple Lie group G. Then the canonical hermitian form η is non-degenerate* ([1], [8]).

By *a symplectic space* (W, j, ρ), we mean a real vector space W endowed with an endomorphism j and a skew-symmetric bilinear form ρ satisfying

$$j^2w=-w, \ \rho(jw, jw')=\rho(w, w'),$$
$$\rho(jw, w)>0 \text{ for any } w\neq 0.$$

A linear endomorphism f of W is called *symplectic* if

$$\rho(fw, w')+\rho(w, fw')=0 \text{ for any } w, w' \in W.$$

Let us denote by $\mathfrak{sp}(W)$ the Lie aqgebra of all symplectic endomorphisms. Then

$$\mathfrak{sp}(W)=\mathfrak{k}(W)+\mathfrak{m}(W), \tag{1.8}$$

where $\mathfrak{k}(W)=\{f \in \mathfrak{sp}(W); f\circ j=j\circ f\}$ and $\mathfrak{m}(W)=\{f \in \mathfrak{sp}(W); f\circ j+j\circ f=0\}$. It is well known that $\mathfrak{k}(W)$ is a maximal compact subalgebra of the semi-simple Lie algebra $\mathfrak{sp}(W)$ and the decomposition (1.8) is a Cartan decomposition. Let $Sp(W)$ and $K(W)$ denote the connected subgroup of $GL(W)$ corresponding to $\mathfrak{sp}(W)$ and $k(W)$ respectively. The homogeneous space $Sp(W)/K(W)$ is a hermitian symmetric space of the non compact type and the complex structure of $Sp(W)/K(W)$ corresponds to the endomorphism I of $\mathfrak{sp}(W)$ given by

$$I(f)=\frac{1}{2}[j, f] \text{ for } f \in \mathfrak{sp}(W).$$

§ 2. A submanifold of a homogeneous Kähler manifold.

Let G/K be a homogeneous Kähler manifold of a connected Lie group G by a closed subgroup K. Let $\mathfrak{r}$ be an abelian ideal of $\mathfrak{g}$ and put

$$\mathfrak{l}=\mathfrak{k}+j\mathfrak{r}+\mathfrak{r}, \quad \mathfrak{l}_0=\mathfrak{k}+j\mathfrak{r}.$$

One can easily see that both $\mathfrak{l}$ and $\mathfrak{l}_0$ are subalgebras of $\mathfrak{g}$. Let L be the connected subgroup of G corresponding to $\mathfrak{l}$. Being a complex submanifold of G/K, $L/L\cap K$ is a homogeneous Kähler submanifold of G/K. Let $\tilde{L}$ be the universal covering group of L and let $\tilde{K}$ be the connected subgroup of $\tilde{L}$ generated by $\mathfrak{k}$. Then $\tilde{L}/\tilde{K}$ is the universal covering space of $L/L\cap K$ and it admits an $\tilde{L}$-invariant Kähler structure so that the canonical projection of $\tilde{L}/\tilde{K}$ onto $L/L\cap K$ is holomorphic and isometric.

We now assume that the sum $\mathfrak{l}=\mathfrak{k}+j\mathfrak{r}+\mathfrak{r}$ is direct. According to [16], we define an affine representation: $u \rightarrow C_u$ of $\mathfrak{l}$ in $\mathfrak{r}^C$ (=the complexification of $\mathfrak{r}$) by

$$C_u(z)=[u, z]+\sqrt{-1}\,x+y \text{ for } z \in \mathfrak{r}^C,$$

where $u=h+jx+y$ ($h \in \mathfrak{k}$, $x, y \in \mathfrak{r}$). This representation induces a homomorphism ϕ of $\tilde{L}$ to the group of affine transformations of $\mathfrak{r}^C$. Clearly $C_u(0)=0$ if and only if $u \in \mathfrak{k}$. Therefore the orbite D^* of $\phi(\tilde{L})$ through the origin 0 is a domain in $\mathfrak{r}^C$ and $\tilde{L}/\tilde{K}$ is the universal covering space of D^*. Since $C_{ju}(0)=\sqrt{-1}\,C_u(0)$ for any $u \in \mathfrak{l}$, the natural projection Φ of $\tilde{L}/\tilde{K}$ onto D^* is holomorphic. Let $\tilde{L}_0$ and $\tilde{R}$ be the connected subgroups of $\tilde{L}$

corresponding to $\mathfrak{l}_0$ and $\mathfrak{r}$ respectively. We then have

$$\tilde{L}=\tilde{R}\cdot\tilde{L}_0,\quad \tilde{R}\cap\tilde{L}_0=\{e\}. \tag{2.1}$$

In fact, the first equality is obvious. Let $a\in\tilde{R}\cap\tilde{L}_0$. There exists $x\in\mathfrak{r}$ such that $a=\exp x$. Then $\phi(a)0=x$. On the other hand, $\phi(\tilde{L}_0)$ leaves the subspece $\sqrt{-1}\,\mathfrak{r}$ invariant. Hence we get $x=0$, proving (2.1). For any $a\in\tilde{L}_0$,, we denote by $\phi_0(a)$ the affine transformation of $\mathfrak{r}$ given by

$$\phi_0(a)v=\frac{1}{\sqrt{-1}}\phi(a)\sqrt{-1}\,v\quad(v\in\mathfrak{r}).$$

Clearly, the assignment: $a\rightarrow\phi_0(a)$ is a homomorphism and it corresponds to the affine representation: $u\rightarrow C_u'$ of $\mathfrak{l}_0$ given by

$$C_u'(r)=[u,r]+x \text{ for } r\in\mathfrak{r},$$

where $u=h+jx$ $(x\in\mathfrak{r},\ h\in\mathfrak{k})$. Let Ω^* be the orbite of $\phi_0(\tilde{L}_0)$ through 0. Then Ω^* is a domain in $\mathfrak{r}$ and ϕ_0 induces a covering projection Φ_0 of $\tilde{L}_0/\tilde{K}$ onto Ω^*. Using (2.1), we obtain

$$D^*=\{z\in\mathfrak{r}^C;\ \mathrm{Im}\ z\in\Omega^*\}.$$

As is mentioned in Shima [13], the following holds:

LEMMA 1. *Ω^* is a convex domain and ϕ_0 is a diffeomorphism of $\tilde{L}_0/\tilde{K}$ onto Ω^*.*

In fact, we can obtain this lemma from a result of Shima [12], noting that $\tilde{L}_0/\tilde{K}$ is a homogeneous Hessian manifold in the sence of [12]. But for the convenience of the readers, we state an outline of the proof by modifying the arguments in [9] and [12].

Let $(z^1,\dots,z^n)$ be the canonical linear coordinate system of $\mathfrak{r}^C$. We may regard $(z^1,\dots,z^n)$ as a local coordinate system of $\tilde{L}/\tilde{K}$ by the map Φ. As usual, we wrire $z^i=x^i+\sqrt{-1}\ y^i$. Then $(y^1,\dots,y^n)$ gives a local coordinate system of $\tilde{L}_0/\tilde{K}$. Let us denote by g the Kähler metric on $\tilde{L}/\tilde{K}$. For any $x\in\mathfrak{r}$, exp x is an isometry and it corresponds to the translation; $z\rightarrow z+x$. Therefore if we express g in terms of the coordinate system $(z^1,\dots,z^n)$ as $g=\sum_{i,j}g_{i\bar{j}}\ dz^i\otimes d\bar{z}^j$, then we have $\frac{\partial g_{i\bar{j}}}{\partial x^k}=0$. Since g is a Kähler metric, this means that $\frac{\partial g_{i\bar{j}}}{\partial y^k}=\frac{\partial g_{k\bar{j}}}{\partial y^i}=\frac{\partial g_{i\bar{k}}}{\partial y^j}$. Let g_0 be the riemannian metric on $\tilde{L}_0/\tilde{K}$ induced from g. If we express g_0 as $g_0=\sum_{i,j}h_{ij}\ dy^i\otimes dy^j$. We then have

$$\frac{\partial h_{ij}}{\partial y^k}=\frac{\partial h_{ik}}{\partial y^j}. \tag{2.2}$$

Let us consider the set Ω_0 given by

$$\Omega_0=\{y\in\mathfrak{r}\ ;\ ty\in\Omega^* \text{ for } 0\leqq t\leqq 1\}.$$

Then Ω_0 is a star-shaped domain in Ω^*. Therefore there exists a diffeomorphism α of Ω_0 onto an open set of $\tilde{L}_0/\tilde{K}$ containing the origin o of $\tilde{L}_0/\tilde{K}$ satisfying $\Phi_0\circ\alpha=1$. Then by the same way as [12], using (2.2) and Poincaré lemma, we can find a function Ψ on Ω_0 so that the induced metric α^*g_0 is expressed as $\alpha^*g_0=\sum_{i,j}\frac{\partial^2\Psi}{\partial y^i\partial y^j}dy^i\otimes dy^j$. Then the function Ψ is convex and it has the following property ([12]): Let y be a point of $\mathfrak{r}\setminus\Omega_0$. If $ty\in\Omega_0$ for $0\leqq t<1$, then $\lim_{t\to 1}\Psi(ty)\to\infty$. From this fact, noting that Ψ is convex and that Ω_0 is star-shaped, one can see that Ω_0 is convex by the similar way as [8]. Let us denote by $\bar{\Omega}_0$ the closure of Ω_0 in Ω^*. Let $y\in\bar{\Omega}_0$. We assert that for any $t\in[0,1)$, $ty\in\Omega_0$. We may assume $t>0$. Let us denote by $B_\delta(x)$ the ball defined by $B_\delta(x)=\{x'\in\mathfrak{r}\ ;\ \|x'-x\|<\delta\}$. Here the norm $\|\ \ \|$ is the usual euclidian norm of $\mathfrak{r}$. If ε is small enough, $B_\varepsilon(0)\subset\Omega_0$. Consider the ball $B_\delta(y)$, where $\delta=\frac{(1-t)}{t}\varepsilon$. Then there exists $y'\in\Omega_0$ such that $y'\in B_\delta(y)$. The line through y' and ty intersects $B_\varepsilon(0)$. Therefore from the convexity of Ω_0, $ty\in\Omega_0$, proving the assertion. Consequently, y belongs to Ω_0 and hence $\bar{\Omega}_0=\Omega_0$. This means that $\Omega^*=\Omega_0$ and therefore we get Lemma 1.

It follows from Lemma 1 that the domain D^* is simply connected. Hence we have

LEMMA 2. *The universal covering space of $L/L\cap K$ is biholomorphic to D^*.*

From Lemma 1, we also know that Ω^* is homeomorphic to $\boldsymbol{R}^n$ and that Ω^* admits an $\tilde{L}_0$-invariant riemannian metric. Moreover every element of $\tilde{L}_0$ acts on Ω^* as an affine transformation. Therefore by the similar arguments as in the proof of Lemma 3.1 in [11], we have

LEMMA 3. *Assume further that G acts on G/K almost effectively. Then by a suitable change of the operator j, $j\mathfrak{r}$ becomes a solvable subalgebra.*

§ 3. j-algebras and associated homogeneous complex manifolds.

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{k}$ be a subalgebra. Let j be an endomorphism of $\mathfrak{g}$ satisfying (1.1), (1.3) and (1.5) and let ω be a linear form on $\mathfrak{g}$. The system $(\mathfrak{g}, \mathfrak{k}, j, \omega)$ or simply $\mathfrak{g}$ is called *a j-algebra* if the following conditions are satisfied;

(3.1) $\omega([\mathfrak{k}, \mathfrak{g}])=0$,

(3.2) $\omega([jx, jy])=\omega([x, y])$ for $x, y \in \mathfrak{g}$,

(3.3) $\omega([jx, x])>0$ if $x \notin \mathfrak{k}$.

Clearly every j-invariant subalgebra of a j-algebra is also a j-algebra. A j-algebra $(\mathfrak{g}, \mathfrak{k}, j, \omega)$ is called *effective* if $\mathfrak{k}$ contains no non-zero ideal of $\mathfrak{g}$. By (3.3), the center of the j-algebra $\mathfrak{g}$ is contained in $\mathfrak{k}$ and it is trivial if $\mathfrak{g}$ is effective.

Let $(\mathfrak{g}, \mathfrak{k}, j, \omega)$ be an effective j-algebra. Let G be a connected Lie group with $\mathfrak{g}$ as its Lie algebra and let K be the connected subgroup of G corresponding to $\mathfrak{k}$. If we regard ω as a left invariant 1-form on G, then K is the identity component of the subgroup given by $\{a \in G; R_a^* \omega=\omega\}$, where R_a denotes the right translation of G defined by $R_a g=ga (g \in G)$. Therefore the group K is closed. The homogeneous space G/K admits a G-invariant complex structure J satisfying (1.4). We call G/K the homogeneous complex manifold associated with the effective j-algebra $(\mathfrak{g}, \mathfrak{k}, j, \omega)$. By (3.1)~(3.3), the symmetric bilinear form $\omega([jx, y])$ on $\mathfrak{g}$ induces a G-invariant Kähler metric on G/K.

Let $\mathfrak{r}$ be an abelian ideal of the effective j-algebra $\mathfrak{g}$. Using (3.1) and (3.3), one can see that the sum

$$\mathfrak{l}=\mathfrak{k}+j\mathfrak{r}+\mathfrak{r}$$

is direct. Let L be the connected subgroup of G corresponding to $\mathfrak{l}$. Then L contains K. Let D^* and Ω^* be as in § 2. We already know from Lemmas 1 and 2 that Ω^* is convex and that the universal covering space of L/K is biholomorphic to D^*. Moreover using the fact stated in § 1, we have from [16]

PROPOSITION 4 ([16]). (1) *Ω^* is a convex domain not containing any entire straight line and hence D^* is biholomorphic to a homogeneous bounded domain.*

(2) *L/K is biholomorphic to D^*.*

REMARK 1. The proof of [16] contains a small gap. But it can be easily corrected by a carefull use of a result of Koszul [9] or by using the fact that Ω^* is simply connected.

An abelian ideal $\mathfrak{r}$ of the effective j-algebra $\mathfrak{g}$ is called of the first kind if there exists $r_0 \in \mathfrak{r}$ such that $[jx, r_0]=x$ for any $x \in \mathfrak{r}$. The element r_0 is called the principal idempotent. Using Proposition 4, one can see from [16] that if there exists a non-trivial abelian ideal $\mathfrak{r}$, then there exists a non-trivial abelian ideal of the first kind contained in $\mathfrak{r}$. In what follows, $\mathfrak{r}$ denotes an abelian ideal of the first kind with the principal idempotent r_0. Let $\mathfrak{l}=\mathfrak{k}+j\mathfrak{r}+\mathfrak{r}$ and let ψ be the Koszul form of $(\mathfrak{l}, \mathfrak{k}, j)$. By Proposition 4, the symmetric bilinear form $\psi([jx, y])$ $(x, y \in \mathfrak{r})$ is positive definite. Therefore there exists a unique $r_0' \in \mathfrak{r}$ such that $\psi(x)=\psi([jx, r_0'])$ for $x \in \mathfrak{r}$. On the one side, $\psi(x)=\psi([jx, r_0])$ for any $x \in \mathfrak{r}$. Therefore $r_0'=r_0$. This means that r_0 is uniquely determined. Using the fact $\psi([\mathfrak{k}, \mathfrak{l}])=0$, one can easily see

$$[\mathfrak{k}, r_0]=0. \tag{3.4}$$

Therefore the condition " $[jx, r_0]=x$ for any $x \in \mathfrak{r}$ " is independent to the choise of the operator j.

From [16], we also know the following

PROPOSITION 5 ([16]). *Let $\mathfrak{r}$ be an abelian ideal of the first kind with the principal idempotent r_0 and let $\mathfrak{g}^{(a)}$ be the largest* ad jr_0*-invariant subspace on which every eigenvalue of* ad jr_0 *has real part a. Then the Lie algebra $\mathfrak{g}$ is decomposed into the sum of subspaces*

$$\mathfrak{g}=\mathfrak{s}+j\mathfrak{r}+\mathfrak{r}+\mathfrak{w}$$

in the following way:

(a) $\mathfrak{g}^{(0)}=\mathfrak{s}+j\mathfrak{r}$, $\mathfrak{g}^{(\frac{1}{2})}=\mathfrak{w}$ *and* $\mathfrak{g}^{(1)}=\mathfrak{r}$.

(b) $\mathfrak{s}$ *is a j-invariant subalgebra containing* $\mathfrak{k}$ *and given by*

$$\mathfrak{s}=\{x \in \mathfrak{g}^{(0)};\ [x, r_0]=0\}.$$

(c) $j\,\mathfrak{w} \subset \mathfrak{w}+\mathfrak{k}$.

(d) *Let $\mathfrak{g}'=\mathfrak{k}+j\mathfrak{r}+\mathfrak{r}+\mathfrak{w}$. Then $\mathfrak{g}'$ is a j-invariant subalgebra. Let G' be the connected subgroup of G generated by $\mathfrak{g}'$. Then the homogeneous complex submanifold G'/K is biholomorphic to a homogeneous Siegel domain of the second kind.*

It should be noted that the above decomposition of $\mathfrak{g}$ is uniquely determined from the abelian ideal $\mathfrak{r}$ and independent to the choise of j. More precisely, let j' be another endomorphism equivalent to j and let $\mathfrak{g}^{(0)\prime}$, $\mathfrak{w}'$, $\mathfrak{s}'$ be the subspaces obtained from the element $j'r_0$. Then $\mathfrak{g}^{(0)\prime}=\mathfrak{g}^{(0)}$, $\mathfrak{w}'=\mathfrak{w}$ and $\mathfrak{s}'=\mathfrak{s}$ ([11]).

We also note

$$[\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \subset \mathfrak{g}^{(0)}, \ [\mathfrak{g}^{(0)}, \mathfrak{w}] \subset \mathfrak{w}, \ [\mathfrak{w}, \mathfrak{w}] \subset \mathfrak{r},$$

because $[\mathfrak{g}^{(a)}, \mathfrak{g}^{(b)}] \subset \mathfrak{g}^{(a+b)}$.

By virture of Proposition 5, the results in §§ 3, 4 of [11] also hold for our homogeneous complex manifold G/K. In particular, we have

PROPOSITION 6 ([11]). (1) *Let S be the connected subgroup of G corresponding to the subalgebra $\mathfrak{s}$. Then S is a closed subgroup of G containing K and the homogeneous space G/S is a cell.*

(2) *For any $s \in \mathfrak{s}$,* $\mathrm{Tr}_{\mathfrak{r}} \mathrm{ad}\, s = 0$.

(3) *If $\mathfrak{r}$ is a maximal abelian ideal of the first kind, then the subalgebra $\mathfrak{s}$ is reductive.*

The assertions (2) and (3) are first proved in [16] for an effective algebraic j-algebra.

By Lemma 3, we may assume that $j\mathfrak{r}$ is a solvable subalgebra. We may also assume that $\mathfrak{w}$ is invariant by j. We set

$$\mathfrak{t}' = j\mathfrak{r} + \mathfrak{r} + \mathfrak{w}.$$

Then $\mathfrak{t}'$ is a j-invariant solvable subalgebra and

$$\mathfrak{g} = \mathfrak{t}' + \mathfrak{s} \text{ (vector space direct sum)}.$$

PROPOSITION 7. *Let T' and S be the connected subgroups of G corresponding to $\mathfrak{t}'$ and $\mathfrak{s}$ respectively. Then $G = T' \cdot S$, $T' \cap S = \{e\}$.*

PROOF Let us denote by T_0, L_0 and G_0 the connected subgroups of G generated by $j\mathfrak{r}$, $j\mathfrak{r} + \mathfrak{k}$, and $\mathfrak{g}^{(0)}$ respectively. Let Ω^* be as before and put $\Omega = \Omega^* + r_0$. Let ξ be the translation of $\mathfrak{r}$ given by $\xi(y) = y + r_0$. Then we have $\Omega = \xi \circ \phi_0(\tilde{L}_0) \circ \xi^{-1}(r_0)$. Using (3.4) and using the hypothesis that $\mathfrak{r}$ is of the first kind, we have $\xi \circ \phi_0(\tilde{L}_0) \circ \xi^{-1} = \{\mathrm{Ad}\, a|_{\mathfrak{r}}\,;\, a \in L_0\}$. Therefore $\Omega = \{\mathrm{Ad}\, a\, r_0\,;\, a \in L_0\} = \{\mathrm{Ad}\, a\, r_0\,;\, a \in T_0\}$. Since $\mathrm{Ad}(\exp tjr_0)\, r_0 = e^t r_0$, Ω is a convex cone not containing any entire straight line. It follows from (2) of Proposition 6 and a result of Vinberg [14] that $\Omega = \{\mathrm{Ad}\, a\, r_0\,;\, a \in G_0\}$ and $S = \{a \in G_0\,;\, \mathrm{Ad}\ a\ r_0 = r_0\}$ (cf. [11], [16]). Therefore $\Omega = G_0/S$ and hence $G_0 = T_0 \cdot S$. Let N be the connected subgroup generated by $\mathfrak{r} + \mathfrak{w}$. Since $\mathfrak{r} + \mathfrak{w}$ is an ideal and since $\mathfrak{g} = \mathfrak{g}^{(0)} + \mathfrak{r} + \mathfrak{w}$, we have $G = N \cdot G_0$. Therefore $G = N \cdot T_0 \cdot S = T' \cdot S$. Since $T' \cap S$ is discrete and since G/S is a cell, we have $T' \cap S = \{e\}$.

q. e. d.

REMARK 2. As is stated before, the assertins (2) and (3) are proved in [16] for an effective algebraic j-algebra. But the proof in [16] implicitly used the following fact (cf. P. 422, Proof of Lemma 1, § 3 and P. 430, [16]) : Let $(\mathfrak{g}, \mathfrak{k}, j, \omega)$ be an effective j-algebra and let $\mathfrak{h}$ be a j-invariant subalgebra of $\mathfrak{g}$ containing $\mathfrak{k}$. Denote by $\mathfrak{n}$ the largest ideal of $\mathfrak{h}$ contained in $\mathfrak{k}$. Then the Lie algebra $\mathfrak{h}/\mathfrak{n}$ equipped with the subalgebra $\mathfrak{k}/\mathfrak{n}$ and the endomorphism induced from j admits a structure of an effective j-algebra. This fact is not so obvious if we do not use Lemma 1. 2 of [11]. Note that the effectiveness assumption on $\mathfrak{g}$ is essential. In fact, one can construct a j-algebra $(\mathfrak{g}, \mathfrak{k}, j, \omega)$ so that $\mathfrak{g}/\mathfrak{n}$ does not admits a structure of a j-algebra, where $\mathfrak{n}$ is the largest ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$.

§ 4. A fibering of the associated homogeneous complex manifold G/K.

Let G/K be the homogeneous complex manifold associated with an effective j-algebra $(\mathfrak{g}, \mathfrak{k}, j, \omega)$. We use the same notations as the previous section. Let $\mathfrak{r}$ be a maximal abelian ideal of the first kind with the principal idempotent r_0. Then the subalgebra $\mathfrak{s}$ is reductive by Proposition 6 and hence it is decomposed as

$$\mathfrak{s} = \mathfrak{c}(\mathfrak{s}) + \mathfrak{s}_1 + \mathfrak{s}_2, \tag{4.1}$$

where $\mathfrak{c}(\mathfrak{s})$ denotes the center of $\mathfrak{s}$, $\mathfrak{s}_2$ denotes the compact semi-simple ideal and $\mathfrak{s}_1$ is the semi-simple ideal having no compact components. Recall that $\mathfrak{c}(\mathfrak{s})$ is contained in $\mathfrak{k}$ (§ 3). Being a complex submanifold of G/K, S/K is a homogeneous Kähler manifold on which the semi-simple part of S acts transitively. It follows from Borel [1] that $\mathfrak{k} = \mathfrak{c}(\mathfrak{s}) + \mathfrak{k} \cap \mathfrak{s}_1 + \mathfrak{k} \cap \mathfrak{s}_2$ and we can assume $j\mathfrak{s}_i \subset \mathfrak{s}_i$ $(i=1, 2)$. Moreover let S_1 denote the connected subgroup of G corresponding to $\mathfrak{s}_1$. Then there exists a closed subgroup U_1 of S_1 containing $K \cap S_1$ and the followings hold ([1]) :

(*a*) *The Lie algebra* $\mathfrak{u}_1$ *of* U_1 *is a maximal compact subalgebra of* $\mathfrak{s}_1$.

(*b*) $j\, \mathfrak{u}_1 \subset \mathfrak{u}_1$ *and* $[jx, u] \equiv j[x, u]$ (mod $\mathfrak{u}_1$) *for any* $x \in \mathfrak{s}_1$, $u \in \mathfrak{u}_1$.

(*c*) *The homogeneous space* S_1/U_1 *admits an* S_1*-invariant complex structure by the property* (*b*). *Equipped with this complex structure,* S_1/U_1 *is biholomorphic to a symmetric bounded domain and the canonical projection of* $S_1/S_1 \cap K$ *onto* S_1/U_1 *is holomorphic.*

(*d*) *The fiber* $U_1/U_1 \cap K$ *is a compact simply connected homogeneous Kähler manifold.*

We set

$$\mathfrak{u} = \mathfrak{c}(\mathfrak{s}) + \mathfrak{u}_1 + \mathfrak{s}_2.$$

Clearly $\mathfrak{u}$ is a j-invariant subalgebra containing $\mathfrak{k}$. Let U denote the connected subgroup of G with $\mathfrak{u}$ as its Lie algebra. Then U is a closed subgroup of S and hence it is also closed in G because of Proposition 6. Clearly U/K is compact. Since U/K admits a U-invariant Kähler structure and since the semi-simple part of U acts on U/K transitively, U/K is simply connected by a result of [1]. We shall prove the following

PROPOSITION 8. *The homogeneous space G/U admits naturally a G-invarianot complex structure with respect to which the projection of G/K onto G/U is holomorphic.*

We can assume that $j\mathfrak{r}$ is a solvable subalgebra and that $j\mathfrak{w}=\mathfrak{w}$. We need the following result.

LEMMA 9 ([11]). *There exists $r_\alpha \in \mathfrak{r}$ $(\alpha=1,\dots, m)$ and the decompositions $\mathfrak{r}=\sum_{\alpha\leqq\beta}\mathfrak{r}_{\alpha\beta}$, $\mathfrak{w}=\sum_\alpha \mathfrak{w}_\alpha$ satisfying the followings:*

(a) $[jr_\alpha, jr_\beta]=0$, $[jr_\alpha, r_\beta]=\delta_{\alpha\beta}r_\beta$ *and* $r_0=\sum_\alpha r_\alpha$.

(b) $\mathfrak{r}_{\alpha\alpha}=\boldsymbol{R}\, r_\alpha$ *and* $j\mathfrak{w}_\alpha=\mathfrak{w}_\alpha$.

(c) $\mathfrak{r}_{\alpha\beta}$, $\mathfrak{w}_\alpha$, *and* $j\mathfrak{r}_{\alpha\beta}$ *are invariant by* ad jr_γ *and the real parts of the eigenvalues of* ad jr_γ *on* $\mathfrak{r}_{\alpha\beta}$, $\mathfrak{w}_\alpha$, *and* $j\mathfrak{r}_{\alpha\beta}$ *are equal to* $\frac{1}{2}(\delta_{\alpha\gamma}+\delta_{\beta\gamma})$, $\frac{1}{2}\delta_{\alpha\gamma}$, *and* $\frac{1}{2}(\delta_{\alpha\gamma}-\delta_{\beta\gamma})$.

Using this lemma, we prove

LEMMA 10. *Let ψ be the Koszul form of* $(\mathfrak{l}, \mathfrak{k}, j)$. *Then*

(1) $\psi([jw, w])>0$ *for every non-zero w of* $\mathfrak{w}$.

(2) $\psi([s, x])=0$ *for any* $s\in\mathfrak{s}$ *and* $x\in\mathfrak{r}$.

PROOF. Let $w\in\mathfrak{w}$. We can write as $w=\sum_\alpha w_\alpha$, where $w_\alpha\in\mathfrak{w}_\alpha$. We then have $[w_\alpha, w_\beta]\in\mathfrak{r}_{\alpha\beta}$. Therefore there exists $c_\alpha\in\boldsymbol{R}$ such that $[jw_\alpha, w_\alpha]=c_\alpha r_\alpha$. We can see $c_\alpha\geqq 0$. In fact, let ψ' be the Koszul form of $(\mathfrak{g}', \mathfrak{k}, j)$. By Proposition 5, $\psi'([jx, x])\geqq 0$ for any $x\in\mathfrak{g}'$ and $\psi'([jx, x])=0$ if and only if $x\in\mathfrak{k}$. Therefore $\psi'(r_\alpha)=\psi'([jr_\alpha, r_\alpha])>0$ and $\psi'([jw_\alpha, w_\alpha])\geqq 0$. Hence $c_\alpha\geqq 0$. Similarly, we have $\psi(r_\alpha)>0$. By a direct computation, we have

$$\psi(x)=2\ \mathrm{Tr}_{\mathfrak{r}}\ \mathrm{ad}\ jx \text{ for } x\in\mathfrak{r}. \tag{4.2}$$

Hence using Lemma 9, we have $\psi(\mathfrak{r}_{\alpha\beta})=0$ for $\alpha<\beta$. It follows that $\psi([jw, w])=\sum_\alpha c_\alpha\psi(r_\alpha)\geqq 0$. Evidently, $c_\alpha=0$ if and only if $w_\alpha=0$, proving (1).

Let $x\in\mathfrak{r}$ and $s\in\mathfrak{s}$. Then $[s, jx]-j[s, x]\in\mathfrak{g}^{(0)}$. By a simple calculation, $[[s, jx]-j[s, x], r_0]=0$. This means

$$[s, jx]-j[s, x]\in\mathfrak{s} \text{ for } s\in\mathfrak{s},\ x\in\mathfrak{r}. \tag{4.3}$$

Using (4.2), (4.3) and (2) of Proposition 6, we have

$$\psi([s, x])=2 \operatorname{Tr}_{\mathfrak{r}} \operatorname{ad} j[s, x]=2 \operatorname{Tr}_{\mathfrak{r}} \operatorname{ad} [s, jx]=0.$$

q. e. d.

We now prove Proposition 8. Define a skew-symmetric bilinear form ρ on $\mathfrak{w}$ by

$$\rho(w, w')=\psi([w, w']) \text{ for } w,\ w'\in\mathfrak{w},$$

where ψ is the Koszul form of $(\mathfrak{l}, \mathfrak{k}, j)$. By Lemma 10, $(\mathfrak{w}, j, \rho)$ is a symplectic space and for every $s\in\mathfrak{s}$, ad s is a symplectic endomorphism of $\mathfrak{w}$. It is easily checked that $\operatorname{ad} js\equiv I(\operatorname{ad} s) \pmod{\mathfrak{k}(\mathfrak{w})}$, where I denotes the endomorphism of $\mathfrak{sp}(\mathfrak{w})$ as in § 1. For each $k\in K$, Ad k is an element of $K(\mathfrak{w})$. Then the natural mapping ζ of S/K to $Sp(\mathfrak{w})/K(\mathfrak{w})$ is holomorphic. Since $Sp(\mathfrak{w})/K(\mathfrak{w})$ is biholomorphic to a bounded domain, $\zeta(U/K)$ must be a point. This implies

$$[u, jw]=j[u, w] \text{ for } u\in\mathfrak{u},\ w\in\mathfrak{w}. \tag{4.4}$$

Let us set

$$\mathfrak{s}_0=\{s\in\mathfrak{g}^{(0)};\ [s, \mathfrak{r}]=0\}.$$

Then $\mathfrak{s}_0$ is an ideal of $\mathfrak{g}^{(0)}$ contained in $\mathfrak{s}$. Since $\mathfrak{s}$ is reductive, so is $\mathfrak{s}_0$. Let Ω be the convex cone as in the proof of Proposition 7. Then $\mathfrak{s}/\mathfrak{s}_0$ is identified with the isotropy subalgebra of a transitive isometric transformation group of Ω. In particular, $\mathfrak{s}/\mathfrak{s}_0$ is reductive and its semi-simple part is compact. Consequently, $\mathfrak{s}_1$ is an ideal of $\mathfrak{s}_0$. It follows that for any $x\in\mathfrak{r}$, $\mathfrak{s}_1$ is invariant by ad jx. Hence there corresponds $s(x)\in\mathfrak{s}_1$ satisfying

$$[s(x), s']=[jx, s'] \text{ for any } s'\in\mathfrak{s}_1. \tag{4.5}$$

We then have

$$[s(x), js']\equiv j[jx, s']\equiv j[s(x), s'] \pmod{\mathfrak{k}\cap\mathfrak{s}_1} \text{ for } x\in\mathfrak{r},\ s'\in\mathfrak{s}_1. \tag{4.6}$$

Since $S_1/S_1\cap K$ is a homogeneous Kähler manifold of the semi-simple Lie group S_1, its canonical hermitian form is non-degenerate (see, § 1). As a result, (4.6) combined with a result of Hano [4] means that $s(x)$ is an element of $\mathfrak{k}\cap\mathfrak{s}_1$. Let $u\in\mathfrak{u}$ and $x\in\mathfrak{r}$. By (4.3), $[u, jx]-j[u, x]\in\mathfrak{s}$. We can see for any $s'\in\mathfrak{s}_1$,

$$[[u, jx]-j[u, x],\ s']=[[u, s(x)]-s([u, x]),\ s'].$$

Therefore the $\mathfrak{s}_1$-component of $[u, jx]-j[u, x]$ with respect to the decomposition (4.1) is equal to $[u, s(x)]-s([u, x])$. Hence we have

$$(4.7) \qquad [u, jx]\equiv j[u, x] \pmod{\mathfrak{u}} \text{ for } u\in\mathfrak{u},\ x\in\mathfrak{r}.$$

It follows from (4.4), (4.7) and the propertiy (b) of the group U_1

$$(4.8) \qquad [u, jx]\equiv j[u, x] \pmod{\mathfrak{u}} \text{ for } u\in\mathfrak{u},\ x\in\mathfrak{g}.$$

Since U is connected, (4.8) means that there corresponds a G-invariant complex structure on G/U. Clearly the projection: $G/K\rightarrow G/U$ is holomorphic. Thus we have proved Proposition 8.

§ 5. Proof of Theorem A.

Let G/K be the homogeneous complex manifold associated with the effective j-algebra $(\mathfrak{g}, \mathfrak{k}, j, \omega)$. We use the same notations as before. Let $s(x)$ be the element of $\mathfrak{k}\cap\mathfrak{s}_1$ given by (4.5) for an element $x\in\mathfrak{r}$. If $x\in\mathfrak{r}_{\alpha\beta}$ $(\alpha<\beta)$, then ad jx is a nilpotent endomorphism of $\mathfrak{g}$. Indeed, let us denote by $\mathfrak{g}_\alpha^{(a)}$ the largest ad jr_α-invariant subspace on which every eigenvalue of ad $j\mathfrak{r}_\alpha$ has real part a. Then $[\mathfrak{g}_\alpha^{(a)}, \mathfrak{g}_\alpha^{(b)}]\subset\mathfrak{g}_\alpha^{(a+b)}$ and $j\mathfrak{r}_{\alpha\beta}\subset\mathfrak{g}_\alpha^{(\frac{1}{2})}$ for $\alpha<\beta$. Hence if $x\in\mathfrak{r}_{\alpha\beta}$ $(\alpha<\beta)$, then ad $s(x)$ is a nilpotent endomorphism of $\mathfrak{s}_1$. On the other hand, since $s(x)\in\mathfrak{k}\cap\mathfrak{s}_1$, ad $s(x)$ is semi-simple. Therefore $s(x)=0$ and we get

$$[j\mathfrak{r}_{\alpha\beta}, \mathfrak{s}_1]=0 \text{ for } \alpha<\beta.$$

We now put

$$j'r_\alpha=jr_\alpha-s(r_\alpha),\ j'x=jx \text{ for } x\in\mathfrak{r}_{\alpha\beta}\ (\alpha<\beta).$$

One can easily see that $j'\mathfrak{r}$ is also a solvable subalgebra. Therefore taking $j'\mathfrak{r}$ instead of $j\mathfrak{r}$, we may assume

$$(5.1) \qquad [j\mathfrak{r}, \mathfrak{s}_1]=0.$$

Since $\mathfrak{u}_1$ is a maximal compact subalgebra of $\mathfrak{s}_1$, there exist a solvable subalgebra $\mathfrak{t}_1$ and an endomorphism j_1 of $\mathfrak{t}_1$ satisfying

$$\mathfrak{s}_1=\mathfrak{u}_1+\mathfrak{t}_1 \text{ (vector space direct sum)},$$
$$j_1x\equiv jx \pmod{\mathfrak{u}_1} \text{ for } x\in\mathfrak{t}_1.$$

Let us set

$$\mathfrak{t}=\mathfrak{t}_1+\mathfrak{t}'\ (\mathfrak{t}'=j\mathfrak{r}+\mathfrak{r}+\mathfrak{w}).$$

Using (5.1), we know that $\mathfrak{t}$ is a solvable subalgebra. Clearly

$$\mathfrak{g}=\mathfrak{t}+\mathfrak{u} \text{ (vector space direct sum).}$$

We define an endomorphism $\hat{j}$ of $\mathfrak{t}$ by

$$\hat{j}x=j_1x \text{ if } x\in\mathfrak{t}_1 \text{ and } \hat{j}x=jx \text{ if } x\in\mathfrak{t}'. \tag{5.2}$$

We then have

$$\hat{j}x\equiv jx \pmod{\mathfrak{u}} \text{ for any } x\in\mathfrak{t}. \tag{5.3}$$

Let T_1 and T be the connected subgroup of G corresponding to $\mathfrak{t}_1$ and $\mathfrak{t}$ respectively. Then $S_1=T_1\cdot U_1$ and hence $S=T_1\cdot U$. Recall that $G=T'\cdot S$ by Proposition 7. Therefore $G=T\cdot U$. This means that the group T acts on G/U transitively. Since $T\cap U$ is discrete, T is a covering space of G/U. As a result, T admits a left invariant complex structure so that the projection of T onto G/U is holomorphic. By (5.3), the corresponding endomorphism of $\mathfrak{t}$ is nothing but the operator $\hat{j}$ given by (5.2).

We shall show that the Lie algebra $\mathfrak{t}$ admits a structure of a j-algebra. Let ψ_1 and ψ' be the Koszul forms of $(\mathfrak{t}_1, 0, j_1)$ and $(\mathfrak{t}', 0, j)$ respectively and define a linear form $\hat{\omega}$ on $\mathfrak{t}$ by

$$\hat{\omega}(x_1)=\psi_1(x_1) \text{ if } x_1\in\mathfrak{t}_1 \text{ and } \hat{\omega}(x')=\psi'(x') \text{ if } x'\in\mathfrak{t}'.$$

Since $\psi'(\mathfrak{w})=0$ and since $[\mathfrak{t}_1,\mathfrak{t}']\subset\mathfrak{w}$, we have $\hat{\omega}([\mathfrak{t}_1,\mathfrak{t}'])=0$. Recalling that both $\mathfrak{t}_1$ and $\mathfrak{t}'$ are the Lie algebras corresponding to homogeneous bounded domains, we have for x_1, $y_1\in\mathfrak{t}_1$ and for x', $y'\in\mathfrak{t}'$

$$\begin{aligned}\hat{\omega}([\hat{j}(x_1+x'), \hat{j}(y_1+y')])&=\psi_1([j_1x_1, jy_1])+\psi'([jx', jy'])\\&=\psi_1([x_1, y_1])+\psi'([x', y'])=\hat{\omega}([x_1+x', y_1+y'])\end{aligned}$$

and

$$\hat{\omega}([\hat{j}(x_1+x'), x_1+x'])=\psi_1([j_1x_1, x_1])+\psi'([jx', x'])>0,$$

if $x_1+x'\neq 0$. Hence $(\mathfrak{t}, 0, \hat{j}, \hat{\omega})$ is a j-algebra. We apply Proposition 5 to the j-algebra $\mathfrak{t}$. Let $\mathfrak{r}_t$ be a maximal abelian ideal of the first kind. Then we have $\mathfrak{t}=\hat{j}\mathfrak{r}_t+\mathfrak{r}_t+\mathfrak{w}_t$, because $\mathfrak{t}$ is solvable and $\mathfrak{k}=0$. It follows from Proposition 5 that the group T is biholomorphic to a homogeneous bounded domain. Since the projection: $T\to G/U$ is a holomorphic covering mapping, G/U itself is biholomorphic to a homogeneous bounded domain. It is now clear that $T\cap U=\{e\}$. Hence we have proved

THEOREM 11. *Let G/K be a homogeneous complex manifold associated with an effective j-algebra $(\mathfrak{g}, \mathfrak{k}, j, \omega)$. Then there exist a closed connected* *reductive subgroup U containing K and a connected solvable subgroup T and the followings hold*:

(a) $G=T\cdot U,\ T\cap U=\{e\}$.

(b) *U/K is a compact simply connected homogeneous complex submanifold of G/K.*

(c) *The homogeneous space G/U admits naturally a G-invariant complex structure with respect to which the projection of G/K onto G/U is holomorphic and G/U is biholomorphic to a homogeneous bounded domain.*

We now set $M_1=G/U$ and $M_2=U/K$. For the proof of Theorem A, it remains to show that G/K is biholomorphic to $M_1\times M_2$. This can be done as follows. Consider the fibering: $G/K\rightarrow M_1$. Then every fiber is biholomorphic to the compact complex manifold M_2. Therefore by a result of Fischer and Grauert [2], this fibering is a holomorphic fiber bundle. Its structure group may be taken to be a complex Lie group. Consequently, as is mentioned in [15], this bundle is holomorphically trivial by a theormem of Grauert [3], because M_1 is topologically trivial. Hence we get Theorem A.

REMARK 3. By Theormem 11, the homogeneous complex manifold G/K associated with an effective j-algebra $(\mathfrak{g}, \mathfrak{k}, j, \omega)$ is simply connected. Let $\tilde{G}$ be the simply connected Lie group with $\mathfrak{g}$ as its Lie algebra and let $\tilde{K}$ be the connected subgroup of $\tilde{G}$ corresponding to $\mathfrak{k}$. We then have $\tilde{G}/\tilde{K}=G/K$ because G/K is simply connected. Therefore the associated homogeneous complex manifold is uniquely determined from the effective j-algebra $\mathfrak{g}$ and independent to the choise of the group G.

§ 6. Some consequences obtained from Theorem 11.

An effective j-algebra $(\mathfrak{g}, \mathfrak{k}, j, \omega)$ is called *proper* if every compact semi-simple j-invariant subalgebra is contained in $\mathfrak{k}$. Let $\mathfrak{u}$ be the Lie algebra of the group U as in Theorem 11. Then the semi-simple part of $\mathfrak{u}$ is a compact j-invariant subalgebra. Therefore if the effective j-algebra $\mathfrak{g}$ is proper, then $\mathfrak{u}$ coincides with $\mathfrak{k}$. Hence we obtain from Theorem 11 the following

THEOREM 12 (Vinverg, Gindikin and Pyatetskii-Shapiro [16]). *Every homogeneous complex manifold associated with an effective proper j-algebra is biholomorphic to a homogeneous bounded domain.*

By using Theorem 11, we can also prove the following theorem of Koszul.

THEOREM 13 (Koszul [10]). *Let G/K be a homogeneous complex manifold with a G-invariant volume element and assume that the canonical hermitian form is positive definite. Then G/K is biholomorphic to a homogeneous bounded domain.*

PROOF. We may assume that the action of G is effective. Let ψ be the Koszul form of $(\mathfrak{g}, \mathfrak{k}, j)$. From the hypothesis, $(\mathfrak{g}, \mathfrak{k}, j, \psi)$ is an effective j-algebra. Let K_0 be the identity component of K. Then G/K_0 is the homogeneous complex manifold associated with $(\mathfrak{g}, \mathfrak{k}, j, \psi)$. Let U be as in Theorem 11 and let $\mathfrak{u}$ be its Lie algebra. We denote by ψ_1(resp. by ψ_2) the Koszul form of $(\mathfrak{g}, \mathfrak{u}, j)$ (resp. of $(\mathfrak{u}, \mathfrak{k}, j)$). We then have for any $u \in \mathfrak{u}$, $\psi([ju, u]) = \psi_1([ju, u]) + \psi_2([ju, u])$. Clearly $\psi_1([ju, u]) = 0$. Since U/K is a compact simply conneced homogeneous Kähler manifold, $\psi_2([ju, u]) \leqq 0$. Therefore $\psi([ju, u]) = 0$ and hence $\mathfrak{u} = \mathfrak{k}$. It follows from Theorem 11 that $G/K_0 (= G/U)$ is a homogeneous bounded domain. Hence we can conclude that G/K itself is a homogeneous bounded domain.

q. e. d.

As an immediate consequence of this theorem, we have

COROLLARY 14. *Every homogeneous Kähler manifold of negative definite Ricci tensor is biholomorphic to a homogeneous bounded domain.*

§ 7. Proof of Theorem B.

Let M_1, M_2 and G be as Theorem B. We denote by K the isotropy subgroup of G at a point $(p_1, p_2) \in M_1 \times M_2$. Every element g of G can be expressed as $g(z_1, z_2) = (g_1(z_1, z_2), g_2(z_1, z_2))$. Then $g_1(z_1, z_2)$ is independent to z_2, because M_2 is compact. Let us define a closed subgroup U by

$$U = \{g \in G\,;\ g_1(p_1) = p_1\}.$$

We then have $G/U = M_1$ and $U/K = M_2$. It should be noted that U is connected because M_1 is simply connected. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively and let j be the endomorphism of $\mathfrak{g}$ corresponding to the complex structure of $M_1 \times M_2$. We also denote by $\mathfrak{u}$ the Lie algebra of U. Clearly $\mathfrak{u}$ is j-invariant. Let ψ, ψ_1 and ψ_2 be the Koszul forms of $(\mathfrak{g}, \mathfrak{k}, j)$, $(\mathfrak{g}, \mathfrak{u}, j)$ and $(\mathfrak{u}, \mathfrak{k}, j)$ given by (1.6). For any $x \in \mathfrak{g}$, ad $jx - j \circ$ad x leaves $\mathfrak{u}$ and $\mathfrak{k}$ invariant. Define a linear form ψ_2' on $\mathfrak{g}$ by

$$\psi_2'(x) = \mathrm{Tr}_{\mathfrak{u}/\mathfrak{k}}(\mathrm{ad}\ jx - j \circ \mathrm{ad}\ x) \quad \text{for } x \in \mathfrak{g}.$$

We then have $\psi = \psi_1 + \psi_2'$ and $\psi_2' = \psi_2$ on $\mathfrak{u}$. Since ψ and ψ_1 satisfy (1.7), we have $\psi_2'([\mathfrak{k}, \mathfrak{g}]) = 0$ and $\psi_2'([jx, jy]) = \psi_2'([x, y])$ for any x, $y \in \mathfrak{g}$. Let us set

$$\omega=\beta\ \psi_1-\psi'_2,$$

where β is a positive number. Then ω satisfies (3.1) and (3.2). Since M_1 is a homogeneous bounded domain, $\psi_1([jx, x])\geqq 0$ for any $x\in\mathfrak{g}$ and the equality holds if and only if $x\in\mathfrak{u}$. On the one side, if $x\in\mathfrak{u}$, then $\psi'_2([jx, x])\leqq 0$ and the equality holds if and only if $x\in\mathfrak{k}$, because U/K is a compact simply connected homogeneous Kähler manifold. Consequently, if we take β large enough, then ω satisfies (3.3). Therefore $(\mathfrak{g}, \mathfrak{k}, j, \omega)$ becomes an effective j-algebra, proving Theorem B.

§ 8. Proof of Theorem C.

In this section, we study the structure of homogeneous Kähler manifolds satisfying (C) and prove Theorem C. The following result is essentially proved in [7].

PROPOSITION 15 (Kodama and Shima [7]). *Let G/K be a homogeneous Kähler manifold satisfying (C). Assume further that G is solvable. Then G/K is holomorphically isomorphic to a homogeneous bounded domain.*

Let G/K be a homogeneous Kähler manifold satisfying (C). We may assume that G acts on G/K effectively. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively and denote by j the endomorphism of $\mathfrak{g}$ corresponding to the complex structure of G/K. From the condition (C), we have (cf. [7], [13])

$$[jx, x]\neq 0 \text{ if } x\notin\mathfrak{k}. \tag{8.1}$$

Let $\mathfrak{r}$ be an abelian ideal of $\mathfrak{g}$. Using (8.1), one can easily see that the sum $\mathfrak{l}=\mathfrak{k}+j\mathfrak{r}+\mathfrak{r}$ is direct. Let L be the connected subgroup corresponding to $\mathfrak{l}$. Then $L/L\cap K$ is a homogeneous Kähler submanifold satisfying (C). By Lemma 3, we may assume that $j\mathfrak{r}$ is a solvable subalgebra. Then the connected subgroup corresponding to the solvable subalgebra $j\mathfrak{r}+\mathfrak{r}$ acts on $L/L\cap K$ transitively. Therefore from Proposition 15, we know that the homogeneous space $L/L\cap K$ is biholomorphic to a homogeneous bounded domain.

By virture of this fact, we can also see by the same method as [11] that if $\mathfrak{r}$ is a non-trivial abelian ideal, then there exists a non-trivial abelian ideal of the first kind which is contained in $\mathfrak{r}$.

Next we shall show that the analogous assertion to Proposition 5 holds. Let $\mathfrak{r}$ be an abelian ideal of the first kind with a principal idempotent r_0 and let $\mathfrak{g}^{(a)}$ be the subspace as in Proposition 5. We define a j-invariant subspace Q by

$$Q=\{x\in\mathfrak{g}\ ;\ [x, r_0]=[jx, r_0]=0\}.$$

One can easily see that $\mathfrak{g}=Q+\mathfrak{l}$, $Q\cap\mathfrak{l}=\mathfrak{k}$ and that Q is invariant by ad jr_0. Therefore Q is decomposed as $Q=\sum_{a\in\boldsymbol{R}} Q^{(a)}$, where $Q^{(a)}=Q\cap\mathfrak{g}^{(a)}$. Using (8. 1), we can show by the same way as [11],

$$\mathfrak{g}^{(1)}=\mathfrak{r},\ \mathfrak{g}^{(0)}=j\mathfrak{r}+Q^{(0)},$$
$$Q^{(a)}=0 \text{ for } a<0 \text{ or } a>\frac{1}{2},$$
$$jQ^{(a)}\subset Q^{(a)}+\mathfrak{k}.$$

Let us set $\mathfrak{g}'=\mathfrak{k}+j\mathfrak{r}+\mathfrak{r}+\sum_{0<a} Q^{(a)}$. Then $\mathfrak{g}'$ is a j-invariant subalgebra. Therefore if we denote by G' the corresponding subgroup, then $G'/G'\cap K$ is a homogeneous Kähler manifold satisfying (C). By changing j suitably, we may assume that $j\mathfrak{r}$ is a solvable subalgebra and that $jQ^{(a)}\subset Q^{(a)}$. Note that $[\mathfrak{g}^{(a)}, \mathfrak{g}^{(b)}]\subset\mathfrak{g}^{(a+b)}$ and that $\mathfrak{g}^{(a)}=Q^{(a)}$ if $a\neq 0$, 1. Then $\mathfrak{t}'=j\mathfrak{r}+\mathfrak{r}+\sum_{0<a} Q^{(a)}$ is a j-invariant solvable subalgebra and the corresponding subgroup acts on $G'/G'\cap K$ transitively. Hence by Proposition 15, $G'/G'\cap K$ is biholomorphic to a homogeneous bounded domain. Let us denote by ψ' the Koszul form of $(\mathfrak{t}', 0, j)$. Then $\psi'(Q^{(a)})=0$ for any $a\neq 0$. If $a\neq 0, \frac{1}{2}$, then $[jQ^{(a)}, Q^{(a)}]\subset Q^{(2a)}$ and hence $\psi'([jQ^{(a)}, Q^{(a)}])=0$. This means that $Q^{(a)}=0$ for $a\neq 0, \frac{1}{2}$, because $\psi'([jx, x])>0$ for $x\neq 0$. Clearly $Q^{(0)}=\{x\in\mathfrak{g}^{(0)}\ ;\ [x, r_0]=0\}$. We now set $\mathfrak{s}=Q^{(0)}$ and $\mathfrak{w}=Q^{(\frac{1}{2})}$. Then the decomposition of $\mathfrak{g}$ stated in Proposition 5 also holds.

Now from [11], we know that Proposition 6 also holds for the homogeneous Kähler manifold G/K. If we assume that $\mathfrak{r}$ is a maximal abelian ideal of the first kind, then the group S corresponddding to $\mathfrak{s}$ is reductive and hence S/K is a homogeneous Kähler manifold on which the semi-simple part of S acts transitively. Therefore by Borel [1], S/K is simply connected. In particular, the group K is connected. Now Theorem C can be proved by the same arguments as in §§ 4 and 5.

REMARK 4. To prove Proposition 15, Kodama and Shima [7] calculated the canonical hermitian form of G/K and showed that it is positive definite. From this fact, combined with the result of [16] (Theorem 12) they conclude that G/K is biholomorphic to a homogeneous bounded domain. We may also apply Theorem 13 and obtain Proposition 15.

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