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LOCAL TOPOLOGICAL MODELS OF ENVELOPES

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1. Introduction

Let X and Y be smooth equidimensional manifolds and let $\Gamma \subset X \times Y$ be a smooth hypersurface with the natural projection $\pi_X : \Gamma \to X$, $\pi_Y : \Gamma \to Y$ submersive. Given $x \in X$ (resp. $y \in Y$) we denote by Γ_x (resp. Γ_y) the smooth submanifold $\pi_X^{-1}(x)$ (resp. $\pi_Y^{-1}(y)$) which we can think of as a smooth hypersurface in Y (resp. X). If $M \subset X$ is a smooth submanifold we can form the envelope $E(M)$ of the Γ_x in Y, for $x \in M$. In [2], Bruce discussed local models for $E(M)$. He has shhown that if dim $Y \leqq 6$ then the envelope $E(M)$ has generic Legendrian singularities for a residual set of embeddings $M \to X$. The stratified equivalence theory will be needed when dim $Y \geqq 7$. But he has remarked that his set up does not connect well with Looijenga's canonical stratification discussed in [8].

In this paper we shall avoid the difficulty by using a modification of Mather's stratification. The main result is the following.

THEOREM (1.1). *For a residual set of embeddings $M \to X$, local pictures of the envelope $E(M)$ are given by critical values of MT-stable map germs. Here, we call a map germ MT-stable if it is transverse to the canonical stratification of a jet space which is introduced in* ([5], [7]).

Of course, the critical value of a MT-stable map germ has the canonical Whitney stratification. Hence, we have a finite number of local models of generic envelopes up to stratified equivalence. In [4], it is proved that generic Legendrian singularities are singularities of MT-stable map germs. Hence, singularities of the generic envelopes and generic Legendrian singularities are in the same class of singularities of smooth mappings. Examples of such envelopes are given in [2].

All map germs and diffeomorphisrs considered here, are differentiable of class C^∞, unless stated otherwise.

2. Formulations (Including a quick reviews of Bruce [2])

In this section we introduce the definition of $E(M)$ and fundamental tools to study local properties of $E(M)$. Given $M\subset X$ we define the sets

$$\tilde{E}(M)=\{(x,y)\in(M\times Y)\cap\Gamma \mid T_{(x,y)}\Gamma\supset T_xM\times\{0\}\},$$
$$E(M)=\{y\in Y\mid (x,y)\in\tilde{E}(M) \text{ for some } x\in M\}.$$

Thus $E(M)$ consists of those y with Γ_y tangent to M.

For any $y_0\in E(M)$ and $(x_0, y_0)\in\tilde{E}(M)$, suppose locally at $x_0\in M$, M is parametrised by a smooth immersion germ $\phi:(\boldsymbol{R}^k,0)\to(X,x_0)$ and Γ is, near (x_0, y_0), the inverse image of the regular value 0 of a smooth function germ $F:(X\times Y,(x_0,y_0))\to(\boldsymbol{R},0)$. The germ $\tilde{E}(M)$ at (x_0,y_0) is $\{(\phi(t),y)\mid F(\phi(t),y)=\frac{\partial F}{\partial t_i}(\phi(t),y)=0,\ 1\leqq i\leqq k\}$. Of course, the choices of ϕ and F are not unique. We note that the local choice of F is well defined up to multiplication by a unit in the ring of smooth germs, and hence is well defined up to contact equivalence. For notions and results about the contact equivalence theory, we refer to [3], [5] and [6]. In [2], Bruce has proved the following transversality theorem.

THEOREM (2.1). *Let $\mathscr{S}$ be a smooth contact invariant stratification of the multijet space ${}_rJ^k(M,\boldsymbol{R})$. Then for a residual set of embeddings $M\to X$ and any $(x_j', y')\in(M\times Y)\cap\Gamma$, $1\leqq j\leqq r$, with x_j' all distinct, the germ*

$${}_rj_1^kF:(M^{(r)}\times Y,(x_1',\dots,x_r',y'))\to{}_rJ^k(M,\boldsymbol{R})$$

is transverse to $\mathscr{S}$. (Here the symbol " F " denotes the local choices for defining an equation of Γ at (x_j', y') and

$${}_rj_1^kF(x_1,\dots,x_r,y)=(j^kF_y(x_1),\dots,j^kF_y(x_r)).$$

COROLLARY (2.2). *Let M and Γ be as above and suppose that $\dim Y\leqq 6$. The envelope $E(M)$ will, for a residual set of embeddings $M\to X$, have generic Legendrian singularities* [1, *p* 143].

If we try to extend the above corollary in the case where dim $Y\geqq 7$, it is natural to apply the Looijenga's canonical stratification discussed in [8]. But it is not contact invariant. We shall use a modification of a stratification induced by Mather; see [3], [7].

3. Proof of the main theorem

In order to prove the main theorem, we shall constract a contact invariant stratification in $j^k(M,\boldsymbol{R})$, which will be given by a slight modification of the Mather's stratification in ([3], [7]).

In this section we shall use notations and results in [3]. Let M be a smooth manifold. Then we have a decomposition of the jet space $j^k(M,\boldsymbol{R})$

$\cong \boldsymbol{R}\times J^k(M, 1)$, where $J^k(M, 1)$ denotes the set of k-jets $j^kf(x)$ with $f(x)=0$. For any $x_0\in M$, we let $\mathscr{E}_{x_0}$ denote the ring of C^∞ function germs at x_0. The unique maximal ideal in $\mathscr{E}_{x_0}$ is denoted by $\mathfrak{m}_{x_0}$. We now define the $\mathscr{E}_{x_0}$-module $\theta(f)_{x_0}$, for any map germ $f:(M, x_0)\to(\boldsymbol{R}, y_0)$, to be the set of germs of vector fiels along f. We also write $\theta(M)_{x_0}=\theta(1_M)_{x_0}$. There is a $\mathscr{E}_{x_0}$-homomorphism $tf:\theta(M)_{x_0}\to\theta(f)_{x_0}$ defined by $tf(\xi)=df\circ\xi$; and there is also a pull back homomorphism $f^*:\mathscr{E}_{y_0}\to\mathscr{E}_{x_0}$ defined by $f^*(h)=h\circ f$. Let $z\in J^k(M, 1)$, and let $f:(M, x_0)\to(\boldsymbol{R}, 0)$ be a function germ such that $j^k_{x_0}f=z$. Define

$$\chi(z)=\dim_{\boldsymbol{R}}\theta(f)_{x_0}/tf(\theta(M)_{x_0})+(f^*(\mathfrak{m}_0)+\mathfrak{m}^k_{x_0})(f)_{x_0}.$$

Let $\mathscr{A}^k(M, 1)$ be the canonical stratification of $J^k(M, 1)$-$W^k(M, 1)$ which has difined in ([3], IV, § 2), where $W^k(M, 1)$ is the set of $z\in J^k(M, 1)$ with $\chi(z)\geqq$ k. We now define a whitney stratification $\mathscr{A}^k_0(M, \boldsymbol{R})$ by

$$\{(\boldsymbol{R}-\{0\})\times(J^k(M, 1)-W^k(M, 1))\}\cup\{\{0\}\times\mathscr{A}^k(M, 1)\},$$

where we have a decomposition $J^k(M, \boldsymbol{R})-W^k(M, \boldsymbol{R})\cong \boldsymbol{R}\times(J^k(M, 1)-W^k(M, 1))$. Since the stratification $\mathscr{A}^k_0(M, \boldsymbol{R})$ is contact invariant, it is enough to prove the following theorem.

THEOREM (3.1) (*The local version of Theorem* (1.1)). *Let* $F:(M\times\boldsymbol{R}^r,(x_0, 0))\to(\boldsymbol{R}, 0)$ *be a smooth map germ such that* $j^k_1F(x_0, 0)\notin W^k(M, 1)$ *and* j^k_1F *is transverse to* $\mathscr{A}^k_0(M, \boldsymbol{R})$. *Then*

1) $F^{-1}(0)$ *is a submanifold (germ).*

2) $\pi_F=\pi_r|F^{-1}(0):(F^{-1}(0), (x_0, 0))\to(\boldsymbol{R}^r, 0)$ *is a MT-stable map germ.*

Since $j^k_1F(x_0, 0)\notin W^k(M, 1)$ and j^k_1F is transverse to $\mathscr{A}^k_0(M, \boldsymbol{R})$, then j^k_1F is transverse to $\{0\}\times J^k(M, 1)$ in $J^k(M, \boldsymbol{R})$. Then $F^{-1}(0)=j^k_1F^{-1}(\{0\}\times J^k(M, 1))$ is a smooth submanifold. This completes the proof of 1).

For the proof of 2), we need some preparations.
Let $F:(M\times\boldsymbol{R}^r, (x_0, 0))\to(\boldsymbol{R}, 0)$ be a smooth map germ such that $j^k_1F(x_0, 0)\notin W^k(M, 1)$. If we put $f=F|M\times\{0\}$, then we have the following:
$\dim_{\boldsymbol{R}}\theta(f)_{x_0}/tf(\theta(M)_{x_0})+(f^*(\mathfrak{m}_0)+\mathfrak{m}^k_{x_0})\theta(f)_{x_0}=s<k$

and

$$\mathfrak{m}^k_{x_0}\theta(f)_{x_0}\subset tf(\theta(M)_{x_0})+f^*(\mathfrak{m}_0)\theta(f)_{x_0}.$$

Hence, there exist $\eta_1,\dots,\eta_s \in \theta(f)_{x_0}$ such that $[\eta_1],\dots,[\eta_s]$ generate $\theta(f)_{x_0}/tf(\theta(M)_{x_0})+f^*(\mathfrak{m}_0)\theta(f)_{x_0}$ over $\boldsymbol{R}$. We now define a smooth map germ

$$\tilde{F}:(M\times\boldsymbol{R}^r\times\boldsymbol{R}^s,(x_0,0))\to(\boldsymbol{R},0)$$

by

$$\tilde{F}(x,u,v)=F(x,u)+v_1\eta_1(x)+\dots+v_s\eta_s(x).$$

We call $\tilde{F}$ an *o-$\mathscr{K}$-versal deformation* of F. By the versality theorem in ([3], III, Theorem 3.4), $(\tilde{F},\pi_{r+s}):(M\times\boldsymbol{R}^r\times\boldsymbol{R}^s,(x_0,0,0))\to(\boldsymbol{R}\times\boldsymbol{R}^r\times\boldsymbol{R}^s,(0,0,0))$ is a stable map germ, where $(\tilde{F},\pi_{r+s})(x,u,v)=(\tilde{F}(x,u,v),u,v)$. By the definition of the unfolding and Theorem 2 in ([5], p 27), the following lemma is easy to prove.

LEMMA (3.2). *Let $\tilde{F}$ be the same as the above. Then*

(1) *$((\tilde{F},\pi_{r+s}),I,J)$ is a stable unfolding of*

$$\pi_{\tilde{F}}=\pi_{r+s}|\tilde{F}^{-1}(0):(\tilde{F}^{-1}(0),(x_0,0,0))\to(\boldsymbol{R}^r\times\boldsymbol{R}^s,(0,0))$$

where

$$I:(\tilde{F}^{-1}(0),(x_0,0,0))\to(M\times\boldsymbol{R}^r\times\boldsymbol{R}^s,(x_0,0,0))$$

and

$$J:(\boldsymbol{R}^r\times\boldsymbol{R}^s,(0,0))\to(M\times\boldsymbol{R}^r\times\boldsymbol{R}^s,(x_0,0,0))$$

are canonical inclusions.

(2) *$(\pi_{\tilde{F}},i,j)$ is a stable unfolding of*

$$\pi_F:(F^{-1}(0),(x_0,0))\to(\boldsymbol{R}^r,0)$$

where

$$i:(F^{-1}(0),(x_0,0))\to(\tilde{F}^{-1}(0),(x_0,0,0))$$

and

$$j:(\boldsymbol{R}^r,0)\to(\boldsymbol{R}^r\times\boldsymbol{R}^s,(0,0))$$

are canonical inclusions.

Let $\tilde{F}:(M\times\boldsymbol{R}^r\times\boldsymbol{R}^s,(x_0,0,0))\to(\boldsymbol{R},0)$ be an o-$\mathscr{K}$-versal deformation of F. We now define a map germ

$$\tilde{J}:(M\times\boldsymbol{R}^r\times\boldsymbol{R}^s,(x_0,0,0))\to J^k(M,\boldsymbol{R})$$

by

$$\tilde{J}(x, u, v)=j^k\tilde{F}_{(u, v)}(x).$$

By the definition of $\mathscr{A}^k(M, 1)$, we can prove that

$$\tilde{J}^{-1}(\boldsymbol{R}\times\mathscr{A}^k(M, 1))=j^k(\tilde{F}, \pi_{r+s})^{-1}(\mathscr{A}^k(M\times\boldsymbol{R}^r\times\boldsymbol{R}^s, \boldsymbol{R}\times\boldsymbol{R}^r\times\boldsymbol{R}^s)),$$

where $\mathscr{A}^k(M\times\boldsymbol{R}^r\times\boldsymbol{R}^s, \boldsymbol{R}\times\boldsymbol{R}^r\times\boldsymbol{R}^s)$ is Mather's canonical stratification in $J^k(M\times\boldsymbol{R}^r\times\boldsymbol{R}^s, \boldsymbol{R}\times\boldsymbol{R}^r\times\boldsymbol{R}^s)$. On the other hand, since j_1^kF is transverse to $\mathscr{A}_0^k(M, \boldsymbol{R})$, then F is non-singular at $(x_0, 0)$ and $j_1^kF\,|\,F^{-1}(0)$ is transverse to $\{0\}\times\mathscr{A}^k(M, 1)$ in $\{0\}\times J^k(M, 1)$.

Let $\tilde{i}:(M\times\boldsymbol{R}^r, (x_0, 0))\to(M\times\boldsymbol{R}^r\times\boldsymbol{R}^s, (x_0, 0, 0))$ be the canonical inclusion. Then $\tilde{J}\circ\tilde{i}\,|\,F^{-1}(0)=j_1^kF\,|\,F^{-1}(0)$ and it follows that $\tilde{J}\circ\tilde{i}\,|\,F^{-1}(0)$ is transverse to $\{0\}\times\mathscr{A}^k(M, 1)$ in $\{0\}\times J^k(M, 1)$. Since $\tilde{F}$ is a $\mathscr{K}$-versal deformation of $f=F\,|\,M\times\{0\}$, then $\tilde{J}$ is transverse to the contact class. Hence, $\tilde{J}$ is also transverse to $\{0\}\times\mathscr{A}^k(M, 1)$ and it follows that $\tilde{J}\,|\,\tilde{F}^{-1}(0)$ is transverse to $\{0\}\times\mathscr{A}^k(M, 1)$ in $\{0\}\times J^k(M, 1)$. By the above argument, we can prove that $\tilde{i}\,|\,F^{-1}(0)$ is transverse to $\tilde{J}^{-1}(\{0\}\times\mathscr{A}^k(M, 1))\cap\tilde{F}^{-1}(0)=(\tilde{J}\,|\,\tilde{F}^{-1}(0))^{-1}(\{0\}\times\mathscr{A}^k(M, 1))$ in $\tilde{F}^{-1}(0)$.

By the way, $\pi_{\tilde{F}}$ is a stable map germ and $(\tilde{F}, \pi_{r+s})$ is a stable unfolding of $\pi_{\tilde{F}}$. Let $\mathscr{A}_\pi$ be the canonical regular stratification of $\pi_{\tilde{F}}$. Then we have $\mathscr{A}_\pi=\mathscr{A}_{(\tilde{F}, \pi_{r+s})}\cap\tilde{F}^{-1}(0)$. Here, $\mathscr{A}_{(\tilde{F}, \pi_{r+s})}$ is the canonical regular stratification of $(\tilde{F}, \pi_{r+s})$. By Proposition (2.3) in ([3], IV), the strata of $\mathscr{A}_{(\tilde{F}, \pi_{r+s})}$ and $j^k(\tilde{F}, \pi_{r+s})^{-1}(\mathscr{A}^k(M\times\boldsymbol{R}^r\times\boldsymbol{R}^s, \boldsymbol{R}\times\boldsymbol{R}^r\times\boldsymbol{R}^s))=\tilde{J}^{-1}(\boldsymbol{R}\times\mathscr{A}^k(M, 1))$ which contain $(x_0, 0, 0)$ are equal. Then we can assert that $i=\tilde{i}\,|\,F^{-1}(0)$ is transverse to the strata of $\mathscr{A}_\pi$ which contains $(x_0, 0, 0)$ in $\tilde{F}^{-1}(0)$. By (2) of Lemma (3.2) and (b) of ([3], IV, Proposition (3.1)), this assertion shows that π_F is the MT-stable map germ.

This completes the proof of Theorem (3.1).

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