Hokkaido Mathematical Journal Vol. 16 (1987), p. 97～107

On the Existence of bounded Analytic Functions in a lacunary End of a Riemann surface.

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(Received March 17, 1986)

If a domain G in a Riemann surface R has a compact relative boundary ∂G, we call G an end. Let G be an end of a Riemann surface $\in O_g$. Suppose G has a boundary component $\mathfrak{p}$. The maximal number of linearly independent H. P. s (positive harmonic functions) vanishing on ∂G is called the H-dim of $\mathfrak{p}$. Let F be a closed set in G such that $G-F$ is connected. Let $G'(z, z_0)$: $z_0 \in G-F$ be a Green function of $G-F$. If $\overline{\lim\limits_{z\to\mathfrak{p}}}\, G'(z, z_0)>0$, we say F is irregular at $\mathfrak{p}$. If there exists a sequence $\{\Gamma_n\}$ such that Γ_n consitst of a finite number of analytic curves separating $\mathfrak{p}$ from ∂G and

$$\overline{\lim_{n}} \operatorname*{Min}_{z\in\Gamma_n} G'(z, z_0)>0,$$

we say F is completely irregular at $\mathfrak{p}$. Further if every Γ_n consists of an analytic curve, we say F is completely thin at $\mathfrak{p}$. Evidently if G is a punctured disk : $\{0<|z|<1\}$, F is completely thin at $z=0$ if and only if F is irregular at $z=0$.

We proved

THEOREM[1] 1. *Let G be an end of a Riemann surface $\in O_g$ with a boundary component $\mathfrak{p}$ of H-dim$=\infty$. If F is completely thin at $\mathfrak{p}$,*

$$G-F \in O_{AB}.$$

For Riemann surfaces $\notin O_g$ analogous theorems[2] are discussed before. For examples.

There exists a Riemann surface $R \notin O_g$ with the following properties :

1) *R has no singular boundary points with respect to Martin's topology.*

2) *There exists a boundary point p which is a singular point of second kind with respect to N-Martin's topology such that*

$$G \overset{N}{\ni} p \text{ implies } G \in O_{AB},$$

where $G \overset{N}{\ni} p$ means G is a fine neighbourhood of p with respect to the

N-Martin's topology.

PROBLEM. *Does there exist an end G of a Riemann surface $\in O_g$ with a boundary component $\mathfrak{p}$ of H-dim=1 such that*

$$G-F \overset{K}{\ni} p \text{ implies } G-F \in O_{AB}$$

for any closed set F in G?
where $G-F \overset{K}{\ni} p$ means that $G-F$ is a fine neighbourhood of p relative to Martin's topology $\mathfrak{M}$ over $\bar{G}$ and p is uniquely determined minimal point of G over $\mathfrak{p}$ with respect to $\mathfrak{M}$.

Now this problem is open. In this paper we shall discuss a boundary component of special type. For latter use we note some remarks.

REMARK 1. Let G be an end of $R \in O_g$ and let $\Delta(G)$ be the set of boundary components of G. Let $U_i(z): i=1, 2, \ldots, i_0$ be an HB (a bounded harmonic function) in $G+\partial G$. Then it is known, $U_i(z)$ is an HBD (a bounded and Dirichlet-bounded harmonic function) in G and there exists a sequence of curves $\{\Gamma_n\}$ in G, consisting of a finite number of analytic curves such that $\Gamma_n \to \Delta(G)$ as $n\to\infty$, Γ_n separates $\Delta(G)$ from ∂G and

$$\int_{\Gamma_n}\left|\frac{\partial}{\partial n}U_i(z)\right|ds \to 0 \text{ as } n\to\infty: i=1, 2, \ldots, i_0.$$

REMARK 2. Let $G(z, z_0)$ be a Green function of G of $R\in O_g$ with a boundary component $\mathfrak{p}$ of $H\text{-}dim=1$. Then by $R\in O_g$, $G(z, z_0) < \max_{z\in\partial G'} G(z, z_0)$ on G': $z_0 \in G-\bar{G}'$ for any subend G' of G, whence $G(z, z_0)$ is an HBD in G' and by Remark 1 $\int_{\partial G}\frac{\partial}{\partial n}G(z, z_0)ds=2\pi$. Let $\{p_i\}$ be a sequence such that $p_i\to\mathfrak{p}$ and $G(z, p_i)\to$an HP. $U(z)$. Then $\int_{\partial G}\frac{\partial}{\partial n}U(z)ds = \lim_i \int_{\partial G}\frac{\partial}{\partial n} G(z, p_i)\, ds =2\pi$. Since $\mathfrak{p}$ is of $H\text{-}dim=1$, for any other sequence $\{q_i\}$, $\lim_i G(z, q_i)$ is also $U(z)$. Hence $\lim_i G(z, p_i)$ exists for any sequence $\{p_i\}$ with $p_i\to\mathfrak{p}$. Put $G(z, \mathfrak{p})=\lim_i G(z, p_i)$. Then any HP function in G vanishing on ∂G is a multiple of $G(z, \mathfrak{p})$.

REMARK 3. As for Martin's topology. Let R be a Riemann surface $\notin$

O_g. Let $U(z)$ be an *SPH* (a positive superharmonic function) in R. Let F be a closed set in R. We denote by $U_F(z)$ the least SPH larger than $U(z)$ on F. Suppose Martin's topology $\mathfrak{M}$ is defined on $\bar{R}=R+\varDelta$ ($\varDelta$ is the boundary of R relative to $\mathfrak{M}$). For a point $p\in\bar{R}$, $U_p(z)$ is defined as $\lim_n U_{\bar{v}_n}(z)$, where v_n is a neighbourhood of p relative to $\mathfrak{M}$. Let $\varDelta_1$ be the set of minimal boundary points. Let $p\in R+\varDelta_1$ and $K(z, p)$ be the kernel of p. If

$$K_F(z, p)<K(z, p),$$

we say F is thin at p (or CF is a fine neighbourhood of p) and we denote by $CF\overset{K}{\ni}p$. Then it is easily seen that F is thin at p if and only if,

$$_p(K_F(z, p))=0.$$

In fact, suppose $CF\overset{K}{\ni}p$. Then $K(z, p)>K_F(z, p)$. Assume $_p(K_F(z, p))>0$, then $_p(K_F(z, p))=aK(z, p)$: $a<1$. $K(z, p)$ is an *HP* and $K_F(z, p)-aK(z, p)$ is an *SPH* and $=(1-a)K(z, p)$ on ∂F. By the definition of $K_F(z, p)$ $K_F(z, p)-aK(z, p)\geqq(1-a)K_F(z, p)$. This implies $aK_F(z, p)\geqq aK(z, p)$. Hence $a=0$ and $_p(K_F(z, p))=0$. Conversely clearly $_p(K_F(z, p))=0$ implies $CF\overset{K}{\ni}p$.

Let $\varDelta_{I,\delta}$ be the set of point q in $\varDelta$ such that

$$\varlimsup_{\substack{\mathfrak{M} \\ z\to q}} G(z, z_0)\geqq\delta>0.$$

We call $\varDelta_I=\bigcup_{\delta>0}\varDelta_{I,\delta}$ the set of irregular points.

If an *HP* $U(z)$ satisfies following conditions:

1) $U(z)$ is a singular function.
2) There exists a const. α not depending on M such that

$$D(\mathrm{Min}(M, U(z)))\leqq M\alpha : M<\infty.$$

We call $U(z)$ a *G. G.* (a generalized Green function). Then we proved[3]. There exists a positive mass μ such that

$$U(z)=\int_{\varDelta I}K(z, p)d\mu(p).$$

REMARK 4. Let $U(z)$ be an HP in G. Let F be a closed set. We denote the greatest HP in $G-F$ which is not larger than $U(z)$ and vanishing on F(except a set of capacity zero) by $\overset{G}{\underset{G-F}{I}}[U(z)]$. Let $V(z)$ be an HP in $G-F$ vanishing on F(except a set of capacity zero). We denote by $\overset{G}{\underset{G-F}{E}}[V(z)]$ the least HP in G which is not smaller than $V(z)$. Then

$$\text{if } \overset{G}{\underset{G-F}{E}}[V(z)]<\infty,\ \overset{G}{\underset{G-F}{I}}[\overset{G}{\underset{G-F}{E}}[V(z)]]=V(z).^{4)}$$

REMARK 5. Apply the results of the Remark 4 to an end G of $R\in O_g$ with a boundary component $\mathfrak{p}$ of $H\text{-}dim=1$. Let F be a closed set. Let $G(z, z_0)$ and $G'(z, z_0)$ be Green functions of G and $G-F$ respectively. We suppose Martin's topologies $\mathfrak{M}$ and $\mathfrak{M}'$ are defined on $\bar{G}$ and $\overline{G-F}$. Since $\mathfrak{p}$ is of $H\text{-}dim=1$, there exists only one minimal point p(relative to $\mathfrak{M}$) over $\mathfrak{p}$ and $K(z, p)$ is a multiple of $G(z, \mathfrak{p})$. Then we have

PROPOSITION 1. *F is irregular at $\mathfrak{p}$ if and only if F is thin at p. If $G-F\overset{K}{\ni}p$, $G(z, \mathfrak{p})-G_F(z, \mathfrak{p})=\overset{G}{\underset{G-F}{I}}[G(z, \mathfrak{p})]$ is minimal and there exists an $\mathfrak{M}'$-minimal point q over $\mathfrak{p}$. Hence let $\{p_i\}$ be a sequence in $G-F$ such that $p_i\to\mathfrak{p}$ and $G'(z, p_i)\to$ an HP. $U(z)$. Then $U(z)$ is a multiple of $\overset{G}{\underset{G-F}{I}}[G(z, \mathfrak{p})]$.*

2. *Further if F is completely irregular at $\mathfrak{p}$, there exists no other $\mathfrak{M}'$-points except q over $\mathfrak{p}$. Hence any singular function in $G'=G-F$ is a multiple of $\overset{G}{\underset{G-F}{I}}[G(z, \mathfrak{p})]$, especially $K'(z, q)$ is also its multiple.*

Proof of 1). Suppose F is irregular at $\mathfrak{p}$, then there exists a sequence $\{p_i\}$ tending to $\mathfrak{p}$ such that $G_F(z, p_i)$ and $G'(z, p_i)$ tend to an SPH $V(z)$ in G and to an HP. $G(z)$ in G' respectively and $G'(p_i, z_0)\geqq\delta>0$, whence $G(z)>0$. Evidently $V(z)=G(z, \mathfrak{p})$ on F. By definition

$$G_F(z, \mathfrak{p})\leqq V(z).$$

$$G(z, \mathfrak{p})-G_F(z, \mathfrak{p})\geqq G(z, \mathfrak{p})-V(z)=\lim_i(G(z, p_i)-G_F(z, p_i))=$$

$\lim_i G'(z, p_i)=G(z)$. Hence

$$G-F\overset{K}{\ni}p.$$

Next suppose $G-F\overset{K}{\ni}p$. Let $\{R_n\}$ be an exhaustion of R, where G is an end of R. Put $F_m=F\cap\bar{R}_m$. Then $G_{F_m}(z, \mathfrak{p})\uparrow G_F(z, \mathfrak{p})\leqq G(z, \mathfrak{p})$ as $m\to\infty$. Since F_m is compact, $G(z, p_i)-G_{F_m}(z, p_i)=G'_m(z, p_i)$: $p_i\in G-F_m$, where $G'_m(z, p_i)$ is a Green function of $G-F_m$ and

$$D(\mathrm{Min}(M, G'_m(z, p_i)))=2\pi M \ : \ M<\infty.$$

By the lower semicontinuity of Dirichlet integrals, by letting $i\to\infty$ and then $m\to\infty$, $D(\mathrm{Min}(M, (G(z, \mathfrak{p})-G_F(z, \mathfrak{p})))\leqq 2\pi M$. Evidently $G(z, \mathfrak{p})-G_F(z, \mathfrak{p})$ is a singular function in G' and is a $G.$ $G.$ in G'. By Remark 3 $0<G(z, \mathfrak{p})-G_F(z, \mathfrak{p})=\int_{\Delta I}K'(z, q)d\mu(q)$, where $K'(z, q)$ is the kernel of q with respect to $\mathfrak{M}'$.

Let I be the set of irregular point of F in G. Then I is of capacty zero. Since $G(z, \mathfrak{p})-G_F(z, \mathfrak{p})<\infty$ in G, μ must be $=0$ on I. Hence μ must lie over $\Delta^{\mathfrak{M}'}(\mathfrak{p})=\mathfrak{M}'$-boundary points over $\mathfrak{p}$. i. e. there exists a point q in Δ_I with reapect to $\mathfrak{M}'$ and there exists a sequence $\{q_i\}$ in G' such that $\overline{\lim_{q_i\to q}}\, G'(q_i, z_0)>0$. Hence F is irregular at $\mathfrak{p}$.

By definition $G(z, \mathfrak{p})-G_F(z, \mathfrak{p})\leqq \overset{G}{\underset{G-F}{I}}[G(z, \mathfrak{p})]$. Put $G_{n+i}=G\cap R_{n+i}$. Let $U_{n, n+i}(z)$ be the solution of the Dirichlet problem $H_{\varphi}^{G_{n+i}-F_n}$ in $G_{n+i}-F_n$ with boundary value $\varphi=G(z, \mathfrak{p})$ on $F_n, =0$ elsewhere. Then $U_{n, n+i}(z)\nearrow G_{F_n}(z, \mathfrak{p})$ as $i\to\infty$ and $G_{F_n}(z, \mathfrak{p})\uparrow G_F(z, \mathfrak{p})$ as $n\to\infty$. Let $V_{n+i}(z)=H_{\psi}^{G_{n+i}-F}$: $\psi=G(z, \mathfrak{p})$ on $\partial G_{n+i}-F=0$ elsewhere. Then $V_{n+i}(z)\downarrow I_{G-F}[G(z, \mathfrak{p})]$ as $n+i\to\infty$. Now

$$G(z, \mathfrak{p})-U_{n, n+i}(z)\geqq V_{n+i}(z).$$

Let $i\to\infty$ and then $n\to\infty$. Then $G(z, \mathfrak{p})-G_F(z, \mathfrak{p})\geqq I[\overset{G}{\underset{G-F}{G}}(z, \mathfrak{p})]$. Thus $G(z, \mathfrak{p})-G_F(z, \mathfrak{p})=\overset{G}{\underset{G-F}{I}}[G(z, \mathfrak{p})]$. The minimality of $G(z, \mathfrak{p})$ implies the minimality of $\overset{G}{\underset{G-F}{I}}[G(z, \mathfrak{p})]$ in G'. Hence the mass μ must be a point mass on the uniquely determined point q in $\Delta^{\mathfrak{M}'}$ over $\mathfrak{p}$. Let $U(z)=\lim_i G'(z, p_i)$. Then $U(z)\leqq G(z, \mathfrak{p})$ and $\overset{G}{\underset{G-F}{E}}[U(z)]<\infty$. Hence $U(z)=\overset{G}{\underset{G'}{I}}[\overset{G}{\underset{G'}{E}}[U(z)]]\leqq\overset{G}{\underset{G'}{I}}[G(z, \mathfrak{p})]$ and $U(z)$ is a multiple of $\overset{G}{\underset{G-F}{I}}[G(z, \mathfrak{p})]$.

2) By the definition, there exists a sequence of curves $\{\Gamma_n\}$ separating $\mathfrak{p}$

from ∂G and $\underset{z\in\Gamma_n}{\mathrm{Min}}\ G'(z, z_0)>\delta>0$. Let q be $\mathfrak{M}'$-minimal point over $\mathfrak{p}$. Then q is $\mathfrak{M}'$-accessible. There exists a curve Λ $\mathfrak{M}'$-tending to q and intersecting Γ_n at q_n.

$$K'(z, q_n)=\frac{G'(z, q_n)}{G'(z_0, q_n)}\leqq\frac{G'(z, q_n)}{\delta}\leqq\frac{G(z, q_n)}{\delta},\ K'(z, q)$$
$$=\lim_n K'(z, q_n)\leqq\frac{G(z, \mathfrak{p})}{\delta} \text{ and } \overset{G}{\underset{G-F}{E}}[K'(z, q)]<\infty.$$

Hence $K'(z, q)=I[E[K'(z, q)]]\leqq\frac{1}{\delta}\overset{G}{\underset{G-F}{I}}[G(z, \mathfrak{p})]$ and $K'(z, q)$ is a multiple of $\overset{G}{\underset{G-F}{I}}[G(z, \mathfrak{p})]$. Hence theree exists only one $\mathfrak{M}'$-minimal point q over $\mathfrak{p}$.

PROBLEM. *Under the condition : $\mathfrak{p}$ is of H-dim$=1$ and F is irregular at $\mathfrak{p}$ (without completely irregularity), can we conclude that there exists only only one $\mathfrak{M}'$-minimal point over $\mathfrak{p}$?*

C-type boundary component. Concentrated rings[5].
Let $\Omega=\{0<|z|<1\}$ be a punctured disk in the z-plane. Let $\mathfrak{R}=\{r\exp(-2\alpha)<|z|<r\exp(2\alpha)\}$: $\alpha>0$ be a ring. Let S_{ij} be a sector such that

$$S_{ij}=\{(j-1)\beta_i\leqq\arg z\leqq j\,\beta_i\}\ :\ \beta_i=2\pi/2^{i-1}:$$
$$j=1, 2, 3, \dots, 2^{i-1},\ i=1, 2, 3, \dots, i_0.$$

i. e. $S_{1,1}$, $S_{2,1}$, S_{22}, S_{31}, S_{32}, S_{33}, S_{34}, $S_{4,1}$, ...

Let L_{ij} be a half line such that

$$L_{ij}=\{\arg z=(j-1+\frac{1}{2})\beta_i\}\ :\ j=1, 2, 3, \dots, 2^{i-1},\ i=1, 2, 3, \dots, i_0.$$

Let $s_{i,k}$ and $s'_{i,k}$ be slits such that, by putting $\gamma=\frac{\alpha}{2i_0}$, $s_{ik}=\{r\exp(2\alpha+(2i-1)\gamma)\leqq|z|\leqq r\exp(-2\alpha+2i\gamma),\ \arg z=k\eta\}$. $s'_{ik}=\{r\exp(2\alpha-2i\gamma))\leqq|z|\leqq r\exp(2\alpha-(2i-1)\gamma),\ \arg z=k\eta\}$. $k=1, 2, 3, 4, \dots, (2^{i_0}m_0)$, $i=1, 2, 3, \dots, i_0$ and $\eta=2\pi/(2^{i_0}\times m_0)$. For an i, identify edges of s_{ik} and s'_{ik} in S_{ij} lying symmetrically with respect to L_{ij}: $j=1, 2, 3, \dots, 2^{i-1}$. Such operation is performed for $i=1, 2, 3, \dots, i_0$. Then we have a generalized ring $\mathfrak{R}'$. We see for any $\varepsilon>0$, we can find i_0 and m_0[5] such that

$$Os\ of\ U(z)\ \text{on}\ \{|z|=\rho\}<\varepsilon\ :\ r\exp(-\frac{3\alpha}{4})\leqq\rho\leqq r\exp(\frac{3\alpha}{4})$$

for any *HB* function $U(z)$ with $|U(z)|\leqq 1$ in $\mathfrak{R}'$, where *Os* means the oscillation of $U(z)$. Let $\{\varepsilon_n\}$ be a sequence. We denote such $\mathfrak{R}'$ by $\mathfrak{R}'(r, \alpha, \varepsilon_n)$ and call it a concentrated ring with deviation ε_n. Let $\Omega=\{0<|z|<1\}$, $a> 4\alpha>0$ and $r_n=\exp(-na)$. Construct a concentrated ring $\mathfrak{R}'(r_n, \alpha, \varepsilon_n)$ from a ring $\mathfrak{R}_n=\{r_n \exp(-2\alpha)<|z|<r_n \exp(2\alpha)\}$, $n=1, 2, 3, \ldots$ and make the part $\Omega-\sum\mathfrak{R}_n$ remain as original. Then $G=\Omega-\sum\mathfrak{R}_n+\sum\overline{\mathfrak{R}}'(r_n, \alpha, \varepsilon_n)$ is an end G of a Riemann surface $\in O_g$. Since every $\mathfrak{R}'(r_n, \alpha, \varepsilon_n)$ contains an ordinay ring with module 2α, G has a boundary component $\mathfrak{p}$ of *H-dim*$=1$ by M. Heins's[6] theorem. In the following we denote by z a point in G also. Then evidently $G(z, \mathfrak{p})=-\log|z|$. We call such boundary component $\mathfrak{p}$ a *C*-type component defined by

$$\mathfrak{R}'_n(\exp(-na), \alpha, \varepsilon_n) : a>4\alpha.$$

THEOREM. *Let $\mathfrak{p}$ be a C-type boundary component of G defined by $\mathfrak{R}'_n(r_n, \alpha, \varepsilon n) : r_n=\exp(-na)$, where $\lim_n \varepsilon_n=0$ and $\lim(n/\log\varepsilon_n)=0$. Let F be a closed set in G such that G−F is connected, F be irregular at $\mathfrak{p}$ and F be so slightly distributed somewhere as there exists a subsequece $\{n'\}$ of $\{n\}$ such that F satisfies the condition A:*

$$\min_{\rho''_n\leqq\rho\leqq\rho'_n}\left(\max_{|z|=\rho}(W(F, z, \mathfrak{R}'_{n'})\right)<\varepsilon_{n'}:$$

$$\rho'_{n'}=r_{n'}\exp\left(\frac{3\alpha}{4}\right),\quad \rho''_{n'}=r_{n'}\exp\left(-\frac{3\alpha}{4}\right),$$

where $W(F, z, \mathfrak{R}'_n)$ is an H. M. (a harmonic measure) of F relative to $\mathfrak{R}'_n(r_n, \alpha, \varepsilon_n)$. Then for any subdend G′ of G we have

$$G'-F\in O_{AB}.$$

PROOF. Assume $G'-F\notin O_{AB}$. Then there exists an *AB* (a bounded analytic function) $w=f(z)$ in $G_\rho-F$ with $|f(z)|\leqq 1$, where $G_\rho=\{|z|<\rho<1\}\subset G'$. From the condition A there exists a sequence of dividing cuts $\Gamma_{n'}=\{|z|=r'_{n'}\} : \rho''_{n'}\leqq r'_{n'}\leqq\rho'_{n'}$ on which $W(F, z, \mathfrak{R}'_{n'})<\varepsilon_{n'}$, where $\{n'\}$ is a subsequence of $\{n\}$. Clearly $\Gamma_{n'}\cap F=\phi$. $\Gamma_{n'}$ divides G_ρ into two parts. We denote by $v_{n'}(\mathfrak{p})$ the part containing $\Gamma_{m'}: m'>n'$. Then $\{v_{n'}(\mathfrak{p})\}$ is a determining sequence of $\mathfrak{p}$. For the simplicity put $G=G_\rho$ and $G'=G_\rho-F$. We remark. Let $U(z)$ be an *HB* in $\mathfrak{R}'_n(r_{n'}, \alpha, \varepsilon_{n'})-F$ such that $|U(z)|\leqq 1$. Let $U'(z)=H_g^{\mathfrak{R}'_{n'}} : g=U(z)$ on $\partial\mathfrak{R}'_{n'}(r_{n'}, \alpha, \varepsilon_{n'})$. Then

$$|U(z)-U'(z)|\leqq W(F, z, \mathfrak{R}'_{n'}).$$

Hence Os *of* $U(s)$ *on* $\Gamma_{n'} \leqq Os$ *of* $U'(z)$ on $\Gamma_{n'}+2 \underset{z\in\Gamma_{n'}}{\mathrm{Max}}\, W(F, z, \varepsilon_{n'})$

$$\leqq 3\varepsilon_{n'} \tag{1}$$

Let $G(z, z_0)$ and $G'(z, z_0): z_0 \in G'$ be Green functions of G and G' respectively. Let n_0 be a number such that $v_{n_0}(\mathfrak{p}) \subset G$ and $z_0 \notin v_{n_0}(\mathfrak{p})$. Since G is an end of a Riemann surface $\in O_g$, by the maximum principle

$$G'(z, z_0) < M = \underset{z\in\partial v_{n_0}(\mathfrak{p})}{\mathrm{Max}}\, G'(z, z_0): z \in (G' \cap v_{n'}(\mathfrak{p})).$$

Hence by (1)

$$Os \text{ of } G'(z, z_0) \text{ on } \Gamma_{n'} < 3\, \varepsilon_{n'} M: n' > n_0+1.$$

Sinc F is irregular at $\mathfrak{p}$, there exists a const. $\delta>0$ such that $\overline{\lim_{z\to\mathfrak{p}}}\, G'(z, z_0) > 2\delta$. Let $N_{n'} = \underset{z\in\Gamma_{n'}}{\mathrm{Min}}\, G'(z, z_0)$. Then

$$G'(z, z_0) < \underset{z\in\Gamma_{n'}}{\mathrm{Max}}\, G'(z, z_0) \leqq N_{n'} + 3M\varepsilon_{n'}: z \in v_{n'}(\mathfrak{p}),\ n' \geqq n_0+1.$$

Now since $\varepsilon_n \to 0$, there exists a number n_1 such that $N_{n'} > \frac{3\delta}{2}: n' > n_1$, i. e.

$$\Gamma_{n'} = \partial v_{n'}(\mathfrak{p}) \subset G^{\delta} = \{z \in G': G'(z, z_0) > \delta\}: n' > n_1.$$

and F is completely thin at $\mathfrak{p}$ and by the proposition, any HP function in $G' = G-F$ vanishing on $\partial G + F$ is a multiple of $G'^{*}(z, \mathfrak{p}) = \overset{G}{\underset{G-F}{I}}[G(z, \mathfrak{p})]$: $G(z, \mathfrak{p}) = -\log |z| + \log \rho$. We show

$$\lim_{\substack{z\to\mathfrak{p}\\ z\in G^{\delta}}} f(z) \text{ exists.}$$

Let $G^{w}(w, q)$ be a Green function of $|w|<1$. Then $G^{w}(f(z_1), f(z_2)) \geqq G'(z_1, z_2)$. Assume there exist sequences $\{p_n^1\}$ and $\{p_n^2\}$ such that $\lim_n f(p_n^1) = w_2 \neq w_2 = \lim_n f(p_n^2)$. Then $G^{w}(f(z), f(p_n^i)) \geqq G'(z, p_n^i): i=1, 2$. Choose subsequence $\{p_{n'}^i\}$ of $\{p_n^i\}$ such that $G'(z, p_{n'}^i) \to$ an HP. $U^i(z): i=1, 2$. By the proposition 1. $U^i(z) = \alpha_i\, G'^{*}(z, \mathfrak{p}): \alpha_i > 0$ by $p_n^i \in G^{\delta}$. Then

$$\infty > \underset{w}{\mathrm{Sup}}(\mathrm{Min}(G^{w}(w, w_1), G^{w}(w, w_2)) \geqq (\mathrm{Min}\,(\alpha_1, \alpha_2))\; G'^{*}(z, \mathfrak{p}) > 0.$$

This is a contradiction. Now $\{p_n^i\}$ is an arbitrary sequence in G^δ. Hence $\lim_{\substack{z\to\mathfrak{p}\\ z\in G^\delta}} f(z)$ exists.

In the following we suppose $\lim_{\substack{z\to\mathfrak{p}\\ z\in G^\delta}} f(z)=0$, $|f(z)|\leqq 1$. We can find a closed set $F'\supset F$ without disturbing the condition A such that $f(z)\neq 0$ on $\partial F'$, $f(z)$ is analytic on $\partial F'$ and every point of $\partial F'$ is regular. We can suppose from the first $f(z)$ and F satisfy the above conditions. Similarly we can suppose also $f(z)\neq 0$ on ∂G. Now

$$\log\frac{1}{|f(z)|}=S(z)+\sum_i G'(z,p_i)+V(z):\ z\in G'.$$

where $S(z)$ is a non negative singular function, p_i is a zero point of $f(z)$ and $V(z)$ is a non negative quasibounded harmonic function. By the assumption of F, zero points of $f(z)$ has no accumulating point in $G-F+\partial G+\partial F$ and $V(z)=H_g^{G'}$: $g=-\log|f(z)|$ on $\partial F+\partial G$. Since G is an end of a Riemann surface $\in O_g$, such $H_g^{G'}$ is uniquely determined and $V(z)=-\log|f(z)|$ on $\partial G+\partial F$, though g is not bounded, by the existence of *SPH*. $-\log|f(z)|$ and by the regularity of ∂F. Hence $S(z)=0$ on $\partial G+\partial F$ and by the proposition $S(z)=b\ G'^*(z,\mathfrak{p}):\ \infty>b\geqq 0$. For any two points z and q in G'

$$\log\left|\frac{1-\bar{f}(q)f(z)}{f(z)-f(q)}\right|=G^w(f(z),\ f(q))\geqq G'(z,q). \tag{2}$$

Now there exists a sequence $\{p_n\}$ in G^δ such that $p_n\to\mathfrak{p}$ and $G'(z,p_n)\to$an *HP*. $U(z)$. Then $f(p_n)\to 0$. By $p_n\in G^\delta$ and by proposition $U(z)=a'\ G'^*(z,\mathfrak{p}):\ G'^*(z,\mathfrak{p})=\underset{G-F}{\overset{G}{I}}\ G(z,\mathfrak{p})$. $a'>0$ and a' depends only on $\{p_n\}$.

Putting $p_n=q$ and then $n\to\infty$. Then by 2) $-\log|f(z)|\geqq a'\ G'^*(z,\mathfrak{p})$. Clearly $G'^*(z,\mathfrak{p})$ is singular, hence

$$S(z)=b\ G'^*(z,\mathfrak{p}):\ b\geqq a'>0. \tag{3}$$

Let $T(z)=\sum_i G(z,p_i)+V(z)$ and $\eta_{n'}=\mathrm{Min}\frac{T(z)}{S(z)}:\ z\in\Gamma_{n'}$. Assume $\eta=\overline{\lim_{n'}}\ \eta_{n'}>0$. Then we have at once $T(z)\geqq\eta\ S(z)$ by $S(z)=S_{\Gamma_{n'}}(z)$ in $G'-v_{n'}(\mathfrak{p})$. This is a contradiction. Hence $\eta=0$ and there exists a number n_2 and a point $z'_{n'}$ on $\Gamma_{n'}=\partial v_{n'}(\mathfrak{p})$ such that

$$T(z'_{n'}\leqq\frac{1}{2}S(z'_{n'}):\ n'\geqq n_2.$$

Now $G'^{*}(z, \mathfrak{p})=\overset{G}{\underset{G-F}{I}}[G(z, \mathfrak{p})]\leqq-\log\ |z|+\log\ \rho\leqq-\log\ |z|$. By $r'_{m'}\geqq r_{n'}\exp(-\frac{3\alpha}{4})=\exp(-na-\frac{3\alpha}{4})$, we have by $-\log\ |f(z'_{n'})|\leqq\frac{3b}{2}G'^{*}(z'_{n'}, \mathfrak{p})$

$$0>\log\ |f(z'_{n'})|\geqq\frac{3b}{2}(-n'a-\frac{3\alpha}{4}):\ n'\geqq n_2. \tag{4}$$

Re $f(z)$ and Im $f(z)$ are *HB.s* in $\mathfrak{R}'_{n'}(r_{n'}, \alpha, \varepsilon_{n'})-F$. Then by (1)

$$|f(z)-f(z'_{n'})|<6\varepsilon_{n'}:\ z\in\Gamma_{n'}.$$

This means that the image $f(\partial v_{n'}(\mathfrak{p}))$ of $\partial v_{n'}(\mathfrak{p})$ is contained in a circle $C_{n'}=\{|w-f(z'_{n'})|<6\varepsilon_{n'}\}$. Put $C^{w}=\{|w-t|<\rho\}:\ 1>|t|+\rho,\ |t|>\rho>0$. Let $W(C^{w}, w)$ be the *H. M* of C^{w} with respect to $\{|w|<1\}$. Then we have by a brief computation

$$W(C^{w}, 0)<2\ \log\ |t|/\log\ \rho:\ |t|<\frac{1}{4}.$$

Hence by (4) the value of *H. M* of $f(\partial v_{n'}(\mathfrak{p}))$ with respect to $|w|<1$ at $w=0\leqq-3b(n'\mathrm{a}+\frac{3\alpha}{4})/\log\ 6\varepsilon_{n'}=l(n')$.

By the assumption $\lim\limits_{n'} n'/\log\ \varepsilon_{n'}=0$, for given i we can find a number $m(i)$ such that $l(n'_{m(i)})<\frac{1}{2^{i}}$. Hence we can find a subsequence $\{n'_i\}$ of $\{n'\}$ satisfying $l(n'_i)<\frac{1}{2^{i}}$. In the sequel we attend to only this subsequence. Then

$$\sum_{i} W(f(\partial v_{n'_i}(\mathfrak{p})),\ 0)<\infty.$$

This measns $\mathfrak{F}=\sum\limits_{i} f(\partial v_{n'_i}(\mathfrak{p}))$ is irregular at $w=0$. Suppose Martin' topology $\mathfrak{M}^{w}$ is defined over $|w|\leqq 1$. Then $\dot{K}(w, 0)=C\ \log\ \frac{1}{|w|}=C\,G^{w}(w, 0)$: C is a const. Let $v_m=\{|w|<\frac{1}{m}\}$. Then $\{v_m\}$ is equivalent to $\{v_m(0)\}$: $v_n(0)=\{w:\mathfrak{M}^{w}\text{-dist}(w, 0)<\frac{1}{m}\}$, where $\mathfrak{M}^{w}$-dist means the distance of $\mathfrak{M}^{w}$. Hence by the Remark 3

$$\lim_{m} {}_{\tilde{v}_m}(G^w_{\mathfrak{F}}(w, 0)) = 0. \tag{5}$$

Compare $G^w(f(z), 0)$ and $S(z)$. Since $f(z) \to 0$ as $z \to \mathfrak{p}$ in G^δ, there exists a number $n(m)$ for any given number m such that

$$f(\partial v_{n'_i}(\mathfrak{p})) \subset v_m: \ n'_i \geqq n(m). \tag{6}$$

Let $\tilde{v}_{n'_i} = f^{-1}(f(\partial v_{n'_i}(\mathfrak{p})))$. Then $\tilde{v}_{n'_i} \supset \partial v_{n'_i}(\mathfrak{p})$. Since $G^w(f(z), 0) \geqq S(z) = bG'^*(z, \mathfrak{p})$ on $\tilde{v}_{n'_i}$ and $f(G-F)$ is contained in $|w|<1$, we have by $\mathfrak{F} \supset f(\tilde{v}_n) = f(\partial v_{n'_i}(\mathfrak{p}))$ and by 3)

$$G^w_{\mathfrak{F}}(f(z), 0) \geqq bG'^*_{v_{n_i}}(z, \mathfrak{p}) \geqq bG'^*_{\partial v_{n_i}}(z, \mathfrak{p}).$$

Since $v_{n'_i}(\mathfrak{p})$ is a determining sequence, $G'^*_{\partial v_{n_i}}(z, \mathfrak{p}) = G'^*(z, \mathfrak{p})$ in $G - v_{n'_i}(\mathfrak{p})$. Hence $G^w_{\mathfrak{F}}(f(z), 0) \geqq bG'^*(z, \mathfrak{p}) > 0$. Also by (6)

$$_{\tilde{v}_m}(G^w_{\mathfrak{F}}(f(z), 0)) \geqq bG'^*_{\partial v_{n_i}}(z, \mathfrak{p}) = bG'^*(z, \mathfrak{p}) > 0. \tag{7}$$

Let $m \to \infty$. Then by (5) $G'^*(z, \mathfrak{p}) = 0$. This is a contradiction. Hence

$$G' - F \in O_{AB}$$

for any subend G' of G.

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