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Solvable Generation of Finite Groups

To Bertram Huppert on his sixtieth birthday

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1. Introduction In their paper [1], Aschbacher and Guralnick proved that any finite group G is generated by a pair of conjugate solvable subgroups. The purpose of this note is to show that we can impose some conditions on how the generating subgroups are embedded in G. More precisely, we will prove the following theorem.

THEOREM *Let G be any finite group. Then, there is a solvable subgroup S such that*

(1) *$G=\langle S, S^{g}\rangle$ for some element g of G,*

(2) *the conjugacy class of the subgroup S is stable under the group* Aut *G of automorphisms of G, and*

(3) $N_{G}(S)=S$.

In this note, a subgroup which satisfies the second condition will be called a (*)-subgroup of G. Thus, a subgroup H is a (*)-subgroup of G if, for any automorphism σ of G, there is an element x, depending on σ, such that

$$\sigma(H)=x^{-1}Hx.$$

The conditions (2) and (3) impose some restrictions on the way the subgroup S is embedded in G. Since every maximal solvable subgroup of any finite group is self-normalizing, to impose the condition (3) alone is trivial, but to put the two conditions (2) and (3) together on S seems to be not so trivial. It may be possible to impose further conditions on the embedding of S or on the properties of the element g.

We add the following remarks. Let G be any finite group. Then, a conjugacy class of solvable subgroups which satisfies the conditions (1), (2), and (3) is not necessarily unique. It follows from elementary group theory ([2], p. 99) that the normalizer of an S_{p}-subgroup is a self-normalizing (*)-subgroup. In particular, let H be the normalizer of an S_{2}-subgroup of G. (If the order $|G|$ is odd, we have $H=G$.) By the Feit-Thompson Theorem, H is solvable. It is fairly obvious that the group G is generated by *all the conjugates of* H, and that there are groups G in which any pair of conjugates of H generates a proper subgroup of G.

2. Preliminaries We can partition the set of nonabelian finite simple groups into mutually disjoint subsets $\mathscr{S}(p)$ where p runs over all prime numbers. If $p>2$, then $\mathscr{S}(p)$ is the totality of simple groups of Lie type which are derived from the Chevalley groups defined over fields of characteristic p, while $\mathscr{S}(2)$ consists of all the remaining nonabelian simple groups of finite order not contained in any $\mathscr{S}(p)$ $(p>2)$. Thus, $\mathscr{S}(2)$ consists of all the sporadic simple groups and the alternating groups A_n for $n \geqq 7$, as well as most of the simple groups of Lie type of characteristic two.

Aschbacher and Guralnick have proved the following result ([1], Lemmas 2. 1, 2. 5, and 2. 9).

LEMMA 1. *Let G be a simple group in $\mathscr{S}(p)$. Then, G is generated by two S_p-subgroups.*

We will prove the following lemma.

LEMMA 2. *Let G be a direct product of nonabelian simple groups. Then, there exists a solvable self-normalizing* (*)*-subgroup H such that*

$$G=<H, H^x>$$

for some element x of G.

PROOF. By assumption, we have

$$G=S_1 \times S_2 \times \,..\, \times S_t$$

where the S_i are simple groups. Then, the set $F=\{S_1, S_2, \ldots, S_t\}$ is uniquely determined, and F is stable under the group Aut G. (Cf. [2], p. 131; or Chap. VI, § 6.) Set

$$G(p)=\Pi_p S_i$$

where the product Π_p is taken over those S_i which are contained in $F \cap \mathscr{S}(p)$. Then, each $G(\mathrm{p})$ is a characteristic subgroup of G, and we have

$$G=\Pi\ G(p)$$

where the product is over all prime numbers. By lemma 1, each group S_i in $F \cap \mathscr{S}(p)$ is generated by two S_p-subgroups. So, there exist an S_p-subgroup P_i of S_i and an element g_i of S_i such that

$$S_i=<P_i, g_i^{-1}P_i g_i>.$$

Put $P=\Pi_p P_i$ and $g(p)=\Pi_p g_i$. Then, P is an S_p-subgroup of $G(p)$. Let $H(p)$ be the normalizer of P in $G(p)$. Then, $H(p)$ is a self-normalizing

(*)-subgroup of $G(p)$. We remark that $H(p)$ is solvable. This is clear if $p=2$. On the other hand, if $p>2$, then $H(p)$ corresponds to the Borel subgroup of the Chevalley group, and so $H(p)$ is solvable. Since

$$g(p)^{-1}Pg(p)=\Pi_p\, g_i^{-1}P_ig_i,$$

we get that $<H(p),\ H(p)^{g(p)}>=G(p)$.

Set $H=\Pi\ H(p)$ and $g=\Pi\ g(p)$. Then, H is a solvable subgroup of G such that

$$<H,\ H^g>=\Pi<H(p),\ H(p)^{g(p)}>=\Pi G(p)=G.$$

It is easy to verify that H is a self-normalizing (*)-subgroup of G. Indeed, if $\sigma\in \operatorname{Aut} G$, then σ leaves every $G(p)$ invariant. So, σ induces an automorphism of $G(p)$. Since $H(p)$ is a (*)-subgroup of $G(p)$, we can find an element $x(p)$ of $G(p)$ such that

$$H(p)^\sigma=H(p)^{x(p)}.$$

Then, $H^\sigma=H^x$ for $x=\Pi\ x(p)$. Thus, H is a (*)-subgroup of G. Clearly, H is self-normalizing because each $H(p)$ is.

3. **Proof of Theorem** We proceed by induction on $|G|$. Let $S(G)$ be the solvable normal subgroup of maximal order in G. Clearly, $S(G)$ is a characteristic subgroup of G. We will prove the existence of a subgroup S satisfying the given requirements as well as the further condition that

$$S(G)\subset S.$$

We will divide the proof into two cases depending on whether or not $S(G)=\{1\}$.

Case 1. First, we assume that $S(G)\neq\{1\}$ and consider the factor group $\bar{G}=G/S(G)$. Since $|\bar{G}|<|G|$, the inductive hypothesis gives us a self-normalizing solvable (*)-subgroup $\bar{S}$ of $\bar{G}$ such that

$$\bar{G}=<\bar{S},\ \bar{g}^{-1}\bar{S}\bar{g}>$$

for some element $\bar{g}$ of $\bar{G}$. Let S be the subgroup of G such that

$$\bar{S}=S/S(G),$$

and let g be an element of G which corresponds to $\bar{g}$ by the canonical map of G onto $\bar{G}$. Since $S(G)$ is solvable, S is a solvable subgroup of G such that

$$G=<S,\ S^g>.$$

It is clear from the correspondence Theorem that S is self-normalizing. It remains to show that S is a (*)-subgroup of G. Let σ be an automorphism of G. Since $S(G)$ is a characteristic subgroup of G, σ leaves $S(G)$ invariant. Thus, σ induces an automorphism τ of $\bar{G}$. By the inductive hypothesis, $\bar{S}$ is a (*)-subgroup of $\bar{G}$. Hence, $\tau(\bar{S})=\bar{x}^{-1}\bar{S}\bar{x}$ for some element x of G. It follows that $\sigma(S)=x^{-1}Sx$. Thus, S is a (*)-subgroup of G. This completes the proof in this case.

Case 2. We assume that $S(G)=\{1\}$. Let $F^*(G)$ be the generalized Fitting subgroup of G (cf. [2], Chap. VI, § 6). Then, $F^*(G)$ is a characteristic subgroup of G. Under the assumption that $S(G)=\{1\}$, $F^*(G)$ coincides with the maximal semisimple normal subgroup E, and it is a direct product of nonabelian simple groups.

By Lemma 2, E contains a self-normalizing solvable (*)-subgroup H such that

(1) $\qquad E=\langle H, H^x\rangle$

for some element x of E. Since H is a (*)-subgroup of $E\lhd G$, we get

(2) $\qquad G=EN_G(H)$.

We have $N_G(H)\cap E=N_E(H)=H$. Set $N=N_G(H)$. Then, we have

(3) $\qquad \{1\}\neq H\subset S(N)$.

It follows that N is a proper subgroup of G. We may apply the inductive hypothesis to N and conclude that there is a solvable self-normalizing (*)-subgroup S of N such that

(4) $\qquad S(N)\subset S$ and $N=\langle S, S^y\rangle$

for some element y of N. We will prove that S satisfies all the requirements.

Let $g=yx$ and set $G_0=\langle S, S^g\rangle$. We will prove that $G_0=G$. By (3) and (4), we have

$$H\subset S.$$

Since $y\in N=N_G(H)$, G_0 contains $H^g=(H^y)^x=H^x$ as well as H. So, by (1), we get

$$E=\langle H, H^x\rangle\subset G_0.$$

Since $x\in E$, G_0 contains $S^y=x(S^g)x^{-1}$. Hence, G_0 contains $\langle S, S^y\rangle=N$, so by (2), we have

$$G_0 = EN = G.$$

(This is the proof of Theorem A in [1].) Thus, Condition (1) is satisfied.

Next, we will show that S is a (*)-subgroup of G. Let $\sigma \in \mathrm{Aut}\ G$. Then, σ leaves E invariant because E char G. Since H is a (*)-subgroup of E, there is an element u of E such that

$$H^{\sigma} = H^{u}.$$

Let $i(u)$ be the inner automorphism of G induced by the element u, and let $\tau = \sigma\ i(u)^{-1}$. Then, τ is an automorphism of G which leaves the subgroup H invariant. Clearly, the automorphism τ leaves the subgroup $N = N_G(H)$ invariant and induces an automorphism of N. Since S is a (*)-subgroup of N, we have

$$S^{\tau} = S^{v}$$

for some element v of N. Thus, we get

$$S^{\sigma} = S^{\tau i(u)} = S^{vu}.$$

This proves that S is a (*)-subgroup of G.

Finally, we will show that S is self-normalizing in G. Clearly, $N_G(S)$ normalizes $E \cap S$. On the other hand,

$$H \subset E \cap S \subset E \cap N_G(H) = H.$$

This proves that $E \cap S = H$. Thus, $N_G(S)$ normalizes H. It follows that

$$N_G(S) = N_N(S) = S$$

because S is self-normalizing in N. This completes the proof.

References

[1] M. ASCHBACHER and R. GURALNICK, Solvable Generation of Groups and Sylow Subgroups of Lower Central Series. J. Algebra 77 (1982), 189–201

[2] M. SUZUKI, Group Theory I (1982), II (1985) Springer-Verlag Berlin-Heidelberg-New York

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