Hokkaido Mathematical Journal vol. 17 (1988), p. 151~195

On differential invariants of integrable finite type linear differential equations

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(Received February 2, 1987, Revised February 2, 1988)

§ 0. Introduction

The behavior of coefficients $a_1(t)$, $a_2(t)$, ..., $a_n(t)$ of a linear ordinary differential equation

$$\mathscr{R}: \left(\frac{d}{dt}\right)^n u + a_1(t)\left(\frac{d}{dt}\right)^{n-1} u + a_2(t)\left(\frac{d}{dt}\right)^{n-2} u + \ldots + a_n(t)u = 0$$

under a change of variables $\bar{t}=\phi(t)$, $\bar{u}=\lambda(t)u$ satisfying $\phi'(t)\neq 0$ and $\lambda(t) \neq 0$, was studied by Laguerre, Forsyth and others in the latter half of the 19 th century. Their fundamental results may be summarized as follows (cf. [5]):

i) There exists a change of variables which transforms $\mathscr{R}$ into a form $a_1(t)=a_2(t)=0$. (Such a form is called a Laguerre-Forsyth's canonical form of $\mathscr{R}$.)

ii) If a change of variables $(t, u) \longrightarrow (\bar{t}, \bar{u})$ transforms a canonical form into a canonical form, then there exists constants a, b, c, d and e such that

$$\phi(t)=\frac{at+b}{ct+d}, \qquad \lambda(t)=\frac{e}{(ct+d)^{n-1}}.$$

iii) For each canonical form of $\mathscr{R}$, let $\theta_p(t)(dt)^p$, $p=3, \ldots, n$ be the tensor fields defined respectively by

$$\theta_p(t)=\frac{(p-2)!\,p!}{(2p-3)!\,n!} \times\left\{\sum_{j=0}^{p-3}(-1)^j\frac{(2p-j-2)!\,(n-p+j)!}{(p-j-1)!\,j!}\left(\frac{d}{dt}\right)^j a_{p-j}(t)\right\}.$$

Then the definition of $\theta_p(t)(dt)^p$ does not depend on the choice of the Laguerre-Forsyth's canonical form of $\mathscr{R}$. Moreover $\theta_p(t)(dt)^p$ $p=3, \ldots, n$ form a fundamental system of invariants of $\mathscr{R}$.

The first purpose of this paper is to reformulate the classical Laguerre-Forsyth's theory of differential invariants of linear ordinary differential equations, by applying the E. Cartan's method. More precisely, we construct a Cartan connection and a complete system of differential invariants associated with each linear ordinary differential equation.

The second purpose of this paper is to give an extension of the Laguerre-Forsyth's theory to integrable finite type linear partial differential equations of a certain class, called the class of type $(\mathfrak{l}, \rho)$.

By a linear differential equation, we mean a triple $\mathscr{R}=(M, E, R^n)$, where M is a manifold and E is a vector bundle over M and R^n is a subbundle of $J^n(E)$, the n-th jet bundle of E. The symbol $g_x=\bigoplus_{p=0}^{n}(g_p)_x$ of $\mathscr{R}$ at $x\in M$ is defined as a subspace of $\bigoplus_{p=0}^{n} S^p(T_x^*)\otimes E_x$.

Now we fix a semisimple graded Lie algebra $\mathfrak{l}=\mathfrak{l}_{-1}\oplus\mathfrak{l}_0\oplus\mathfrak{l}_1$ of the first kind and a representation $\rho: \mathfrak{l}\longrightarrow gl(S)$ of $\mathfrak{l}$ on a finite dimensional vector space S. Under the condition (4.3.1), S is decomposed into a direct sum $S=\bigoplus_{p\in \boldsymbol{Z}} S_p$ such that $S_p\neq 0$ if and only if $p=0, 1, \dots, n-1$ for some positive integer n, and S_p is imbedded into $S^p(V^*)\otimes W$ in a natural manner, where $V=\mathfrak{l}_{-1}$ and $W=S_0$. The subspace $S=\bigoplus_{p=0}^{n} S_p$ of $\bigoplus_{p=0}^{n} S^p(V^*)\otimes W$ thus obtained is called the typical symbol of type $(\mathfrak{l}, \rho)$, and a linear differential equation $\mathscr{R}$ is said to be of type $(\mathfrak{l}, \rho)$, if there exist a linear isomorphism z_V of V onto T_x and a linear isomorphism z_W of W onto E_x such that the induced isomorphism ${}^t z_V^{-1}\otimes z_W: S^p(V^*)\otimes W\longrightarrow S^p(T_x^*)\otimes E_x$ maps S_p onto $(g_p)_x$ for every p. If $\mathscr{R}$ is of type $(\mathfrak{l}, \rho)$ at each point of M, then we say that $\mathscr{R}$ is of type $(\mathfrak{l}, \rho)$.

In the special case where $\mathfrak{l}\simeq sl(2)$ and ρ is the irreducible representation with dim $S=n$, a linear differential equation of type $(\mathfrak{l}, \rho)$ is nothing but a linear ordinary differential equation of order n.

Now let us proceed to the description of the main results of this paper. Let $\mathscr{R}=(M, E, R^n)$ be an integrable linear differential equation of type $(\mathfrak{l}, \rho)$. Let $\mathscr{F}(R^n)$ be the frame bundle of the vector bundle R^n over M. Since dim $S=$rank R^n, each element of $\mathscr{F}(R^n)_x$, $x\in M$, can be regarded as a linear isomorphism of S onto R_x^n. From the integrability of $\mathscr{R}$, there is a fiat connection ∇ in the vector bundle R^n such that for any solution s of $\mathscr{R}$, the n-th jet extension $j^n(s)$ is parallel. Let $\tilde{\omega}$ denote the connection form on $\mathscr{F}(R^n)$ corresponding to the flat connection ∇ in R^n. The 1-form $\tilde{\omega}$ can be considered as a $gl(S)$ valued 1-form on $\mathscr{F}(R^n)$.

For each integer $p\leqq n$, let $(R_p^n)_x$, $x\in M$, be the kernel of the projection $\pi_{p-1}^n: R_x^n\longrightarrow R_x^{p-1}$. It is easily shown that the associated graded vector space $\bigoplus_{p=0}^{n} gr(R_p^n)_x$ of the filtration $\{(R_p^n)_x | p=0, \dots, n\}$ of R_x^n is naturally isomorphic to the symbol $g_x = \bigoplus_{p=0}^{n} (g_p)_x$ of $\mathscr{R}$ at x. Let $\{S^{(p)} | p=0, \dots, n\}$ be the filtration of S defined by $S^{(p)} = \bigoplus_{q=p}^{n} S_q$. Clearly its associated graded vector space is isomorphic to the typical symbol $S = \bigoplus_{p=0}^{n} S_p$. We first construct the reduction $P(\mathscr{R})$ of $\mathscr{F}$ (R^n) which consists of all frames z such that i) $z(S^{(p)}) = (R_p^n)_x$ for every p, and ii) the induced isomorphisms $gr(z): S = \bigoplus_{p=0}^{n} S_p \longrightarrow g_x = \bigoplus_{p=0}^{n} (g_p)_x$ of the associated graded vector spaces are expressed as $gr(z) = {}^t z_V^{-1} \otimes z_W$, where z_V is a linear isomorphism of V onto T_x and z_W is a linear isomorphism of W onto E_x. Let ω denote the pullback of $\tilde{\omega}$ to the reduction $P(\mathscr{R})$.

Our main theorems may be roughly stated as follows: there is a unique normal reduction $(Q(\mathscr{R}), \chi)$ of $(P(\mathscr{R}), \omega)$, and the $gl(S)$ valued 1-form χ decomposes into the two components $\chi_{\mathfrak{g}}$ and $\chi_{\mathfrak{g}^\perp}$. The former gives a Cartan connection in the principal bundle $Q(\mathscr{R})$ and the latter gives a fundamental system of invariants of $\mathscr{R}$. In the construction of the reduction $Q(\mathscr{R})$, the harmonic theory of the cochain complex $(\bigoplus_q C^q, \partial)$ associated with the adjoint representation of $\mathfrak{l}_{-1}$ on $gl(S)$ plays an important role. The 1-form $\chi_{\mathfrak{g}^\perp}$ induces a C^1 valued function $c_{\mathfrak{g}^\perp}$. By the definition of the normal reduction, the function $c_{\mathfrak{g}^\perp}$ satisfies $\partial^* c_{\mathfrak{g}^\perp} = 0$. The harmonic part $Hc_{\mathfrak{g}^\perp}$ of $c_{\mathfrak{g}^\perp}$ gives the fundamental system of invariants of $\mathscr{R}$.

In the last part of this paper, we construct a fundamental system of invariants of linear ordinary differential equations. Our definition of invariants is slightly different from that of Laguerre-Forsyth. In Example 5. 8. 2, we show the relation between the two definitions of invariants in the case of order 6.

In the forthcoming paper, we will apply our theory to various kinds of linear partial differential equations.

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Preliminary remarks

1. In this paper we consider in either real C^∞ category or complex analytic category.

2. Let $\boldsymbol{R}$ and $\boldsymbol{C}$ denote the fields of real numbers or complex numbers respectively, and let $\boldsymbol{F}$ denote either $\boldsymbol{R}$ or $\boldsymbol{C}$ according as we consider in real C^∞ category or complex analytic category.

3. For any vector bundle E, $\underline{E}$ denotes the sheaf of germs of local cross sections of the vector bundle E.

4. As to Lie groups and principal bundles, we use the standard notations and terminolgy as in [1]. Especially let G be a Lie group and P be a principal G bundle over a base manifold M. For $a \in G$, R_a denote the right translation induced by a. Let $\mathfrak{g}$ be the Lie algebra of G. For $X \in \mathfrak{g}$, X^* denotes the vertical vector field on P induced by the 1-parameter group of right transllations $R_{\exp(tx)}$. The vector field X^* is called the fundamental vector field corresponding to X.

§ 1. Linear differential equations.

1.1. Jet bundles. Let M be a manifold. We denote by T and T^* the tangent bundle of M and the cotangent bundle of M respectively. Let E be a vector bundle over M. For every nonnegative integer p, let $J^p(E)$ denote the p-th jet bundle of E. As usual, we identify $J^0(E)$ with E and for convenience, we put $J^p(E) = M \times \{0\}$ for every negative integer p. For each $s \in \underline{E}_x$, $x \in M$, we denote by $j_x^p(s)$ the p-th jet of s at the point $x \in M$.

For each pair of integers p, q such that $p > q$, π_q^p denotes the natural projection of $J^p(E)$ onto $J^q(E)$. For every $p \geqq 1$, we define a bundle homomorphism $\varepsilon_p: S^p(T^*) \otimes E \longrightarrow J^p(E)$ in the following manner: For every $\theta^1, \dots, \theta^p \in T_x^*$ and $e \in E_x$, $x \in M$, we take functions $f^1, \dots, f^p$ and a cross section $s \in \underline{E}_x$ so that $f^i(x) = 0$, $(df^i)_x = \theta^i$, $i = 1, \dots, p$ and $s(x) = e$. Then we put

$$\varepsilon_p(\theta^1 \dots \theta^p \otimes e) = j_x^p(f^1 \dots f^p s).$$

It is well known that the sequence

$$0 \longrightarrow S^p(T^*) \otimes E \xrightarrow{\varepsilon_p} J^p(E) \xrightarrow{\pi_{p-1}^p} J^{p-1}(E) \longrightarrow 0$$

is exact. In the following, we will regard $S^p(T^*) \otimes E$ as a subbundle of $J^p(E)$.

1.2. Spencer operator D. As in [3], we introduce the first order differential operater $D: J^p(E) \longrightarrow T^* \otimes J^{p-1}(E)$ which is characterized by the following properties:

(1.2.1) $D(f\sigma) = df \otimes \pi_{p-1}^p \sigma + f \cdot D\sigma$, $\sigma \in \underline{J^p(E)}$, $f \in \underline{\mathscr{F}(M)}$;

(1.2.2) $0 \longrightarrow \underline{E} \xrightarrow{j^p} \underline{J^p(E)} \xrightarrow{D} \underline{T^* \otimes J^{p-1}(E)}$ (exact);

(1.2.3) For every pair of integers p, q such that $p > q$, the following diagram commutes:

$$\begin{array}{ccc} \underline{J^p(E)} & \xrightarrow{D} & \underline{T^*\otimes J^{p-1}(E)} \\ \pi^p_q \downarrow & & \downarrow Id\otimes\pi^{p-1}_{q-1} \\ \underline{J^q(E)} & \xrightarrow{D} & \underline{T^*\otimes J^{q-1}(E)}. \end{array}$$

Let $x^1, \dots, x^m$ be a local coordinate system in an open subset U of M and $y^1, \dots, y^r$ be a fiber coordinate system in the open subset $\pi^{-1}(U)$ of E. To express the operater D in the local coordinate system $x^1, \dots, x^m$, $y^1, \dots, y^r$, we adopt the multi-index notation. For every multi-index $\alpha=(\alpha_1, \dots, \alpha_m)$ such that all α_i are nonnegative integers, we put

$$|\alpha|=\alpha_1+\dots+\alpha_m$$
$$\alpha\,!=\alpha_1\,!\dots\alpha_m\,!$$

and

$$\alpha+(i)=(\alpha_1, \dots, \alpha_{i-1}, \ \alpha_i+1, \ \alpha_{i+1}, \dots, \alpha_m).$$

Furthermore, for $x=(x^1, \dots, x^m)\in \boldsymbol{R}^m$ and $x_0=(x_0^1, \dots, x_0^m)\in\boldsymbol{R}^m$, we put

$$(x-x_0)^\alpha=(x^1-x_0^1)^{\alpha_1}\dots(x^m-x_0^m)^{\alpha_m}.$$

If we express $\sigma\in\underline{J^p(E)}$ as

$$\sigma(x_0)=\sum_{|\alpha|\leq p}\frac{1}{\alpha\,!}\,a_\alpha(x_0)(x-x_0)^\alpha,$$

where $a_\alpha(x_0)\in\boldsymbol{R}^r$, then we have

$$D\sigma=\sum_{i=1}^m\sum_{|\alpha|\leq p-1}dx^i\otimes\frac{1}{\alpha\,!}\left\{\frac{\partial a_\alpha}{\partial x^i}(x_0)-a_{\alpha+(i)}(x_0)\right\}(x-x_0)^\alpha. \tag{1.2.4}$$

Let $\delta: S^p(T_x^*)\otimes E_x\longrightarrow T_x^*\otimes S^{p-1}(T_x^*)\otimes E_x$ be the linear map defined by

$$\delta(\xi\otimes e)(x)=i(X)\xi\otimes e,$$

where $\xi\in S^p(T_x^*)$, $e\in E_x$, $X\in T_x$ and $i(X)$ stands for the inner multiplication. From (1.2.4), we easily see that the following diagram commutes:

$$\begin{array}{ccc} \underline{S^p(T^*)\otimes E} & \xrightarrow{-\delta} & \underline{T^*\otimes S^{p-1}(T^*)\otimes E} \\ \varepsilon_p \downarrow & & \downarrow Id\otimes\varepsilon_{p-1} \\ \underline{J^p(E)} & \xrightarrow{D} & \underline{T^*\otimes J^{p-1}(E)}. \end{array} \tag{1.2.5}$$

1.3. Linear differential equations. By a linear differential equation of order n, we mean a triple $\mathscr{R}=(M, E, R^n)$, *where* M is a manifold, E is a vector bundle over M and R^n is a subbundle of the n-th jet bundle $J^n(E)$ of E. By a solution of $\mathscr{R}$, we mean a cross section s of E over an open subest U of M which satisfies $j_x^n(s)\in R_x^n$ at all $x\in U$. We denote by $\mathscr{S}ol(\mathscr{R})$ the sheaf of germs of the solutions of $\mathscr{R}$.

Let $\mathscr{R}=(M, E, R^n)$ be a linear differential equation of order n. For each nonnegative integer $p\leqq n$, and each point $x\in M$, we define a subspace R_x^p of $J^p(E)_x$ by

(1.3.1) $R_x^p=\pi_p^n(R_x^n)$,

and a subspace $(g_p)_x$ of $S^p(T_x^*)\otimes E_x$ by

(1.3.2) $(g_p)_x=R_x^p\cap S^p(T_x^*)\otimes E_x$.

The subspace $(g_p)_x$ of $S^p(T_x^*)\otimes E_x$ (resp. the direct sum $g_x=\bigoplus_{p=0}^{n}(g_p)_x$) will be called the p-th symbol (resp. the symbol) of the differential equation $\mathscr{R}$ at the point $x\in M$. Here we notice that $\dim R_x^n=\sum_{p=0}^{n}\dim(g_p)_x=\dim g_x$.

Let us give another description of the symbol g_x. Let $(R_p^n)_x$ be the subspace of R_x^n defined by

(1.3.3) $(R_p^n)_x=R_x^n\cap\ker(\pi_{p-1}^n)_x$.

Then the family of subspaces $\{(R_p^n)_x|p=0, \dots, n+1\}$ of R_x^n gives a filtration of R_x^n. Let $gr(R^n)_x=\bigoplus_{p=0}^{n} gr(R_p^n)_x$ denote the associated graded vector space, i. e.,

$$gr(R_p^n)_x=(R_p^n)_x/(R_{p+1}^n)_x.$$

Now we recall the natural linear isomorphisms

$$\ker(\pi_{p-1}^n)_x/\ker(\pi_p^n)_x\cong\ker(\pi_{p-1}^p)_x=S^p(T_x^*)\otimes E_x.$$

By using these isomorphisms, we regard $gr(R_p^n)_x$ as a subspace of $S^p(T_x^*)\otimes E_x$. Then it is easy to see that $gr(R_p^n)_x\cong(g_p)_x$ for all p and $x\in M$. We will denote by $\hat{\pi}_p$ the projection of $(R_p^n)_x$ onto $(g_p)_x$.

1.4. Isomorphisms of linear differential equations. Let M (resp. M') be a manifold of dimension m and let E (resp. E') be a vector bundle over M (resp. over M') of rank r. Let ϕ be a bundle isomorphism of E onto E'. We denote by ϕ_M the diffeomorphism of M onto M' induced by ϕ. For any cross section s' of E', ϕ^*s' denotes the cross section of E defined by

$$\phi^*s'(x)=(\phi\circ S'\circ\phi_M^{-1})(x).$$

The cross section $\phi^* s'$ will be called the pull back of s'.

Let $J^p(E)$ (resp. $J^p(E')$) denote the p-th jet bundle of E (resp. of E'). For any bundle isomorphism $\phi: E \longrightarrow E'$, there corresponds the natural bundle isomorphism $J^p(\phi): J^p(E) \longrightarrow J^p(E')$ such that

$$J^p(\phi)(j^p(s)) = j^p((\phi^{-1})^* s),$$

where $s \in E$. It is easy to verify that the following diagram commutes:

$$\text{(1.4.1)} \quad \begin{array}{ccc} J^p(E) & \xrightarrow{J^p(\phi)} & J^p(E') \\ \pi^p_q \downarrow & & \downarrow \pi^{p\prime}_q \\ J^q(E) & \xrightarrow{J^q(\phi)} & J^q(E') \end{array},$$

$$\text{(1.4.2)} \quad \begin{array}{ccc} S^p(T^*)\otimes E & \xrightarrow{(\phi_M^{-1})^* \otimes \phi} & S^p(T'^*)\otimes E' \\ \varepsilon_p \downarrow & & \downarrow \varepsilon_p' \\ J^p(E) & \xrightarrow{J^p(\phi)} & J^p(E') \end{array},$$

$$\text{(1.4.3)} \quad \begin{array}{ccc} \underline{J^p(E)} & \xrightarrow{J^p(\phi)} & \underline{J^p(E')} \\ D \downarrow & & \downarrow D' \\ \underline{T^*\otimes J^{p-1}(E)} & \xrightarrow{Id \otimes J^{p-1}(\phi)} & \underline{T^*\otimes J^{p-1}(E')} \end{array},$$

where T'^* stands for the cotangent bundle of M'.

A bundle isomorphism of E onto itself is called a bundle automorphism of E. A vector field on E is called an infinitesimal bundle automorphism if it generates a local 1-parameter group of bundle automorphisms of E. If we take a local coordinate system in E as in 1.2, every infinitesimal bundle automorphism can be expressed in the form:

$$\text{(1.4.4)} \quad \sum_{i=1}^{m} f^i(x) \frac{\partial}{\partial x_i} + \sum_{j=1}^{r} \sum_{k=1}^{r} g_j^k(x) y^j \frac{\partial}{\partial y^k}$$

where $f^i(x)$, $1 \leqq i \leqq m$ and $g_j^k(x)$, $1 \leqq j, k \leqq r$ are functions on U.

Let $\mathscr{R} = (M, E, R^n)$ (resp. $\mathscr{R}' = (M', E', R^{n\prime})$) be a linear differential equation of order n and let $g = \bigoplus_{p=0}^{n} g_p$ (resp. $g' = \bigoplus_{p=0}^{n} g_p'$) be the symbol of $\mathscr{R}$ (resp. of $\mathscr{R}'$). A bundle isomorphism ϕ of E onto E' is said to be an isomorphism of the differential equation $\mathscr{R}$ onto the differential equation $\mathscr{R}'$ if $J^n(\phi)$ maps R^n onto $R^{n\prime}$.

For every isomorphism ϕ of $\mathscr{R}$ onto $\mathscr{R}'$, we will denote by $R^n(\phi)$ the restriction of $J^n(\phi)$ to R^n. One should note that $R^n(\phi)$ is a bundle isomorphism of R^n onto $R^{n\prime}$. From (1.4.1) and (1.4.2), we see that $R^n(\phi)_x$ maps $(R^n_p)_x$ onto $(R^{n\prime}_p)_{x'}$ and $(\phi_M^{-1})^* \otimes \phi$ maps $(g_p)_x$ onto $(g'_p)_{x'}$, where $x' = \phi_M(x)$. Moreover, we have the following commutative diagram

$$
(1.4.5) \quad \begin{array}{ccc}
(R^n_p)_x & \xrightarrow{\hat{\pi}_p} & (g_p)_x \\
{\scriptstyle R^n(\phi)_x}\Big\downarrow & & \Big\downarrow{\scriptstyle (\phi_M^{-1})^* \otimes \phi_x} \\
(R^{n\prime}_p)_{x'} & \xrightarrow{\hat{\pi}_p} & (g'_p)_{x'}
\end{array},
$$

An isomorphism of $\mathscr{R}$ onto itself is called an automorphism of $\mathscr{R}$. An infinitesimal bundle automorphism of E is called an infinitesimal automorphism of $\mathscr{R}$ if it generates a local 1-parameter group of automorphisms of $\mathscr{R}$.

EXAMPLE 1.4.1. The Lie algebra of infinitesimal automorphisms of the linear ordinary differential equation $\left(\frac{d}{dt}\right)^n u = 0$ $(n \geqq 2)$. Let $X = f(t)\frac{\partial}{\partial t} + g(t)u\frac{\partial}{\partial u}$ be an infinitesimal automorphism of the differential equation $\left(\frac{d}{dt}\right)^n u = 0$. For any function $u(t)$, we have $(Xu)(t) = f(t)u'(t) + g(t)u(t)$, and hence

$$\left(\frac{d}{dt}\right)^n (Xu) = \sum_{p=0}^{n} \binom{n}{p} f^{(p)} u^{(n-p+1)} + \sum_{p=0}^{n} \binom{n}{p} g^{(p)} u^{(n-p)}.$$

Since, for every $1 \leqq p \leqq n$, the coefficient of $u^{(n-p)}$ of the right hand side of this equality must be equal to 0, it follows that

$$(n-p) f^{(p+1)} + (p+1) g^{(p)} = 0 \text{ for every } 1 \leqq p \leqq n.$$

Putting $p = 1, 2$, we obtain;

$$(n-1) f'' + 2g' = 0, \quad (n-2) f''' + 3g'' = 0.$$

This implies that $f''' = 0$ and $g'' = 0$. Therefore we see that

$$X = (a + 2bt + ct^2)\frac{\partial}{\partial t} + \{d - (n-1)ct\}u\frac{\partial}{\partial u},$$

where a, b, c, d are constants. It should be noted that the Lie algebra of infinitesimal automorphisms of the equation $\left(\frac{d}{dt}\right)^n u = 0$ is isomorphic to the Lie algebra $\mathrm{sl}(2, \boldsymbol{F}) \otimes \boldsymbol{F}$, under the correspondence

$$X \longrightarrow \left(\begin{pmatrix} b & c \\ -a & -b \end{pmatrix}, d\right) \in sl(2, \boldsymbol{F}) \oplus \boldsymbol{F}.$$

1.5. Flat connection ∇. A linear differential equation $\mathscr{R}=(M, E, R^n)$ of order n is said to be of finite type if $(g_n)_x=0$ at all $x \in M$. A finite type linear differential equation $\mathscr{R}=(M, E, R^n)$ is said to be integrable if there exists a solution $s \in \mathscr{S}ol(\mathscr{R})_x$ such that $j^n_x(s)=\eta$ for every point $x \in M$ and every vector $\eta \in R^n_x$. It should be noted that such a solution s is uniquely determined by the initial condition η.

PROPOSITION 1.5.1. *Let $\mathscr{R}=(M, E, R^n)$ be an integrable finite type linear differential equation. Assume that R^p is a subbundle of $J^p(E)$ for every $p \leqq n-1$. Then the Spencer operator $D: \underline{J^p(E)} \longrightarrow \underline{T^* \otimes J^{p-1}(E)}$ maps R^p into $T^* \otimes R^{p-1}$ for every $p \leqq n$.*

PROOF. Since $\mathscr{R}$ is integrable, the $\underline{\mathscr{F}(M)_x}$ module $\underline{R^p_x}$ *is generated by the set* $\{j^n(s) | s \in \mathscr{S}ol(\mathscr{R})_x\}$. Therefore by (1.2.1) and (1.2.2), we have $D(\underline{R^p_x}) \subset \underline{T^*_x \otimes R^{p-1}_x}$ *for every* $p \leqq n$. q.e.d.

COROLLALLY 1.5.2. *Under the same assumption as in Proposition 1.5.1, δ maps $(g_p)_x$ into $T^*_x \otimes (g_{p-1})_x$ for every $x \in M$*

PROOF. This follows directly from the diagram (1.2.5). q.e.d.

For each integrable finite type linear differential equation $\mathscr{R}=(M, E, R^n)$, *we introduce the flat connection ∇ in the vector bundle R^n* by the following diagram

$$\text{(1.5.1)} \quad \begin{array}{ccc} \underline{R^n} & \xrightarrow{\ \nabla\ } & \underline{T^* \otimes R^n} \\ & {\scriptstyle D} \searrow \quad \swarrow {\scriptstyle Id \otimes \pi^n_{n-1}:\ \text{isomorphism}} & \\ & \underline{T^* \otimes R^{n-1}}. & \end{array}$$

PROPOSITION 1.5.3. *Under the same assumption as in Proposition 1.5.1, we have the following diagrams:*

$$0 \longrightarrow \mathscr{S}ol(\mathscr{R}) \xrightarrow{\ j^n\ } \underline{R^n} \xrightarrow{\ \nabla\ } \underline{T^* \otimes R^n} \ (exact);$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{R^n_{p+1}} & \longrightarrow & \underline{R^n_p} & \xrightarrow{\ \hat{\pi}_p\ } & \underline{g_p} & \longrightarrow & 0 \ (exact) \\ & & \downarrow \nabla & & \downarrow \nabla & & \downarrow -\delta & & \\ 0 & \longrightarrow & \underline{T^* \otimes R^n_p} & \longrightarrow & \underline{T^* \otimes R^n_{p-1}} & \xrightarrow{\ Id \otimes \hat{\pi}_{p-1}\ } & \underline{T^* \otimes g_{p-1}} & \longrightarrow & 0 \ (exact). \end{array}$$

PROOF. These diagrams follow from (1.2.2), (1.2.3) and (1.2.5).

q. e. d.

1.6. Characterization of the bundle isomorphisms $R^n(\phi)$. Lt $\mathscr{R}=(M, E, R^n)$ and $\mathscr{R}'=(M', E', R^{n'})$ be integrable finite type linear differential equations. For every integer p such that $0 \leqq p \leqq n$ and every point $x \in M$ (resp. $x' \in M'$), let $(R^n_p)_x$ (resp. $(R^{n'}_p)_{x'}$) be the subspace of R^n_x (resp. of $R^{n'}_{x'}$) defined as in 1.3. Let ∇ (resp. ∇') be the flat connection in R^n (resp. in $R^{n'}$) defined by (1.5.1). Let ψ be a bundle isomorphism of R^n onto $R^{n'}$ which induces a diffeomorphism ψ_M of M onto M'. Then we say that ψ is filtration-preseving if ψ maps $(R^n_p)_x$ onto $(R^{n'}_p)_{\psi_M(x)}$ for every p, and ψ is connection-preserving if the following diagram commutes:

$$\begin{array}{ccc}
\underline{R^n} & \xrightarrow{\nabla} & \underline{T^*\otimes R^n} \\
\psi \downarrow & & \downarrow (\psi_M^{-1})^*\otimes\psi \\
\underline{R^{n'}} & \xrightarrow{\nabla} & \underline{T'^*\otimes R^{n'}}
\end{array}$$

PROPOSITION 1.6.1. *For every isomorphism ϕ of $\mathscr{R}$ onto $\mathscr{R}'$, the induced bundle isomorphism $R^n(\phi): R^n \longrightarrow R^{n'}$ is filtration-preserving and connection-preserving. Conversely, for every filtration-preserving and connection-preserving bundle isomorphism $R^n(\phi)$, there exists a unique isomorphism ϕ of $\mathscr{R}$ onto $\mathscr{R}'$ which induces the given $R^n(\phi)$.*

PROOF. The first assertion follows from (1.4.1) and (1.4.3). Let $R^n(\phi)$ be a filtration-preserving and connection-preserving bundle isomorphism of R^n onto $R^{n'}$ which induces a diffeomorphism ϕ_M of M onto M'. Since $R^n(\phi)$ is filtration-preserving, there exists a unique bundle isomorphism $\phi: E \longrightarrow E'$ such that $\phi \circ \pi^n_0 = \pi^{n'}_0 \circ R^n(\phi)$. We claim that $R^n(\phi) = J^n(\phi)|R^n$. For each $\eta \in J^n(E)_x, x \in E$, let $s \in \mathscr{S}ol(\mathscr{R})_x$ be the solution of $\mathscr{R}$ such that $\eta = j^n_x(s)$. Since $j^n(s)$ is a flat cross section of R^n, $R^n(\phi)(j^n(s))$ is a fiat cross section of $R^{n'}$. Hence there exists a unique solution $s' \in \mathscr{S}ol(\mathscr{R}')_{x'}$ such that $j^n(s') = R^n(\phi)(j^n(s))$. Then we have

$$s' = \pi^{n'}_0(j^n(s')) = \pi^{n'}_0 R^n(\phi)(j^n(s)) = \phi\pi^n_0(j^n(s)) = (\phi^{-1})^* s.$$

Therefore we see that

$$\begin{aligned} R^n(\phi)(\eta) &= R^n(\phi)(j^n_x(s)) = j^n_{x'}(s') = j^n_{x'}((\phi^{-1})^* s) \\ &= J^n(\phi)(j^n_x(s)) = J^n(\phi)(\eta). \end{aligned}$$

q. e. d.

1.7. Reduced equations. Let E be a vector bundle over a manifold M

and let E' be a subbundle of E. We remark that the p-th jet bundle $J^p(E')$ of E' can be regarded as a subbundle of the p-th jet bundle $J^p(E)$ of E.

PROPOSITION 1.7.1. *Let $\mathscr{R}=(M, E, R^n)$ be an integrable finite type linear differential equation on E. Assume that 0-th symbol g_0 is a subbundle of E, then $R^n \subset J^n(g_0)$.*

PROOF. For each $\eta \in R^n_x$, $x \in M$, we take the solution $s \in \mathscr{Sol}(\mathscr{R})_x$ such that $j^n_x(s)=\eta$. Then it follows that $s=\pi^n_0(j^n(s)) \subset \pi^n_0(\underline{R^n_x})=(\underline{g_0})_x$, *and hence* $\eta=j^n_x(s) \in J^n(g_0)_x$. q. e. d.

Let M and M' be manifolds and let $\phi: M \longrightarrow M'$ be a submersion. For each linear differential equation $\mathscr{R}'=(M', E', R^{n'})$, we define the linear differential equatilon $\phi^*\mathscr{R}'=(M, \phi^*E', \phi^*R^{n'})$ as follows. First we define a vector bundle ϕ^*E' over M by setting $(\phi^*E)_x=E'_{\phi(x)}$, $x \in M$. For each cross section s' of E', we assign the cross section ϕ^*s' by setting $\phi^*s'(x)=s'(\phi(x))$, $x \in M$. Since ϕ is a submersion, for every point $x \in M$, the assignment $s' \in \underline{E'_{\phi(x)}} \longrightarrow \phi^*s' \in \underline{\phi^*E'_x}$ *induces an injective linear map* $J^n(\phi^*)$ of $J^n(E')_{\phi(x)}$ into $J^n(E)_x$. Now we define the subbundle $\phi^*R^{n'}$ of $J^n(\phi^*E')$ by

$$(\phi^*R^{n'})_x=J^n(\phi^*)(R^{n'}_{\phi(x)}), \quad x \in M.$$

The differential equation $\phi^*\mathscr{R}'$ will be called the pull back of $\mathscr{R}'$.

Conversely let $\mathscr{R}=(M, E, R^n)$ be a linear differential equation of order n. For each point $x \in M$, let $g^\perp_x$ be the subspace of T_x defined by

$$g^\perp_x=\{X \in T_x | \delta(X \otimes \xi)=0 \text{ for all } \xi \in g_x\}.$$

We set $g^\perp=\bigcup_{x \in M} g^\perp_x$.

PROPOSITION 1.7.2. *Let $\mathscr{R}=(M, E, R^n)$ be an integrable finite type linear differential equation. Assume that $g_0=E$ and* $\dim g^\perp$ *is constant on M. Then,*

(1) *For every $X \in g^\perp_x$, $x \in M$, $\nabla_X(\underline{R^n_p})_x \subset (\underline{R^n_p})_x$, $p=0, \dots, n$.*

(2) *The distribution $g^\perp$ is completely integrable, i. e., $[g^\perp, g^\perp] \subset g^\perp$.*

(3) *For each integral manifold N of $g^\perp$, there exists a unique flat connection ∇ in the vector bundle $E|N$ over N such that the following diagram commutes:*

$$\begin{array}{ccc}
\underline{R_x} & \xrightarrow{\nabla_X} & \underline{R_x} \\
\pi^n_0 \downarrow & & \downarrow \pi^n_0 \\
\underline{(E|N)_x} & \xrightarrow{\nabla_X} & \underline{(E|N)_x},
\end{array}$$

where $X \in g_x^{\perp}$, $x \in N$.

(4) *If there exist a manifold* M' *and a submersion* $\phi : M \longrightarrow M'$ *such that all fibers* $\phi^{-1}(x')$, $x' \in M'$ *are simply connected integral manifolds of the distribution* $g^{\perp}$, *then there exists an integrable finite type linear differential equation* $\mathscr{R}' = (M', E', R^{n\prime})$ *such that the pull back* $\phi^* \mathscr{R}'$ *s isomorphic to* $\mathscr{R}$.

PROOF. (1) follows from (2) of Proposition 1.5.3.

(2) Since ∇ is a flat connection in R^n, we have

$$\nabla_{[X,Y]}\eta = \nabla_X(\nabla_Y \eta) - \nabla_Y(\nabla_X \eta) \in (\underline{R_p^n})_x,$$

for every X, $Y \in g_x^{\perp}$ and $\eta \in (\underline{R_p^n})_x$. By (2) of Proposition 1.5.3, we have $[X, Y] \in g_x^{\perp}$.

(3) follows from (1).

(4) For each point $x' \in M'$, let $E'_{x'}$ be the space of all flat cross sections of the vector bundle $E|\phi^{-1}(x')$ over $\phi^{-1}(x')$. Since $\phi^{-1}(x')$ is simply connected, the assignment $s' \in E'_{x'} \longrightarrow s'(x) \in E_x$ gives a linear isomorphism for every point $x \in \phi^{-1}(x')$. We define the vector bundle E' over M' by setting $E' = \bigcup_{x' \in M'} E'_{x'}$. Clearly the vector bundle E can be identified with the vector bundle $\phi^* E'$.

We define the subbundle $R^{n\prime}$ of $J^n(E')$ as follows. For each point $x' \in M'$, choose an arbitrary point $x \in \phi^{-1}(x')$. For every solution $s \in \mathscr{Sol}(\mathscr{R})_x$, $j^n(s)$ is a flat cross section of R^n, and hence the restriction $s|\phi^{-1}(y')$ of s to each fiber $\phi^{-1}(y')$, $y' \in M'$ is a flat cross section of the vector bundle $E|\phi^{-1}(y')$. Therefore there exists a unique cross section $s' \in \underline{E_{x'}}$ *such that* $s'(y') = s|\phi^{-1}(y')$, $y' \in M'$. It can be easily verified that $\phi^* s' = s$ and the assignment $s \in \mathscr{Sol}(\mathscr{R})_x \longrightarrow s' \in \underline{E_{x'}}$ *induces an injective linear map of* R_x^n into $J^n(E')_{x'}$. Now we define $R_{x'}^{n\prime}$ to be the imege of this linear map. It is obvious that the definition of $R_{x'}^{n\prime}$ does not depend on the choice of the point $x \in \phi^{-1}(x')$. We easily verify that the pull back of $\mathscr{R}' = (M', E', R^{n\prime})$ can be identified with $\mathscr{R} = (M, E, R^n)$.

With the preceeding propositions in mind, we say that an integrable finite type linear differential equation $\mathscr{R} = (M, E, R^n)$ is reduced if $g_0 = E$ and $g^{\perp} = 0$.

§ 2. Model equations.

2.1. Differential equations of type S. Let $\mathscr{R} = (M, E, R^n)$ be a linear differential equation and let $g = \bigoplus_{p=0}^{n} g_p$ be the symbol of $\mathscr{R}$. Let V (resp. W) be a vector space over the field $\boldsymbol{F}$ with $\dim V = \dim M$ (resp. with $\dim W = \operatorname{rank} E$). Let S be a subspace of $\bigoplus_{p=0}^{n} S^p(V^*) \otimes W$ such that $S = \bigoplus_{p=0}^{n} S_p$, where

$S_p = S \cap S^p(V^*) \otimes W$. Then we say that the linear differential equation $\mathscr{R}$ is of type S, if there exist linear isomorphisms $z_T: V \cong T_x$ and $z_E: W \cong E_x$ such that the induced isomorphism $({}^t z_T^{-1}) \otimes z_E: S^p(V^*) \otimes W \cong S^p(T_x^*) \otimes E_x$ sends S_p onto $(g_p)_x$ for every p and every $x \in M$. In this case R^p, g_p and R_p^n are all vector bundles. We call S the typical symbol of $\mathscr{R}$.

Taking account of the arguments in 1.5 and 1.7, we assume the following conditions:

(2.1.1) $S_n = 0$,
(2.1.2) All inner multiplications $i(v)$, $v \in V$ leave S invariant,
(2.1.3) $S_0 = W$ and V acts on S faithfully, i.e., if $i(v)|S = 0$, $v \in V$, then $v = 0$.

In the following, we set $S_p = 0$ for $p < 0$ or $p > n$ and regard $S = \bigoplus_{p \in \boldsymbol{Z}} S_p$ as a graded subspace of $S(V^*) \otimes W = \bigoplus_{p \in \boldsymbol{Z}} S^p(V^*) \otimes W$.

The purpose of this section is to construct the model equation $\mathscr{R}_S = (G/G', \mathrm{E}_S, R_S^n)$ of type S for each subspace $S = \bigoplus_{p=0}^{n} S_p$ of $\bigoplus_{p=0}^{n} S(V^*) \otimes W$ satisfying the above conditions.

2.2. The Lie algebras $\mathfrak{g}$, $\mathfrak{g}'$, $\mathfrak{n}$ and the Lie group G_0. According to the formula (1.4.4), we introduce the infinite dimensional graded Lie algebra $\tilde{\mathfrak{g}} = \bigoplus_p \tilde{\mathfrak{g}}_p$ modeled after the Lie algebra of infinitesimal bundle automorphisms of the trivial bundle $V \otimes W$ over V, by setting

$$\tilde{\mathfrak{g}}_p = S^{p+1}(V^*) \otimes V \oplus S^p(V^*) \otimes gl(W),$$

for each $p \in \boldsymbol{Z}$. The natural bracket operation on $\tilde{\mathfrak{g}}$ is defined by:

$$\begin{aligned}
&[f \otimes v, f' \otimes v'] = -f \cdot (i(v)f') \otimes v' + f' \cdot (i(v')f) \otimes v, \\
&[f \otimes A, f' \otimes v'] = f' \cdot (i(v')f) \otimes A, \\
&[f \otimes A, f' \otimes A'] = f \cdot f' \otimes [A, A'],
\end{aligned}$$

where f, $f' \in S(V^*)$, v, $v' \in V$ and A, $A' \in gl(W)$. It is easy to see that the Lie algebra $\tilde{\mathfrak{g}} = \bigoplus_p \tilde{\mathfrak{g}}_p$ contains the ideal $\tilde{\mathfrak{n}} = \bigoplus_p \tilde{\mathfrak{n}}_p$ defined by

$$\tilde{\mathfrak{n}}_p = S^p(V^*) \otimes gl(W).$$

The graded Lie algebra $\tilde{\mathfrak{g}}$ acts naturally on the space $S(V^*) \otimes W$ which can be regarded as the space of cross sections of the trivial bundle $V \times W$:

$$\begin{aligned}
&(f' \otimes v)(f \otimes w) = -f' \cdot (i(v)f) \otimes w, \\
&(f' \otimes A)(f \otimes w) = f' \cdot f \otimes Aw,
\end{aligned}$$

where $f, f' \in S(V^*)$, $A \in gl(W)$, $v \in V$ and $w \in W$. In particular $v \in \tilde{\mathfrak{g}}_{-1}$ $(=V)$ acts on $S(V^*) \otimes W$ by $-\mathrm{i}(\mathrm{v})$. It is clear that $\tilde{\mathfrak{g}}_p(S^q(V^*) \otimes W) \subset S^{p+q}(V^*) \otimes W$ for all p, q.

Let $\mathfrak{g}$ be the subalgebra of $\tilde{\mathfrak{g}}$ which consists of all vectors $X \in \tilde{\mathfrak{g}}$ leaving $S = \bigoplus_p S_p$ invariant and let $\mathfrak{n}$ be the ideal of $\mathfrak{g}$ defined by $\mathfrak{n} = \mathfrak{g} \cap \tilde{\mathfrak{n}}$. For each integer p, we put $\mathfrak{g}_p = \mathfrak{g} \cap \tilde{\mathfrak{g}}_p$ and $\mathfrak{n}_p = \mathfrak{n} \cap \tilde{\mathfrak{n}}_p$. It is obvious that $\mathfrak{g} = \bigoplus_p \mathfrak{g}$, $\mathfrak{n} = \bigoplus_p \mathfrak{n}_p$, $\mathfrak{g}_{-1} = \tilde{\mathfrak{g}}_{-1} = V$. We define a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ by $\mathfrak{g}' = \bigoplus_{p \geq 0} \mathfrak{g}_p$.

Under the identification of $\bigoplus_{p=0}^{n} S^p(V^*) \otimes W$ with $J^n(V \times W)$, the subspace S defines a linear differential equation with constant coefficients. It should be noted that each element X in $\mathfrak{g}$ can be regarded as an infinitesimal automorphism of this equation.

Now we consider the direct product $GL(V) \times GL(W)$ of the Lie groups $GL(V)$, $GL(W)$, which can be regarded as a Lie group with Lie algebra $\tilde{\mathfrak{g}}_0$. The group $GL(V) \times GL(W)$ acts both on $S(V^*) \otimes W$ and $\tilde{\mathfrak{g}}$ on the left in natural manners, and we have the identity $(aX)s = (a \circ X \circ a^{-1})s$, where $a \in GL(V) \times GL(W)$, $X \in \tilde{\mathfrak{g}}$ and $s \in S(V^*) \otimes W$. We denote by G_0 the subgroup of $GL(V) \times GL(W)$ which consists of all $a \in GL(V) \times GL(W)$ leaving S invariant.

For every integer p, let $gl(S)_p$ be the subspace of $gl(S)$ consisting of all $X \in gl(S)$ such that $X(S_q) \subset S_{p+q}$ for all $q \in \boldsymbol{Z}$. It is easy to see that $gl(S) = \bigoplus_p gl(S)_p$ becomes a graded Lie algebra. Let $GL(S)_0$ be the subgroup of $GL(S)$ which consists of all elements $a \in GL(S)$ satisfying $a(S_q) = S_q$ for all $q \in \boldsymbol{Z}$.

For each $X \in \mathfrak{g}$, we denote by $r_S(X)$ the restriction of X to S, and for each $a \in G_0$, we denote by $r_S(a)$ the restriction of a to S. It is clear that $r_S(X) \in gl(S)_p$ for $X \in \mathfrak{g}_p$, and $r_S(a) \in GL(S)_0$ for $a \in G_0$.

PROPOSITION 2.2.1. (1) *The asignment $X \longrightarrow r_S(X)$ gives an injective homomorphism of the graded Lie algebra $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ into the graded Lie algebra $gl(S) = \bigoplus_p gl(S)_p$.*

(2) $\mathfrak{g}_p = 0$ *for $p \geq n$, hence $\mathfrak{g}$ is finite dimensional.*

(3) *The assignment $a \longrightarrow r_S(a)$ gives an injective homomorphism of the Lie group G_0 into the Lie group $GL(S)_0$.*

PROOF. (1) It is clear that $r_S : \mathfrak{g} \longrightarrow gl(S)$ is a homomorphism of graded Lie algebras. It suffices to show that $r_S : \mathfrak{g}_p \longrightarrow gl(S)_p$ is injective for all $p \in \boldsymbol{Z}$. The assertion for $p = -1$ follows directly from (2.1.3), and hence we may assume that $p \geq 0$.

For this purpose, we take a basis $e_1, \dots, e_m$ of V and the dual basis $e^1, \dots, e^m \in V^*$ of $e_1, \dots, e_m$. Then, each $X \in \mathfrak{g}_p$ can be expressed in the form:

$$X = \frac{1}{(p+1)!} \sum_{i, i_0, \dots, i_p} a^i_{i_0 \dots i_p} e^{i_0} \dots e^{i_p} \otimes e_i + \frac{1}{p!} \sum_{i_1, \dots, i_p} e^{i_1} \dots e^{i_p} \otimes A_{i_1 \dots i_p},$$

where $a^i_{i_0 \dots i_p} \in \boldsymbol{F}$, $A_{i_1 \dots i_p} \in gl(W)$, and $a^i_{i_0 \dots i_p}$ is symmetric with respect to the induces $i_0, \dots, i_p$ and $A_{i_1 \dots i_p}$ is symmetric with respect to the induces $i_1 \dots i_p$. For every $v = \sum_{j=1}^m x^j e_j \in \mathfrak{g}_{-1}$, $x^1, \dots, x^m \in \mathbf{F}$, we have

$$(2.2.1) \quad ad(v)^p X = (-1)^p \sum_{i, i_0, \dots, i_p} a^i_{i_0 \dots i_p} x^{i_1} \dots x^{i_p} \cdot e^{i_0} \otimes e_i + (-1)^p \sum_{i_1, \dots, i_p} x^{i_1} \dots x^{i_p} \otimes A_{i_1 \dots i_p},$$

$$(2.2.2) \quad ad(v)^{p+1} X = (-1)^{p+1} \sum_{i, i_0, \dots, i_p} a^i_{i_0 \dots i_p} x^{i_0} \dots x^{i_p} \otimes e_i.$$

Assume that $r_S(X)=0$. Since r_S is a homomorphism, we have

$$(2.2.3) \quad r_S(ad(v)^p X) = 0,$$
$$(2.2.4) \quad r_S(ad(v)^{p+1} X) = 0.$$

By (2.1.3) and (2.2.4), we have $ad(v)^{p+1} X = 0$ for all $v \in \mathfrak{g}_{-1}$, and hence by (2.2.2), we have $a^i_{i_0 \dots i_p} = 0$ for all $i, i_0, \dots, i_p$. By using (2.1.3), (2.2.1) and (2.2.3), we also have $\mathrm{A}_{i_1 \dots i_p} = 0$ for all $i_1, \dots, i_p$. Therefore we have $X = 0$.

(2) follows from (1) and the fact that $S_p = 0$ for $p \geqq n$.

(3) Let $a \in G_0$. Assume that $r_S(a) = Id$. We express a in the form:

$$a = (a_V, a_W),$$

where $a_V \in GL(V)$ and $a_W \in GL(W)$. By (2.1.3), we have $a_W = Id$. Since $(av)s = (a \circ v \circ a^{-1})s = vs$ for all $v \in \mathfrak{g}_{-1}$ and all $s \in S$, we also have $a_V = Id$. Therefore we have $a = Id$. q. e. d.

In view of this proposition, we will regard $\mathfrak{g} = \bigoplus_p \mathfrak{g}_p$ as a subalgebra of $gl(S) = \bigoplus_p gl(S)_p$ and G_0 as a subgroup of $GL(S)_0$. Let $N(\mathfrak{g})$ denote the normalizer of $\mathfrak{g}$ in $GL(S)$.

PROPOSITION 2.2.2. (1) $G_0 = N(\mathfrak{g}) \cap GL(S)_0 = N(\mathfrak{g}_{-1}) \cap GL(S)_0$.

(2) $\mathfrak{g}_p = \{X \in gl(S)_p \mid [\mathfrak{g}_{-1}, X] \subset \mathfrak{g}_{p-1}\}$,
$\mathfrak{n}_p = \{X \in gl(S)_p \mid [\mathfrak{g}_{-1}, X] \subset \mathfrak{n}_{p-1}\}$,

for every $p \geqq 0$.

PROOF. (1) It is clear that $G_0 \subset N(\mathfrak{g}) \cap GL(S)_0 \subset N(\mathfrak{g}_{-1}) \cap GL(S)_0$. Hence it suffices to show that $G_0 \supset N(\mathfrak{g}_{-1}) \cap GL(S)_0$. For each $a \in N(\mathfrak{g}_{-1}) \cap GL(S)_0$, we define $a_V \in GL(V)$ and $a_W \in GL(W)$ respectively by

$$a_V v = Ad(a)v, \quad v \in V(=\mathfrak{g}_{-1}),$$
$$a_W w = aw, \quad w \in W(=S_0).$$

We put $\tilde{a} = (a_V, a_W) \in GL(V) \times GL(W)$. Then we have

$$v^q(as) = a(Ad(a^{-1})v)^q s = a_W(a_V^{-1}v)^q s = v^q(\tilde{a}s).$$

for every $s \in S_q$ and every $v \in V$. This means that $a = r_S(\tilde{a}) \in G_0$.

(2) Let $\hat{\mathfrak{g}}_p$ be the subspace of $gl(S)_p$ consisting of all $X \in gl(S)_p$ such that $[\mathfrak{g}_{-1}, X] \subset \mathfrak{g}_{p-1}$. Cleary we have $\mathfrak{g}_p \subset \hat{\mathfrak{g}}_p$. We must show that $\hat{\mathfrak{g}}_p \subset \mathfrak{g}_p$. Let $X \in \hat{\mathfrak{g}}_p$. Then we have

$$[v_0, [v_1, \ldots, [v_p, X] \ldots]] \in V(=\mathfrak{g}_{-1}),$$

for $v_0, \ldots, v_p \in V(=\mathfrak{g}_{-1})$. Since $\mathfrak{g}_{-1}$ is abelian, $[v_0, [v_1, \ldots, [v_p, X] \ldots]]$ is symmetric with respect to $v_0, \ldots, v_p$. Hence we can find a unique $X_V \in S^{p+1}(V^*) \otimes V$ such that

$[v_0, [v_1, \ldots, [v_p, X_V] \ldots]] = [v_0, [v_1, \ldots, [v_p, X] \ldots]]$ for all $v_0, \ldots, v_p \in V$. We also define $X_W \in S^p(V^*) \otimes gl(W)$ by

$$X_W w = Xw \text{ for all } w \in W(=S_0).$$

We set $\tilde{X} = X_V + X_W$.

We claim that $Xs = \tilde{X}s$ for all $s \in S$. For every $v \in V$ and every $s \in S_q$, we have $ad(v)^k X = 0$ for $k \geqq p+2$ and $v^{p+q-k}s = 0$ for $k \leqq p-1$. Hence it follows that

$$\begin{aligned} v^{p+q}Xs &= \sum_{k=0}^{p+q} \binom{p+q}{k} (ad(v)^k X) v^{p+q-k} s \\ &= \binom{p+q}{p} (ad(v)^p X) v^q s + \binom{p+q}{p+1} (ad(v)^{p+1} X) v^{q-1} s. \end{aligned}$$

Similarly we obtain

$$v^{p+q} X_V s = \binom{p+q}{p+1} (ad(v)^{p+1} X_V) v^{q-1} s$$
$$v^{p+q} X_W s = \binom{p+q}{p} (ad(v)^p X_W) v^q s.$$

Since $v^q s \in S_0$, we have

$$(ad(v)^p X)v^q s = v^p X v^q s = v^p X_W v^q s = (ad(v)^p X_W)v^q s.$$

Therefore we obtain

$$v^{p+q} X s = v^{p+q}(X_V + X_W)s = v^{p+q}\tilde{X}s.$$

This means that $X = r_S(\tilde{X}) \in \mathfrak{g}_p$. The proof of the second assertion is quite similar. q. e. d.

2.3. The Lie groups G, G'. Let G^0 be the connected Lie subgroup of $GL(S)$ with Lie algebra $\mathfrak{g}$ and let G_0 be the Lie subgroup of $GL(S)$ defined as in 2.2. Since $G_0 \subset N(\mathfrak{g})$, we have $G_0 \cdot G^0 = G^0 \cdot G_0$. We put $G = G_0 \cdot G^0$. Then it is easily checked that G is a Lie subgroup of $GL(S)$ with Lie algebra $\mathfrak{g}$.

PROPOSITION 2.3.1. *For every $a \in G$, $Ad(a)\mathfrak{g} = \mathfrak{g}$ and $Ad(a)\mathfrak{n} = \mathfrak{n}$.*

PROOF. The first assertion is obvious. The second assertion follows from the facts that $[\mathfrak{g}, \mathfrak{n}] \in \mathfrak{n}$ and $Ad(a_0)\mathfrak{n} = \mathfrak{n}$ for every $a_0 \in G_0$. q. e. d.

Let $\{S^{(p)} | p \in \boldsymbol{Z}\}$ be the filtration of S defined by $S^{(p)} = \bigoplus_{q \geq p} S_q$. Let $GL(S)^{(0)}$ be the Lie subgroup of $GL(S)$ which consists of all $a \in GL(S)$ preserving the filtration $\{S^{(p)} | p \in \boldsymbol{Z}\}$, that is, $a(S^{(p)}) = S^{(p)}$ for all $p \in \boldsymbol{Z}$. Clearly we have $GL(S)^{(0)} \supset GL(S)_0$. It is easy to see that every $a \in GL(S)^{(0)}$ can be written uniquely in the form:

$$(2.3.1) \quad a = a_0 exp(X_1) \dots exp(X_{n-1}),$$

where $a_0 \in GL(S)_0$, $X_p \in gl(S)_p$ for $p = 1, \dots, n-1$. Furthermore the assignment $a \longrightarrow a_0$ gives a homomorphism of $GL(S)^{(0)}$ onto $GL(S)_0$.

We set $G' = G \cap GL(S)_0$. It is clear that G' is a closed Lie subgroup of G with Lie algebra $\mathfrak{g}' = \bigoplus_{p \geq 0} \mathfrak{g}_p$ and contains G_0 as Lie subgroup of G'.

PROPOSITION 2.3.2. *For every $a \in GL(S)^{(0)}$ in the form (2.3.1), $a \in G'$ if and only if $a_0 \in G_0$, $X_p \in \mathfrak{g}_p$, $p = 1, \dots, n-1$.*

PROOF. Assume that $a \in G'$. We first remark that $Ad(a)v \in \mathfrak{g}$ for all $v \in V(=\mathfrak{g}_{-1})$ and

$$Ad(a)v \equiv Ad(a_0)v \pmod{\bigoplus_{p \geq 0} gl(S)_p}.$$

By (1) of Proposition 2.2.2, this means that $a_0 \in N(\mathfrak{g}_{-1}) \cap GL(S)_0 = G_0$. Therefore, without loss of generalities, we may assume that $a_0 = Id$. Then, for every $v \in V$, we have

$$As(a)v \equiv v + [X_1, v] \pmod{\bigoplus_{p \geq 1} gl(S)_p}.$$

and hence $[X_1, v] \in \mathfrak{g}_0$. From (2) of Proposition 2.2.2, we conclude that $X_1 \in \mathfrak{g}_1$. By the same argument, we can prove that $X_p \in \mathfrak{g}_p$ for every $p \geqq 2$. The converse is obvious q. e. d.

2.4. Model equations. Now we will define the model of the differential equation of type S. Let $\pi_W : S \longrightarrow W (=S_0)$ denote the natural projection. Let $\rho_W : G' \longrightarrow GL(W)$ be the representation of G' on W defined by

$$\rho_W(a')\pi_W(s) = \pi_W(a's),$$

where $a' \in G'$ and $s \in S$. The Lie group G' acts on $G \times W$ on the right by

$$(a, w)a' = (aa', \rho_W(a')^{-1}w),$$

where $a \in G$, $a' \in G'$ and $w \in W$. Let E_S be the vector bundle over G/G' defined by $E_S = G \times W/G'$. Let $\pi_1 : G \longrightarrow G/G'$ and $\pi_2 : G \times W \longrightarrow E_S$ denote the natural projections.

Let $\mathscr{F}(G, W)_{G'}$ denote the space of all W valued functions f *on* G satisfying $f(aa') = \rho_W(a')^{-1}f(a)$ for all $a \in G$ and $a' \in G'$. We assign to each $f \in \mathscr{F}(G, W)_{G'}$ the cross section $\sigma_f \in \Gamma(E_S)$ defined by

$$(2.4.1) \quad \sigma_f(\pi_1(a)) = \pi_2(a, f(a)).$$

As usual, we will identify $\mathscr{F}(G, W)_{G'}$ with $\Gamma(E_S)$ through the assignment $f \longrightarrow \sigma_f$.

The Lie group G acts on E_S on the left by

$$\hat{a}\pi_2(a, w) = \pi_2(\hat{a}a, w),$$

where $\hat{a}$, $a \in G$ and $w \in W$. Hence G acts both on $\Gamma(E_S)$ and $J^n(E_S)$. Under the identification $\mathscr{F}(G, W)_{G'} \cong \Gamma(E_S)$, G acts on $\mathscr{F}(\mathrm{G}, \mathrm{W})_{G'}$ by

$$(\hat{a}f)(a) = f(\hat{a}^{-1}a),$$

where a, $\hat{a} \in G$ and $f \in \mathscr{F}(G, W)_{G'}$.

For each $s \in S$, we define a W valued function f_s on G by

$$f_s(a) = \pi_W(a^{-1}s),$$

where $a \in G$. It is easy to verify that $f_s \in \mathscr{F}(G, W)_{G'}$ and $f_{as} = af_s$ for all $a \in G$ and $s \in S$. We simply denote by σ_s the cross section of E_S corresponding to f_s.

For each point $x \in G/G'$, let $(R_S^n)_x$ be the subspace of $J_x^n(E_S)$ defined by

$$(R_S^n)_x = \{j_x^n(\sigma_s) | s \in S\}.$$

It is clear that $a(R_S^n)_x=(R_S^n)_{ax}$ for every $a\in G$ and $x\in G/G'$. Let R_S^n be the subbundle of $J^n(E_S)$ defined by $R_S^n=\bigcup_{x\in M}(R_S^n)_x$.

PROPOSITION 2.4.1. *$\mathscr{R}_S=(G/G', E_S, R_S^n)$ is an integrable linear differential equation of type S and $\mathscr{S}\!\mathscr{A}(\mathscr{R}_S)_x=\{\sigma_s|s\in S\}$ for every point $x\in G/G'$.*

PROOF. Let $\varkappa$ be the map of V into G/G' defined by $\varkappa(v)=\pi_1(exp(v))$. Then there exist a neighborhood V_0 of $0\in V$ and an open subset U of G/G' such that $\varkappa$ is a diffeomorphism of V_0 onto U (In the complex analytic category, $\varkappa$ is further biholomorphic). Now we identify the direct product $V_0\times W$ with $E_S|U$ through the correspondence $(v, w)\in V_0\times W\longrightarrow \pi_2(exp(v), w)$.

For every $f\in\mathscr{F}(G, W)_{G'}$, we assign a W valued function $\hat{f}$ on V_0 by $\hat{f}(v)=f(exp(v))$, $v\in V_0$. From (2.4.1), we see that σ_f corresponds to $\hat{f}$ under the trivialization $E|S\cong V_0\times W$. In particular, for every $s\in S$, σ_S corresponds to the function

$$\hat{f}_s(v)=\pi_W(exp(-v)s),\ \ v\in V_0.$$

Now we choose a basis $e_1, \dots, e_m$ of V. Then we have

$$\begin{aligned}\hat{f}(x+v)&=\pi_W(exp(-x)exp(-v)s)\\&=\sum_{q=0}\frac{(-1)^q}{q\,!}\sum_{i_1,\dots,i_q}x^{i_1}\dots x^{i_q}\pi_W(e_{i_1}\dots e_{i_q}exp(-v)s)\\&=\sum_{q=0}^{n-1}\frac{1}{q\,!}\sum_{i_1,\dots,i_q}x^{i_1}\dots x^{i_q}\pi_W(i(e_{i_1})\dots i(e_{i_q})exp(-v)s),\end{aligned}$$

where $x=\sum_{i=1}^m x^i e_i$. This shows that $j_v^n(\hat{f}_s)$ corresponds to $exp(-v)s\in S$ at the point $v\in V_0$, under the canonical identification $J^n(V_0\times W)_v\cong\bigoplus_{p=0}^n S^p(V^*\otimes W$. Hence we see that $\mathscr{R}_S$ is of type S on U. Since G leaves R_S^n invariant, $\mathscr{R}_S$ is of type S on G/G'.

By the definition of R_S^n, σ_s, s$\in$S are solutions of $\mathscr{R}_S$. Hence $\mathscr{R}_S$ is integrable. The last assertion follows from the definition of R_S^n. q.e.d.

It should be remarked that the equation $\mathscr{R}_S$ is locally isomorphic to the linear differential equation with constant coefficients defined by S under the canonical identification $J^n(V\times W)\cong\bigoplus_{p=0}^n S^p(V^*)\otimes W$.

EXAMPLE 2.4.2. The model equation of a linear ordinary differential equation $\mathscr{R}:\left(\frac{d}{dt}\right)^n u+\sum_{p=1}^n a_p(t)\left(\frac{d}{dt}\right)^{n-p}u=0$. For 1-dimensional vector spaces V and W, we construct the vector space $S=\bigoplus_{p=0}^{n} S_p$ by putting $S_p=S^p(V^*)\otimes W$ for $0\leqq p\leqq n-1$ and $S_n=0$. It is easy to see that $\mathscr{R}$ is of type S. It is also easily verified that $\mathfrak{g}\cong sl(2, \boldsymbol{F})\oplus\boldsymbol{F}$ (cf. Example 1.4.1), and that G/G' is isomorphic to the projective space $P^1(\boldsymbol{F})$.

§ 3. Canonical $\mathbf{G}^{(0)}$ reductions.

3.1. The Lie subgroup $G^{(0)}$ of $GL(S)$. Throughout this section, we fix a subspace $S=\bigoplus_{q=0}^{n} S_q$ of $S(V^*)\otimes W=\bigoplus_{q=0}^{n} S^q(V^*)\otimes W$ satisfying (2.1.1), (2.1.2) and (2.1.3).

Let $\{S^{(p)}|p\in\boldsymbol{Z}\}$ be the filtration of S defined as in 2.3. We remark that the associated graded vector space $gr(S)=\bigoplus_{q\in\boldsymbol{Z}} S^{(q)}/S^{(q+1)}$ can be identified with $\bigoplus_{q\in\boldsymbol{Z}} S_q$. As in 2.3, let $GL(S)^{(0)}$ denote the subgroup of $GL(S)$ which consists of all elements $a\in GL(S)$ preserving the filtration $\{S^{(p)}|p\in\boldsymbol{Z}\}$. For any element $a\in GL(S)^{(0)}$, we denote by $gr(a)\in GL(S)_0$ the induced automorphism of the graded vecor space $S=\bigoplus_{q\in\boldsymbol{Z}} S_q$. Clearly the assignment $a\in GL(S)^{(0)}\longrightarrow gr(a)\in GL(S)_0$ gives a homomorphism of $GL(S)^{(0)}$ onto $GL(S)_0$. Let $G^{(0)}$ be the subgroup of $GL(S)^{(0)}$ which consists of all elements $a\in GL(S)^{(0)}$ such that $gr(a)\in G_0$. We denote by $\mathfrak{g}^{(0)}$ the Lie algebra of $G^{(0)}$. The following lemma is obvious.

LEMMA 3.1.1. *For every element $a\in GL(S)^{(0)}$ in the form (2.3.1), $a\in G^{(0)}$ if and only if $a_0\in G^{(0)}$. Hence $\mathfrak{g}^{(0)}=\mathfrak{g}_0\oplus(\bigoplus_{p=1}^{n-1} gl(S)_p)$, and both G_0 and G' are subgroups of $G^{(0)}$.*

3.2. Canonical $G^{(0)}$ reductions. Let $\mathscr{R}=(M, E, R^n)$ be an integrable linear differential equation of type S. Since $\dim S=\operatorname{rank} R^n$, we may regard S as a typical fiber of the vector bundle R^n. For any point $x\in M$, let $\mathscr{F}(R^n)_x$ be the set of all linear isomorphisms of S onto R^n. We set $\mathscr{F}(R^n)=\bigcup_{x\in M}\mathscr{F}(R^n)_x$. It is obvious that $\mathscr{F}(R^n)$ is a principal $GL(S)$ bundle over M. We call $\mathscr{F}(R^n)$ the frame bundle of R^n.

Let $\tilde{\omega}$ be the connection form on $\mathscr{F}(R^n)$ corresponding to the flat connection ∇ in R^n defined as in 1.5. We may regard $\tilde{\omega}$ as a $gl(S)$ valued 1-form on $\mathscr{F}(R^n)$.

For $x\in M$, let $\{(R^n_p)_x|p\in\boldsymbol{Z}\}$ be the filtration of R^n_x defined by (1.3.3). As we have remarked in 1.3, the associated graded vector space $gr(R^n_x)=\bigoplus_p gr(R^n_p)_x$ is isomorphic to the symbol $g_x=\bigoplus_p (g_p)_x$ at x. Let $P(\mathscr{R})_x$ *be the subset of* $\mathscr{F}(R^n)_x$ which consists of all $z\in\mathscr{F}(R^n)_x$ satisfying the following conditions:

(3.2.1) z is an isomorphism of filtered vector spaces, namely, $z(S^{(p)})=(R_p^n)_x$ *for all* $p\in\boldsymbol{Z}$;
(3.2.2) There exist linear isomorphisms $z_V:\ V\cong T_x$ and $z_W:\ W\cong E_x$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r] \arrow[d, "gr(z)"'] & S(V^*)\otimes W \arrow[d, "{}^t z_V^{-1}\oplus z_W."] \\ g_x \arrow[r] & S(T_x^*)\otimes E_x \end{tikzcd}$$

We set $P(\mathscr{R})=\bigcup_{x\in M} p(\mathscr{R})_x$.

PROPOSITION 3.2.1. *$P(\mathscr{R})$ is a principal $G^{(0)}$ budle over M.*

PROOF. It suffices to show that $P(\mathscr{R})_x\neq\phi$ for every point $x\in M$. Since R^n is of type S, there exist linear isomorphisms $z_V:\ V\longrightarrow T_x$ and $z_W:\ W\longrightarrow E_x$ such that ${}^t z_V^{-1}\otimes z_W$ maps S onto g_x. Then we can choose a linear isomorphism z of S onto R_x^n so that $z(S_p)\subset(R_p^n)_x$ and the diagram

$$\begin{tikzcd} S_p \arrow[rr, "z"] \arrow[dr, "{}^t z_V^{-1}\otimes z_W"'] & & (R_p^n)_x \arrow[dl, "\hat{\pi}_p"] \\ & (g_p)_x & \end{tikzcd}$$

commutes for every p. q.e.d.

Let ω be the pullback of $\tilde{\omega}$ to $P(\mathscr{R})$, and for every p, let ω_p be the $gl(S)_p$-component of ω.

PROPOSITION. 3.2.2. (1) *$R_a{}^*\omega=Ad(a)^{-1}\omega$ for every $a\in G^{(0)}$.*

(2) *$\omega(X^*)=X$ for every $X\in\mathfrak{g}^{(0)}$, where X^* stands for the fundamental vector field corresponding to X.*

(3) $d\omega+\dfrac{1}{2}\omega\wedge\omega=0$.

(4) *ω_{-1} is a $\mathfrak{g}_{-1}$ valued basic form, that is, for every point $z\in P(\mathscr{R})$, $\mathrm{Ker}(\omega_{-1})_z=V(P(\mathscr{R}))_z$ and $(\omega_{-1})_z$ gives a linear isomorphism of the quotient space $T(P(\mathscr{R}))_z/V(P(\mathscr{R}))_z$ onto the vector space $\mathfrak{g}_{-1}$, where $V(P(\mathscr{R}))_z$ stands for the vertical tangent space of $P(\mathscr{R})$ at z.*

(5) $\omega_q=0$ *for* $q\leqq-2$.

PROOF. (1) and (2) follows from general properties of connection forms. (3) follows from the fact that ∇ is flat. To show (4) and (5), we take an arbitrary cross section σ of $P(\mathscr{R})$. For any point $x\in M$, we take linear isomorphisms $\sigma_V(x): V\longrightarrow T_x$ and $\sigma_W(x): W\longrightarrow E_x$ satisfying $gr(\sigma(x))s={}^t\sigma_V(x)^{-1}\otimes\sigma_W(x)s$ for all $s\in S$.

For any $s\in S$, let σ_s be the cross section of R^n defined by $\sigma_s(x)=\sigma(x)s$, $x\in M$. Then we have

$$\sigma(x)^{-1}(\nabla_X\sigma_s)=(\sigma^*\omega)(X)s,$$

where $x\in M$ and $X\in T_x$. From the definition of $P(\mathscr{R})$, it follows that if $s\in S_q$, then $\sigma_s(x)\in(R^n_q)_x$ and $\hat{\pi}_q(\sigma_s(x))={}^t\sigma_V(x)^{-1}\otimes\sigma_W(x)s$ for every point $x\in M$. We assert that

(3.2.3) $(\nabla_X\sigma_s)(x)\in(R^n_{q-1})_x$.

(3.2.4) $\hat{\pi}_q((\nabla_X\sigma_s)(x))=-i(X)({}^t\sigma_V(x)^{-1}\otimes\sigma_W(x))s,$

for every $X\in T_x$. In fact, by (1.2.3), we have

$$\pi^n_{q-2}(\nabla_X\sigma_s)=\pi^{n-1}_{q-2}(D_X\sigma_s)=D_X(\pi^n_{q-1}s)=0,$$

which shows (3.2.3). By (1.2.3) and (1.2.5), we also have

$$\hat{\pi}_q(\nabla_X\sigma_s)=D_X(\hat{\pi}_q\sigma_s)=-i(X)({}^t\sigma_V(x)^{-1}\otimes\sigma_W(x))s,$$

which shows (3.2.4).

Now (5) follows immediately from (3.2.3). On the other hand, by (3.2.4), we have

$$(\sigma^*\omega_{-1})(\sigma_V(x)v)=-i(v)\ \text{for all}\ v\in V,$$

which means (4). q. e. d.

We call the pair $(P(\mathscr{R}),\omega)$ *the canonical* $G^{(0)}$ reduction of $(\mathscr{F}(R^n),\tilde{\omega})$.

3.3. Isomorphisms of canonical $G^{(0)}$ reductions. Let $\mathscr{R}=(M,E,R^n)$ (resp. $\mathscr{R}'=(M',E',R^{n'})$) be a linear differential equation of type S. Let $\mathscr{F}(R^n)$ (resp. $\mathscr{F}(R^{n'})$) be the frame bundle of R^n (resp. of $R^{n'}$) and let $\tilde{\omega}$ (resp. $\tilde{\omega}'$) be the connection form on $\mathscr{F}(R^n)$ (resp. on $\mathscr{F}(R^{n'})$) corresponding to the flat connection in R^n (resp. in $R^{n'}$). Let $(P(\mathscr{R}),\omega)$ (resp. $(P(\mathscr{R}'),\omega')$) be the canonical $G^{(0)}$ reduction of $(\mathscr{F}(R^n),\tilde{\omega})$ (resp. of $(\mathscr{F}(R^{n'}),\tilde{\omega}')$).

For every isomorphism ϕ of $\mathscr{R}$ onto $\mathscr{R}'$, let $\mathscr{F}(\phi)$ be the bundle isomorphism of $\mathscr{F}(R^n)$ onto $\mathscr{F}(R^{n'})$ defined by

(3.3.1) $\mathscr{F}(\phi)(z)=R^n(\phi)\circ z,$

where $z \in \mathscr{F}(R^n)$. Since the bundle isomorphism $R^n(\phi): R^n \longrightarrow R^{n'}$ maps R^n_p onto $R^{n'}_p$ and the diagram (1.4.5) commutes, $\mathscr{F}(\phi)$ maps $P(\mathscr{R})$ onto $P(\mathscr{R}')$. We denote by $P(\phi)$ the restriction of $\mathscr{F}(\phi)$ to $P(\mathscr{R})$. Clearly $P(\phi)$ is a bundle isomorphism of $P(\mathscr{R})$ onto $P(\mathscr{R}')$. Since $R^n(\phi)$ is connection preserving, we have $P(\phi)^*\omega' = \omega$.

PROPOSITION 3.3.1. *For every isomorphism ϕ of $\mathscr{R}$ onto $\mathscr{R}'$, the bundle isomorphism $P(\phi): P(\mathscr{R}) \longrightarrow P(\mathscr{R}')$ satisfies $P(\phi)^*\omega' = \omega$. Conversely, for every bundle isomorphism $P(\phi)$ of $P(\mathscr{R})$ onto $P(\mathscr{R}')$ such that $P(\phi)^*\omega' = \omega$, there exists a unique isomorphism ϕ of $\mathscr{R}$ onto $\mathscr{R}'$ which induces the given $P(\phi)$.*

PROOF. We must show the converse. Let $P(\phi)$ be a bundle isomorphism of $P(\mathscr{R})$ onto $P(\mathscr{R}')$ such that $P(\phi)^*\omega' = \omega$. Then $P(\phi)$ can be extended to the unique bundle isomorphism $\mathscr{F}(\phi)$ of $\mathscr{F}(R^n)$ onto $\mathscr{F}(R^{n'})$. Let $R^n(\phi)$ be the bundle isomorphism of R^n onto $R^{n'}$ defined by (3.3.1). Since $P(\phi)^*\omega' = \omega$, $R^n(\phi)$ is connection-preserving. On the other hand, we have

$$R^n(\phi)(R^n_p)_x = (R^n(\phi) \circ z)(S^{(p)}) = (P(\phi))(z))(S^{(p)}) = (R^{n'}_p)_{x'}$$

where $x \in M$, $z \in P(\mathscr{R})_x$, $x' = \phi_M(x)$, ϕ_M being a diffeomorphism of M onto M' induced by $R^n(\phi)$. Hence $R^n(\phi)$ is filtration preserving. Therefore, by Proposition 1.6.1, there exists an isomorphism ϕ of $\mathscr{R}$ onto $\mathscr{R}'$ which induces the given $P(\phi)$. q.e.d.

3.4. The canonical G' reduction $(Q(\mathscr{R}_S), \chi_S)$. *As in* 2.4, *let* $\mathscr{R}_S = (G/G', E_S, R^n_S)$ be the model equation of type S. We denote by $\mathscr{F}(R^n_S)$ the frame bundle of R^n_S and by $\tilde{\omega}_S$ the connection form on $\mathscr{F}(R^n_S)$ corresponding to the flat connection in R^n_S. Let $(P(\mathscr{R}_S), \omega_S)$ be the canonical $G^{(0)}$ reduction of $(\mathscr{F}(R^n_S), \tilde{\omega}_S)$. We will use the same notation as in 2.4.

Let o denote the origin of G/G', that is, $o = \pi_1(e)$, e being the unit element of G. Let z_o be the linear isomorphism of S onto $(R^n_S)_o$ defined by $z_o(s) = j^n_o(\sigma_s)$. In the proof of Proposition 2.4.1, we show that

$$\tilde{f}_s(x) = \sum_q \frac{1}{q!} \sum_{i_1, \dots, i_q} x^{i_1} \dots x^{i_q} \pi_W(i(e_{i_1}) \dots i(e_{i_q})s),$$

where $X = \sum_{i=1}^m x^i e_i \in V$ and hence $z_o \in P(\mathscr{R}_S)$. Since G acts on E_S as automorphisms of $\mathscr{R}_S$, G acts also on $P(\mathscr{R}_S)$. Let $Q(\mathscr{R}_S)$ be the G-orbit in $P(\mathscr{R}_S)$ through the point $z_o \in P(\mathscr{R}_S)$. It is easy to see that $Q(\mathscr{R}_S)$ is a principal G' bundle over G/G' and that $Q(\mathscr{R}_S)$ is diffeomorphic to G (in the complex analytic category, $Q(\mathscr{R}_S)$ is further biholomorphic to G). Let χ_S be the pullback of ω_S to $Q(\mathscr{R}_S)$.

PROPOSITION 3.4.1. *χ_S is a $\mathfrak{g}$ valued 1-form on $Q(\mathscr{R}_S)$. Furthermore the pair $(Q(\mathscr{R}_S), \chi_S)$ is a flat Cartan connection of type G/G', namely:*

(i) $R_a{}^*\chi_S = Ad(a)^{-1}\chi_S$, $a \in G'$;

(ii) $\chi_S(X^*) = X$, $X \in \mathfrak{g}'$;

(iii) *For every point $z \in Q(\mathscr{R}_S)$, $(\chi_S)_z$ gives a linear isomorphism of $T(Q(\mathscr{R}_S))_z$ onto $\mathfrak{g}$;*

(iv) $d\chi_S + \frac{1}{2}\chi_S \wedge \chi_S = 0$.

PROOF. Let κ be the diffeomorphism of the neighborhood V_o of the origin of V onto the neighborhood U of the origin of G/G' defined as in the proof of Proposition 2.4.1. We identify V_o with the open subset U of G/G', through the map κ.

Let τ be the cross section of the principal bundle $P(\mathscr{R}_S)$ defined by $\tau(v) = P(exp(v))(z_o)$, $v \in V_o$. We first show that $\tau^*\omega$ is $\mathfrak{g}_{-1}$ valued at the origin of V_o. For this purpose, we take a cross section $\hat{\tau}$ of the frame bundle $\mathscr{F}(R^n_S)$ defined by

$$\hat{\tau}(v)s = j^n_v(\sigma_s),$$

where $v \in V_o$ and $s \in S$. Then we have

$$\tau(v)s = P(exp(v))(z_o)s = R^n(exp(v))(j^n_o(\sigma_s)) = j^n_v(\sigma_{exp(v)s})$$

for every $v \in V_o$ and $s \in S$. Hence we have

$$(3.4.1) \quad \tau(v) = \hat{\tau}(v) \cdot exp(v).$$

Since $\hat{\tau}(v)s$, $s \in S$ are flat cross sections of R^n, we have

$$(3.4.2) \quad \hat{\tau}^*\tilde{\omega} = 0.$$

To calculate $\tau^*\omega(v)$, $v \in V (= T(V_o)_o)$, we take the line $v(t) = tv$ passing through the origin of V. Then, by (3.4.1) and (3.4.2), we have

$$\tau^*\omega(v) = \tau^*\tilde{\omega}(v) = \hat{\tau}^*\tilde{\omega}(v) + v = v \in \mathfrak{g}_{-1}.$$

From (2) of Proposition 3.2.2 and the assertion just proved, we see that ω is $\mathfrak{g}$ valued at the point $z_o \in Q(\mathscr{R}_S)$. Since $P(a)^*\omega = \omega$ for every $a \in G$, we obtain the first assertion. The second assertion follows from Proposition 3.2.2 immediately. q. e. d.

§ 4. Typical symbol of type $(\mathfrak{l}, \rho)$.

As to § 4 and § 5, the reader may refer to Tanaka [4].

4.1. Semisimple graded Lie algebras of the first kind. A semisimple graded Lie algebra of the first kind over the field $\mathbf{F}$ is, by definition, a graded Lie algebra $\mathfrak{l}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{l}_p$ over $\boldsymbol{F}$ satisfying the following conditions:

(4.1.1)
(i) $\mathfrak{l}$ is finite dimensional and semisimple;
(ii) $\mathfrak{l}_{-1}\neq 0$ and $\mathfrak{l}_p=0$ for $p\leqq -2$;
(iii) If $X\in\mathfrak{l}_0$ *and* $[X,\mathfrak{l}_{-1}]=0$, then $X=0$.

Let $B_{\mathfrak{l}}$ denote the Killing form of the Lie algebra $\mathfrak{l}$. The following properties of $\mathfrak{l}=\bigoplus_{p\in \boldsymbol{Z}}\mathfrak{l}_p$ are well known.

LEMMA 4.1.1. (1) *There exists a unique element Z in $\mathfrak{l}_0$ such that*

$$\mathfrak{l}_p=\{X\in\mathfrak{l}\,|\,[Z, X]=pX\} \text{ for all } p.$$

(2) *$B_{\mathfrak{l}}(\mathfrak{l}_p, \mathfrak{l}_q)=0$ for $p+q\neq 0$, and the restriction of the Killing form $B_{\mathfrak{l}}$ to $\mathfrak{l}_p\times\mathfrak{l}_{-p}$ is nondegenerate. In particular $\mathfrak{l}_p=0$ for $p\geqq 2$.*

(3) *The case $\boldsymbol{F}=\boldsymbol{R}$. There is an involutive automorphism θ of $\mathfrak{l}$ satisfying the following properties:*

(i) $\theta(\mathfrak{l}_p)=\mathfrak{l}_{-p}$;

(ii) $B_{\mathfrak{l}}(X, \theta(x))<0$ *for* $X\neq 0$.

(4) *The cace $\boldsymbol{F}=\boldsymbol{C}$. There is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ such that $Z\in\mathfrak{h}\subset\mathfrak{l}_0$, and there is an involutive automorphism θ of $\mathfrak{l}$ as a Lie algebra over $\boldsymbol{R}$ having the following properties:*

(i) $\theta(\mathfrak{l}_p)=\mathfrak{l}_{-p}$;

(ii) $\theta(\lambda \mathrm{X})=\bar{\lambda}\theta(\mathrm{X})$ *for* $\lambda\in\boldsymbol{C}$ *and* $X\in\mathfrak{l}$. *Hence the bilinear form $B_{\mathfrak{l}}(X, \theta(Y))$, X, $Y\in\mathfrak{l}$ is hermitian;*

(iii) $B_{\mathfrak{l}}(X, \theta(X))<0$ *for* $X\neq 0$.

Now let us consider the case $\boldsymbol{F}=\boldsymbol{C}$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{l}$ such that $Z\in\mathfrak{h}\subset\mathfrak{l}_0$. Let Φ denote the set of nonzero roots of $\mathfrak{l}$ with respect to $\mathfrak{h}$. For each $\alpha\in\Phi$, we denote by $\mathfrak{l}^\alpha$ the root space attached to the root α and by $\mathfrak{h}_\alpha$ the vector in $\mathfrak{h}$ defined by $B_{\mathfrak{l}}(h_\alpha, h)=\alpha(h)$, $h\in\mathfrak{h}$. For each $p\in\boldsymbol{Z}$, let Φ_p denote the set of nonzero roots α satisfying $\mathfrak{l}^\alpha\subset\mathfrak{l}_p$. It is easy to see that

$$\Phi_p=\{\alpha\in\Phi\,|\,\alpha(Z)=p\},$$
$$\Phi_{-1}=-\Phi_1,\ \Phi_0=-\Phi_0,$$
$$\Phi=\Phi_{-1}\cup\Phi_0\cup\Phi_1 \text{ (disjoint union).}$$

The next lemma is also well known (cf. Matsushima-Murakami [2, Lemma 4.2]).

LEMMA 4.1.2. (1) *There is an ordering in Φ such that all roots belonging to Φ_1 are positive.*

(2) *$\{h_\alpha | \alpha \in \Phi\}$ spannes $\mathfrak{h}$.*

(3) $Z = 2 \sum_{\alpha \in \Phi_1} h_\alpha$.

(4) *The involutive automorphism θ in Lemma* 4.1.1 *can be taken so that $\theta(h_\alpha) = -h_\alpha$, for all $\alpha \in \Phi$.*

4.2. Associated gradation of $\mathfrak{l}$ modules. Let $\mathfrak{l} = \bigoplus_p \mathfrak{l}_p$ be a semisimple graded Lie algebra of the first kind over $\boldsymbol{F}$ and $\rho : \mathfrak{l} \longrightarrow gl(S)$ a representation of $\mathfrak{l}$ on a finite dimensional vector space S over $\boldsymbol{F}$. For each $\mu \in \boldsymbol{R}$, we set

$$S_{(\mu)} = \{s \in S | \rho(Z)s = \mu s\}.$$

It is easy to see that $\rho(\mathfrak{l}_p) S_{(\mu)} \subset S_{(p+\mu)}$ for all $\mu \in \boldsymbol{R}$ and $p \in \boldsymbol{Z}$.

PROPOSITION 4.2.1. *Assume that ρ is irreducible, then $\rho(Z)$ is a semisimple endomorphism of S with real eigenvalues. Furthermore let λ be the minimal eigenvalue of $\rho(Z)$ and n the positive integer such that $S_{(\lambda+q)} \neq 0$ for $q = 0, 1, \ldots, n-1$ and $S_{(\lambda+n)} = 0$. Then,*

$$S = \bigoplus_{q=0}^{n-1} S_{(\lambda+q)},$$
$$S_{(\lambda)} = \{s \in S | \rho(\mathfrak{l}_{-1}) s = 0\}.$$

PROOF. We first consider the case $\boldsymbol{F} = \boldsymbol{C}$. The first assertion follows from (3) of Lemma 4.1.2 and the weight space decomposition of S. It is obvious $\bigoplus_{q=0}^{n-1} S_{(\lambda+q)}$ is a submodule of S. Since ρ is irreducible, we have $S = \bigoplus_{q=0}^{n-1} S_{(\lambda+q)}$.

Let S' be the subspace of S consisting of all $s \in S$ such that $\rho(\mathfrak{l}_{-1}) s = 0$. We claim that S' is $\rho(z)$ invariant. Indeed, for every $s \in S'$, $X \in \mathfrak{l}_{-1}$, we have

$$\rho(X)\rho(Z)s = \rho(Z)\rho(X)s + \rho([X, Z])s = 0,$$

which shows the assertion. Hence we have

$$S' = \bigoplus_{q=0}^{n-1} S'_{(\lambda+q)},$$

where $S'_{(\lambda+q)}=S'\cap S_{(\lambda+q)}$. Cleary we have $S'_{(\lambda)}=S_{(\lambda)}$. Suppose that there exists $q_0\geqq 1$ such that $S'_{(\lambda+q_0)}\neq 0$. Then the submodule generated by $S'_{(\lambda+q_0)}$ is contained in the subspace $\bigoplus_{q=q_0}^{n-1} S_{(\lambda+q)}$, which is a contradiction.

Next we consider the case $\boldsymbol{F}=\boldsymbol{R}$. Let $\mathfrak{l}^C$, S^C and ρ^C be the complexifications of $\mathfrak{l}$, S and ρ respectively. If ρ^C is irreducible, then we have the direct sum decomposition $S^C=\bigoplus_{q=0}^{n-1} S^C_{(\lambda+q)}$ as above. Since $\lambda+q$ is a real number, we have

$$S^C_{(\lambda+q)}=\overline{S^C_{(\lambda+q)}}\,.$$

This means $S=\bigoplus_{q=0}^{n-1} S_{(\lambda+q)}$.

If ρ^C is not irreducible, then there exists a complex structure I of S such that $\rho(X)I=I\rho(X)$ for all $X\in\mathfrak{l}$. Let S^+ (resp. S^-) be the subspace of S^C consisting of all vectors $s\in S^C$ such that $Is=\sqrt{-1}s$ (resp. $Is=-\sqrt{-1}s$). Then both S^+ and S^- are irreducible $\mathfrak{l}^C$ modules, and $S^C=S^+\oplus S^-$. Let $S^+=\bigoplus_{q=0}^{n^+-1} S^+_{(\lambda^++q)}$ (resp. $S^-=\bigoplus_{q=0}^{n^--1} S^-_{(\lambda_-+q)}$) be the decomposition of S^+ (resp. of S^-) defined as above. Then we see that $\lambda^+=\lambda^-$ and that $\overline{S^+_{(\lambda^++q)}}=S^-_{(\lambda^-+q)}$ for all q. This means $S=\bigoplus_{q=0}^{n-1} S_{(\lambda+q)}$, where $\lambda=\lambda^+=\lambda^-$ and $n=n^+=n^-$. q. e. d.

COROLLARY 4. 2. 2. *Under the same assumption as in Proposition* 4. 2. 1, *put* $S_q=S_{(\lambda+q)}$, $q\in\boldsymbol{Z}$. *Then we have a direct sum decomposition* $S=\bigoplus_{q=0}^{n-1} S_q$ *satisfying the following conditions*:

(i) $\rho(\mathfrak{l}_p)S_q\subset S_{p+q}$ *for all* p, q;

(ii) $S_0=\{s\in S|\rho(\mathfrak{l}_{-1})s=0\}$.

4. 3. Typical symbol of type $(\mathfrak{l},\rho)$. Let $\mathfrak{l}=\bigoplus_p \mathfrak{l}_p$ be a semisimple graded Lie algebra of the first kind and let $\rho^i:\mathfrak{l}\longrightarrow gl(S^i)$, $i=1,\dots,l$ be irreducible representations of $\mathfrak{l}$ on finite dimensional vector spaces S^i respectively. We define a representation $\rho:\mathfrak{l}\longrightarrow gl(S)$ by setting $S=\bigoplus_{i=l}^{l} S^i$ and $\rho=\bigoplus_{i=l}^{l}\rho^i$.

For each $i=1,\dots,l$, let λ^i be the minimal eigenvalue of the endomorphism $\rho^i(Z)$ of S^i and let $S^i=\bigoplus_{q=0}^{n^i-1} S^i_q$ be the direct sum decomposition of S^i defined as in Corollary 4. 2. 2. In the following we assume the condition:

(4.3.1) $\lambda^1=\dots=\lambda^l$, and all ρ^i are faithfull.

We denote by λ the common value of $\lambda^1,\dots,\lambda^l$ and put $n=\mathrm{Max}(n^1,\dots,n^l)$

and $S_q=\bigoplus_{i=1}^{l} S_q^i$ for each q. Then, we have

$$S=\bigoplus_{q=0}^{n-1} S_q,$$
$$\rho(\mathfrak{l}_p)S_q\subset S_{p+q},$$
$$S_0=\{s\in S|\rho(\mathfrak{l}_{-1})s=0\}.$$

Now we put $V=\mathfrak{l}_{-1}$ and $W=S_0$. Then, we define a linear map $i: S\longrightarrow S(V^*)\otimes W$ by

$$i(s)(v_1, \dots, v_q)=(-1)^q\rho(v_1)\dots\rho(v_q)s,$$

where $s\in S_q$ and $v_1, \dots, v_q\in V$.

PROPOSITION 4.3.1. (1) *$i(v)\cdot i(s)=-i(\rho(v)s)$ for every $v\in V$ and $s\in S$.*
(2) *$i: S\longrightarrow S(V^*)\otimes W$ is injective.*

PROOF. (1) is obvious. To prove (2), it suffices to show that $i: S_q\longrightarrow S^q(V^*)\otimes W$ is injective for each nonnegative integer q. We proceed by induction on q. The case $q=0$ is obvious. Assume that $q\geqq 1$ and the assertion is valid for $q-1$. Let s be a vector in S_q such that $i(s)=0$. Then we have $i(\rho(v)s)=-i(v)\cdot i(s)=0$ for all $v\in V$. By induction assumption, we have $\rho(v)s=0$ for all $v\in V$. Hence we obtain $s\in S_0\cap S_q=0$. q. e. d.
With this proposition in mind, we will regard S as a subspace of $S(V^*)\otimes W$ through the map i. The subspace S of $S(V^*)\otimes W$ thus obtained is called the typical symbol of type $(\mathfrak{l}, \rho)$. Let $\mathfrak{g}$ be the Lie algebra defined as in 2.2, which can be considered as a Lie subalgebra of $\mathfrak{gl}(S)$. Since $\rho: \mathfrak{l}\longrightarrow gl(S)$ is faithful, we can also regard $\mathfrak{l}$ as a Lie subalgebra of $gl(S)$.

PROPOSITION 4.3.2. (1) *The typical symbol of type $(\mathfrak{l}, \rho)$ satisfies the conditions* (2.1.1), (2.1.2) *and* (2.1.3).
(2) *$\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$.*

PROOF. (1) The condition (2.1.1.) is obvious, and the condition (2.1.2) follows from (1) of Proposition 4.3.1. The condition (2.1.3) follows from the definition of W and the fact that ρ is faithful. (2) follows from (2) of Proposition 2.2.2. q. e. d.

EXAMPLE 4.3.3. Typical symbol of type $(sl(2, \boldsymbol{F}), \rho)$. Let $\mathfrak{l}$ denote the simple Lie algebra $sl(2, \boldsymbol{F})$ and X, Y, Z be the basis of $sl(2, \boldsymbol{F})$ defined by

$$X=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\quad Y=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\quad Z=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Let $\mathfrak{l}_{-1}$, $\mathfrak{l}_0$, $\mathfrak{l}_1$ be the 1-dimensional subspaces of $\mathfrak{l}$ spanned respectively by Y, Z, X. It clear that $\mathfrak{l}=\mathfrak{l}_{-1}\oplus\mathfrak{l}_0\oplus\mathfrak{l}_1$becomes a simple graded Lie algebra of the first kind.

For nonnegative integer n, we put $S=S^{n-1}(\boldsymbol{F}^2)$. *Let* $\rho : \mathfrak{l}\longrightarrow gl(S)$ be the natural irreducible representation of $\mathfrak{l}$ on S. By using the canonical basis e_0, e_1 of $\boldsymbol{F}^2$, we introduce a basis $s_0, s_1, \dots, s_{n-1}$ of S by setting $s_p=(e_0)^p (e_1)^{n-p-1}$. Then we see that $\rho(Z)s_p=(-\frac{n-1}{2}+p)s_p$, $S_p=\boldsymbol{F}s_p$ for $p=0, 1, \dots, n-1$ and $S_p=0$ otherwise. Therefore the typical symbol of type $(sl(2, \boldsymbol{F}), \rho)$ is isomorphic to the typical symbol of the ordinary differential equation considered in Example 2. 4. 2.

4. 4. Structure theory of the Lie algebras $\mathfrak{g}$ and $gl(S)$. Let S be the typical symbol of type $(\mathfrak{l}, \rho)$. Let Z be the vector in $\mathfrak{l}_0$ given as in (1) of Lemma 4. 1. 1. From the definition of the direct sum decomposition $S=\bigoplus_{q=0}^{n-1} S_q$, it follows that

$$gl(S)_p=\{X\in gl(S)|[Z, X]=pX\}.$$

Let $\mathfrak{z}(\mathfrak{l})$ be the centralizer of $\mathfrak{l}$ in $gl(S)$. Since $\mathfrak{l}$ is semisimple, we have $\mathfrak{l}\cap\mathfrak{z}(\mathfrak{l})=0$. We define a bilinear form Tr on $gl(S)$ by

$$Tr(Y, Y')=\text{Trace of the endomorphism } Y\cdot Y' \text{ of } S,$$

where Y, $Y'\in gl(S)$. One should remark that Tr is $Ad(G)$ invariant and nondegenerate. Let $\mathfrak{g}^{\perp}$ be the orthogonal complement of $\mathfrak{g}$ in $gl(S)$ with respect to the bilinear form Tr.

PROPOSITION 4. 4. 1. (1) $\mathfrak{g}=\mathfrak{l}\oplus\mathfrak{z}(\mathfrak{l})$ *(direct sum of Lie algebras). More precisely,* $\mathfrak{g}_{-1}=\mathfrak{l}_{-1}$, $\mathfrak{g}_0=\mathfrak{l}_0\oplus\mathfrak{z}(\mathfrak{l})$, $\mathfrak{g}_1=\mathfrak{l}_1$ *and* $\mathfrak{g}_p=\mathfrak{l}_p=0$ *for* $|p|\geqq 2$.

(2) $\mathfrak{n}=\mathfrak{z}(\mathfrak{l})$ *and hence* $\mathfrak{g}=\mathfrak{l}\oplus\mathfrak{n}$ *and* $\mathfrak{l}\cong\mathfrak{g}/\mathfrak{n}$.

(3) $gl(S)=\mathfrak{l}\oplus\mathfrak{n}\oplus\mathfrak{g}^{\perp}$ *(orthogonal decomposition).*

(4) $Ad(G)$ *leaves* $\mathfrak{g}$, $\mathfrak{l}$, $\mathfrak{n}$ *and* $\mathfrak{g}^{\perp}$ *invariant.*

PROOF. The case $\boldsymbol{F}=\boldsymbol{R}$ can be easily reduced to the case $\boldsymbol{F}=\boldsymbol{C}$, by taking the complexifications of $\mathfrak{l}$, S and ρ. Hence it suffices to show the case $\boldsymbol{F}=\boldsymbol{C}$. We fix a Cartan subalgebra $\mathfrak{h}$ such that $Z\in\mathfrak{h}\subset\mathfrak{l}$ and an ordering in the set of nonzero roots as in (1) of Lemma 4. 1. 2. We denote by Φ^+ the set of all positive roots with respect to this ordering.

Since $Z\in\mathfrak{l}$, we have $\mathfrak{z}(\mathfrak{l})\subset gl(S)_0$. By (2) of Proposition 2. 2. 2, we have $\mathfrak{z}(\mathfrak{l})\subset\mathfrak{n}_0$. Considering $\mathfrak{g}$ as an $\mathfrak{l}$ module with respect to the adjoint action, we write

$$\mathfrak{g}=\mathfrak{l}\oplus\mathfrak{z}(\mathfrak{l})\oplus\sum_{\gamma}m_{\gamma}U^{-\gamma} \text{ (direct sum of } \mathfrak{l} \text{ modules)},$$

where γ ranges over all nonzero dominant integral forms on $\mathfrak{h}$ and $U^{-\gamma}$ stands for the irreducible module with lowest weight $-\gamma$ and m_{γ} is the multiplicity of $U^{-\gamma}$ in $\mathfrak{g}$. Suppose that $m_{\gamma}>0$ for some γ. Since $ad(Z)U^{-\gamma}\subset U^{-\gamma}$, we have $U^{-\gamma}=\bigoplus_{p}U_{p}^{-\gamma}$, where $U_{p}^{-\gamma}=U^{-\gamma}\cap\mathfrak{g}_{p}$. Furthermore, since $\mathfrak{g}_{p}=\mathfrak{l}_{p}$ for all $p<0$, we have $U_{p}^{-\gamma}=0$ for all $p<0$. Let $v^{-\gamma}$ be a weight vector of weight $-\gamma$ in $U^{-\gamma}$. Since $ad(Z)v^{-\gamma}=-\gamma(Z)v^{-\gamma}$, we have $v^{-\gamma}\in U_{-\gamma(Z)}^{-\gamma}$. Therefore we have $\gamma(Z)\leqq 0$. By (3) of Lemma 4. 1. 2, we have

$$(4.4.1)\qquad \sum_{\alpha\in\Phi_{1}}\gamma(h_{\alpha})\leqq 0$$

Since $\Phi_{1}\subset\Phi^{+}$ and γ is dominant, we have $\gamma(h_{\alpha})\geqq 0$. Thus (4. 4. 1) holds if and only if $\gamma(h_{\alpha})=0$ for all $\alpha\in\Phi_{1}$. From (2) of Lemma 4. 1. 2, it follows that $\gamma=0$, which is a contradiction. Hence we have (1) and (2).

Since $\mathfrak{l}$ is semisimple, Tr is nondegenerate on $\mathfrak{l}$. We clain that $\mathfrak{l}$ is orthogonal to $\mathfrak{n}$. Indeed, since $\mathfrak{n}=\mathfrak{z}(\mathfrak{l})$, we have

$$Tr([X_{1}, X_{2}], Y)=Tr(X_{1}, [X_{2}, Y])=0,$$

for every X_{1}, $X_{2}\in\mathfrak{l}$ and every $Y\in\mathfrak{n}$. Since $[\mathfrak{l}, \mathfrak{l}]=\mathfrak{l}$, we have $Tr(\mathfrak{l}, \mathfrak{n})=0$.

Next we will show that Tr is nondegenerate on $\mathfrak{n}$. Let $\mathfrak{n}^{\perp}$ be the orthogonal complement of $\mathfrak{n}$ in $gl(S)$. We take a $\mathfrak{l}$ submodule $\mathfrak{n}'$ of $gl(S)$ such that $gl(S)=\mathfrak{n}'\oplus\mathfrak{n}^{\perp}$. Then we have

$$Tr([Y', X], Y)=Tr(Y', [X, Y])=0,$$

for every $Y'\in\mathfrak{n}'$, $Y\in\mathfrak{n}$ and $X\in\mathfrak{l}$. Since the restriction of Tr to $\mathfrak{n}'\times\mathfrak{n}$ is nondegenerate, we obtain $[Y', X]=0$. This means that $\mathfrak{n}'\subset\mathfrak{z}(\mathfrak{l})=\mathfrak{n}$. Since $\dim\mathfrak{n}'=\dim\mathfrak{n}$, we have $\mathfrak{n}'=\mathfrak{n}$, and hence Tr is nondegenerate on $\mathfrak{n}$. Therefore we have (3).

Finally (4) follows from (3) and Proposition 2. 3. 1. q. e. d.

COROLLARY 4. 4. 2. *Every element a of G' can be written uniquely in the form :*

$$a=a_{0}\,exp(X),$$

where $a_{0}\in G_{0}$, and $X\in\mathfrak{l}_{1}$.

4. 5. The cochain complex $(C=\bigoplus_{q}C^{q}, \partial)$. Here we will introduce the cochain complex associated with the adjoint representation of $\mathfrak{l}_{-1}$ on $gl(S)$. Let $(C=\bigoplus_{q}C^{q}, \partial)$ be the cochain complex defined by

$$C^q=\mathrm{Hom}(\wedge^q\mathfrak{l}_{-1}, gl(S)) \quad (=\wedge^q\mathfrak{l}_{-1}{}^*\otimes gl(S)),$$

where the coboundary operator $\partial: C^q \longrightarrow C^{q+1}$ is defined by

$$(\partial c)(v_0, \dots, v_q)=\sum_{i=0}^{q}(-1)^i[v_i, c(v_0, \dots, \hat{v}_i, \dots, v_q)],$$

for $c\in C^q$ and $v_0, \dots, v_q\in\mathfrak{l}_{-1}$.

For each $a\in G^{(0)}$ and each $c\in C^q$, we define $ac\in C^q$ by

$$(ac)(v_1, \dots, v_q)=Ad(a)c(Ad(a_0^{-1})v_1, \dots, Ad(a_0^{-1})v_q),$$

where we write a as in (2. 3. 1). We remark that the assignment $a\in G^{(0)}\longrightarrow a_0\in G_0$ is a homomorphism. Hence the group $G^{(0)}$ acts on C on the left. In particular, the group G_0 acts on C. The following lemma is easily proved.

LEMMA 4. 5. 1. *The action of G_0 on $C=\bigoplus_q C^q$ is compatible with the operation of ∂, that is,*

$$\partial(ac)=a(\partial c), \quad a\in G_0, \quad c\in C.$$

For each integer p, let $C^{p,q}$ be the subspace of C^q defined by

$$C^{p,q}=\mathrm{Hom}(\wedge^q\mathfrak{l}_{-1}, gl(S)_{p-1}) \quad (=\wedge^q\mathfrak{l}_{-1}{}^*\otimes gl(S)_{p-1}).$$

Then we have $C^q=\bigoplus_p C^{p,q}$, $\partial(C^{p,q})\subset C^{p-1,q+1}$.

For each integer q, let $C^q(\mathfrak{g})$ and $C^q(\mathfrak{g}^\perp)$ be the subspaces of C^q respectively defined by

$$C^q(\mathfrak{g})=\mathrm{Hom}(\wedge^q\mathfrak{l}_{-1}, \mathfrak{g}) \quad (=\wedge^q\mathfrak{l}_{-1}{}^*\otimes\mathfrak{g}),$$
$$C^q(\mathfrak{g}^\perp)=\mathrm{Hom}(\wedge^q\mathfrak{l}_{-1}, \mathfrak{g}^\perp) \quad (=\wedge^q\mathfrak{l}_{-1}{}^*\otimes\mathfrak{g}^\perp),$$

Since $\mathfrak{g}$ and $\mathfrak{g}^\perp$ are $\mathfrak{l}$ submodules of $gl(S)$, both $(C(\mathfrak{g})=\bigoplus_q C^q(\mathfrak{g}), \partial)$ and $(C(\mathfrak{g}^\perp)=\bigoplus_q C^q(\mathfrak{g}^\perp), \partial)$ are subcomplex of $(C=\bigoplus_q C^q, \partial)$. We put $C^{p,q}(\mathfrak{g})=C^q(\mathfrak{g})\cap C^{p,q}$, $C^{p,q}(\mathfrak{g}^\perp)=C^q(\mathfrak{g}^\perp)\cap C^{p,q}$. Then we have $\partial(C^{p,q}(\mathfrak{g}))\subset C^{p-1,q+1}(\mathfrak{g})$ and $\partial(C^{p,q}(\mathfrak{g}^\perp))\subset C^{p-1,q+1}(\mathfrak{g}^\perp)$.

4. 6. The adjoint operator ∂^*. In the following, we fix an involutive automorphism θ of $\mathfrak{l}$ having the properties in Lemma 4. 1. 1. We define an inner product $(\ ,\ )_\mathfrak{l}$ in $\mathfrak{l}$ by

$$(X_1, X_2)_\mathfrak{l}=-B_\mathfrak{l}(X_1, \theta(X_2)), \quad X_1, X_2\in\mathfrak{l}.$$

It is easily checked that

$$([X_1, X_2], X_3)_\mathfrak{l}+(X_2, [\theta(X_1), X_3])_\mathfrak{l}=0,$$

where $X_1, X_2, X_3 \in \mathfrak{l}$. It should be noted that if $\boldsymbol{F}=\boldsymbol{C}$, then the inner product $(\ ,\ )_{\mathfrak{l}}$ is hermitian.

Next we define an inner product $(\ ,\ )$ in $gl(S)$ as follows. First we recall that there is an inner product $(\ ,\ )_S$ in S having the property

$$(Xs, s')_S+(s, \theta(X)s')_S=0, \quad s, s' \in S, X \in \mathfrak{l}.$$

Then we set

$$(Y_1, Y_2)=\text{Trace of } Y_1 \cdot Y_2^*, \quad Y_1, Y_2 \in gl(S),$$

where Y_2^* stands for the adjoint of Y_2 with respect to the inner product $(\ ,\ )_S$. It is easy to see that

$$([X, Y_1], Y_2)+(Y_1, [\theta(X), Y_2])=0,$$

where $X \in \mathfrak{l}$, $Y_1, Y_2 \in gl(S)$. We should remark that in the case $\boldsymbol{F}=\boldsymbol{C}$, $(\ ,\ )_S$ can be chosen so that

$$(\lambda s, s')_S=(s, \bar{\lambda} s')_S=\lambda(s, s')_S,$$

where $s, s' \in S$, $\lambda \in \boldsymbol{C}$, and hence $(\ ,\ )$ is hermitian.

The inner product $(\ ,\ )$ in $gl(S)$ together with the inner product $(\ ,\ )_{\mathfrak{l}}$ in $\mathfrak{l}$ induces an inner product in C^q in a natural manner. Namely, let $e_1, \ldots, e_m$ be an orthonormal basis of $\mathfrak{l}_{-1}$: $(e_i, e_j)_{\mathfrak{l}}=\delta_{ij}$. Then

$$(c, c')=\frac{1}{q!} \sum_{i_1, \ldots, i_q}(c(e_{i_1}, \ldots, e_{i_q}), c'(e_{i_1 \mathfrak{l}}, \ldots, e_{i_q})),$$

where $c, c' \in C^q$.

Now we define the operator $\partial^*: C^{q+1} \longrightarrow C^q$ in the following way. Let $e_1, \ldots, e_m$ be a basis of $\mathfrak{l}_{-1}$ and let $e^1, \ldots, e^m$ be a basis of $\mathfrak{l}_1$ defined by $B_{\mathfrak{l}}(e_i, e^j)=\delta_{ij}$. Then,

$$(\partial^* c)(v_1, \ldots, v_q)=\sum_{i=1}^m [e^i, c(e_i, v_1, \ldots, v_q)],$$

where $c \in C^{q+1}$ and $v_1, \ldots, v_q \in \mathfrak{l}_{-1}$. It is easily verified that the definition of ∂^* does not depend on the choice of the basis $e_1, \ldots, e_m$ of $\mathfrak{l}_{-1}$ and that $\partial^*(C^{p,q+1}) \subset C^{p+1,q}$, $\partial^*(C^{p,q+1}(\mathfrak{g})) \subset C^{p+1,q}(\mathfrak{g})$ and $\partial^*(C^{p,q+1}(\mathfrak{g}^{\perp})) \subset C^{p+1,q}(\mathfrak{g}^{\perp})$. Furthermore, if $e_1, \ldots, e_m$ is an orthonormal basis of $\mathfrak{l}_{-1}$, then

$$(\partial^* c)(v_1, \ldots, v_q)=-\sum_{i=1}^m [\theta(e_i), c(e_i, v_1, \ldots, v_q)],$$

LEMMA 4.6.1. *The operator ∂^* defined above is the adjoint operator of ∂ with respect to the inner product $(\ ,\ )$, i.e.,*

$$(\partial c, c')=(c, \partial^* c'), \quad c\in C^q, \ c'\in C^{q+1}.$$

PROOF. We will prove this lemma in the case $q=1$. The general case can be proved quite similarly. Let $e_1, \dots, e_m$ be an orthonormal basis of $\mathfrak{l}_{-1}$. Then we have

$$\begin{aligned}(\partial c, c')&=\frac{1}{2}\sum_{i,j}((\partial c)(e_i, e_j), c'(e_i, e_j))\\&=\frac{1}{2}\sum_{i,j}(([e_i, c(e_j)], c'(e_i, e_j))\\&\qquad-\frac{1}{2}\sum_{i,j}([e_j, c(e_i)], c'(e_i, e_j))\\&=-\frac{1}{2}\sum_{i,j}(c(e_j), [\theta(e_i), c'(e_i, e_j)])\\&\qquad+\frac{1}{2}\sum_{i,j}(c(e_i), [\theta(e_j), c'(e_i, e_j)])\\&=-\sum_j(c(e_j), \sum_i[\theta(e_i), c'(e_i, e_j)])\\&=(c, \partial^* c').\end{aligned}$$

q. e. d.

LEMMA 4.6.2. *The action of G' on $C=\bigoplus_q C^q$ is compatible with the operation of ∂^*, that is,*

$$\partial^*(ac)=a(\partial^* c), \quad a\in G', \ c\in C.$$

PROOF. Let $e_1, \dots, e_m$ be a basis of $\mathfrak{l}_{-1}$ and let $e^1, \dots, e^m$ be a basis of $\mathfrak{l}_1$ such that $B_{\mathfrak{l}}(e_i, e^j)=\delta_{ij}$. From (4) of Proposition 4.4.1, it follows that $Ad(a)\mathfrak{l}=\mathfrak{l}$ and hence $Ad(a)\mathfrak{l}_1=\mathfrak{l}_1$. In particular, $Ad(a)e^j\in\mathfrak{l}_1$. On the other hand, since $Ad(a)$ is an automorphism of $\mathfrak{l}$, $Ad(a)$ keeps the Killing form $B_{\mathfrak{l}}$ invariant. Therefore, we have

$$\begin{aligned}B_{\mathfrak{l}}(Ad(a_0)e_i, Ad(a)e^j)&=B_{\mathfrak{l}}(Ad(a)e_i, Ad(a)e^j)\\&=B_{\mathfrak{l}}(e_i, e^j)=\delta_{ij},\end{aligned}$$

where we write $a\in G'$ as in Corollary 4.4.2. Now the assertion follows immediately. q. e. d.

For any $X\in gl(S)_p$, let $[\partial X]^{\perp}$ denote the $C^{p,1}(\mathfrak{g}^{\perp})$ component of ∂X.

LEMMA 4.6.3. *If $p\geqq 0$ and $\partial^*([\partial X]^{\perp})=0$, then $X\in\mathfrak{g}_p$.*

PROOF. Let $X^{\perp}$ be the $\mathfrak{g}^{\perp}$ component of X. Then we have $\partial^*\partial X^{\perp}=\partial^*([\partial X]^{\perp}))=0$. By (2) of Proposition 2.2.2, we have $X^{\perp}=0$. q. e. d.

4.7. The harmonic projection. As usual, the operator $\Delta=\partial\partial^*+\partial^*\partial$: $C\longrightarrow C$ is called the Laplacian, and a form $c\in C$ is called harmonic if $\Delta c=$

0. Clearly, c is harmonic if and only if $\partial c=\partial^* c=0$. We denote by $\mathscr{H}$ the space of all harmonic forms in C. It is obvious that $\mathscr{H}=\bigoplus_q \mathscr{H}_q$, where $\mathscr{H}^q=\mathscr{H}\cap C^q$. Let $H: C\longrightarrow\mathscr{H}$ be the projection of C onto $\mathscr{H}$ with respect to the orthogonal decomposition $C=\mathscr{H}\oplus\mathrm{Im}\ \Delta$. We recall that the action of G' on C leaves both Ker ∂^* and Im ∂^* invariant.

LEMMA 4.7.1. *For every $c\in Ker\ \partial^*$ and every $a\in G'$,*

$$H(ac)=a_0(Hc),$$

where we write $a\in G'$ as in Corollary 4.4.2.

PROOF. For each $X\in\mathfrak{l}_1$ and each $c\in C^q$, we define a q-form $Xc\in C^q$ by

$$(Xc)(v_1, \dots, v_q)=[X, c(v_1, \dots, v_q)],$$

where $v_1, \dots, v_q\in\mathfrak{l}_{-1}$. We also define a 1-form $\alpha_X\in\mathfrak{l}_{-1}{}^*$ by

$$\alpha_X(v)=B_{\mathfrak{l}}(X, v),$$

where $v\in\mathfrak{l}_{-1}$. We will show the following equality

$$(4.7.1)\quad Xc=\partial^*(\alpha_X\wedge c)+\alpha_X\wedge\partial^* c.$$

In fact, we have

$$\begin{aligned}
&\partial^*(\alpha_X\wedge c)(v_1, \dots, v_q)\\
&=\sum_{i=1}^m[e^i, (\alpha_X\wedge c)(e_i, v_1, \dots, v_q)]\\
&=\sum_{i=1}^m[e^i, \alpha_X(e_i)c(v_1, \dots, v_q)]\\
&\qquad+\sum_{i=1}^m\sum_{j=1}^q(-1)^j[e^i, \alpha_X(v_j)(e_i, v_1, \dots, \hat{v}_j, \dots, v_q)]\\
&=\sum_{i=1}^m[\alpha_X(e_i)e^i, c(v_1, \dots, v_q)]\\
&\qquad+\sum_{j=1}^q(-1)^j\alpha_X(v_j)\sum_{i=1}^m[e^i, c(e_i, v_1, \dots, \hat{v}_j, \dots, v_q)]\\
&=\sum_{i=1}^m[B_{\mathfrak{l}}(X, e_i)e^i, c(v_1, \dots, v_q)]\\
&\qquad+\sum_{j=1}^q(-1)^j\alpha_X(v_j)(\partial^* c)(v_1, \dots, \hat{v}_j, \dots, v_q)\\
&=[X, c(v_1, \dots, v_q)]-(\alpha_X\wedge\partial^* c)(v_1, \dots, v_q),
\end{aligned}$$

which shows the assertion.

From (4.7.1), we see that if $\partial^* c=0$, then $Xc=\partial^*(\alpha_X\wedge c)$ and hence $H(Xc)=0$. Therefore we have

$$H(exp(X)c)=\sum_k \frac{1}{k!}H(X^k c)=H(c).$$

On the other hand, it is clear that the action of G_0 on C is compatible with the projection H, i.e.,

$$H(a_0 c)=a_0 H(c),$$

where $a_0 \in G_0$ and $c \in C$. Therefore we have

$$H(a_0 exp(X)c)=a_0 H(exp(X)c)=a_0 H(c). \qquad \text{q. e. d.}$$

§ 5. Normal G' reductions.

5.1. Normal G' reductions. Throughout this section, we fix a typical symbol S of type $(\mathfrak{l}, \rho)$. We say that a linear differential equation $\mathscr{R}=(M, E, R^n)$ is of type $(\mathfrak{l}, \rho)$ if it is of type S.

Let $\mathscr{R}=(M, E, R^n)$ be an integrable linear differential equation of type $(\mathfrak{l}, \rho)$. Let $(P(\mathscr{R}), \omega)$ be the canonical $G^{(0)}$ reduction of $(\mathscr{F}(R^n), \tilde{\omega})$ defined as in § 3. Let $Q(\mathscr{R})$ be a G' reduction of the principal $G^{(0)}$ bundle $P(\mathscr{R})$ and χ be the pull back of the $gl(S)$ valued 1-form ω to $Q(\mathscr{R})$. We denote by $\chi_{\mathfrak{g}}$ (resp. by $\chi_{\mathfrak{g}^\perp}$) the $\mathfrak{g}$ component (resp. the $\mathfrak{g}^\perp$ component) of χ with respect to the direct sum decomposition $gl(S)=\mathfrak{g}\oplus\mathfrak{g}^\perp$. Furthermore, for each p, we denote respectively by χ_p, $(\chi_{\mathfrak{g}})_p$ and $(\chi_{\mathfrak{g}^\perp})_p$ the $gl(S)_p$ component of χ, $\chi_{\mathfrak{g}}$ and $\chi_{\mathfrak{g}^\perp}$.

PROPOSITION 5.1.1. (1) *The pair $(Q(\mathscr{R}), \chi_{\mathfrak{g}})$ is a Cartan connection of type G/G', that is,*

(i) $R_a^* \chi_{\mathfrak{g}}=Ad(a^{-1})\chi_{\mathfrak{g}}$, $a \in G'$;

(ii) $\chi_{\mathfrak{g}}(X^*)=X$, $X \in \mathfrak{g}'$;

(iii) *For every point $z \in Q(\mathscr{R})$, $(\chi_{\mathfrak{g}})_z$ gives a linar isomorphism of $T(Q(\mathscr{R}))_z$ onto $\mathfrak{g}$.*

(2) *$\chi_{\mathfrak{g}^\perp}$ is a tensorial 1-form on $Q(\mathscr{R})$, that is,*

(iv) $R_a^* \chi_{\mathfrak{g}^\perp}=Ad(a^{-1})\chi_{\mathfrak{g}^\perp}$, $a \in G'$;

(v) $\chi_{\mathfrak{g}^\perp}(X^*)=0$, $X \in \mathfrak{g}'$.

(3) $(\chi_{\mathfrak{g}^\perp})_p=0$ *for* $p \leqq -1$.

PROOF. (1) We first recall that $Ad(G')$ leaves both $\mathfrak{g}$ and $\mathfrak{g}^\perp$ invariant. The assertions (i) and (ii) follow from Proposition 3.2.2. Let $Y \in T(Q(\mathscr{R}))_z$, $z \in Q(\mathscr{R})$ be a tangent vector such that $\chi_{\mathfrak{g}}(Y)=0$. The we have $\omega_{-1}(Y)=(\chi_{\mathfrak{g}})_{-1}(Y)=0$. From (4) of Proposition 3.2.2, we see that Y is vertical and hence there exists a vector $X \in \mathfrak{g}'$ such that $Y=X_z^*$. By (ii), we have $X=\chi_{\mathfrak{g}}(Y)=0$. Since dim $\mathfrak{g}$=dim $Q(\mathscr{R})$, we have (iii). (2)

and (3) also follow from Proposition 3. 2. 2. q. e. d.

For any $X \in \mathfrak{g}$, let X^* be the vector field on $Q(\mathscr{R})$ defined by

$$\chi_{\mathfrak{g}}(X_z^*) = X \text{ for all } z \in Q(\mathscr{R}).$$

We define a $C^1(\mathfrak{g}^\perp)(=\mathrm{Hom}(\mathfrak{l}_{-1}, \mathfrak{g}^\perp))$ valued function $c_{\mathfrak{g}^\perp}$ on $Q(\mathscr{R})$ by

$$c_{\mathfrak{g}^\perp}(z)(v) = (\chi_{\mathfrak{g}^\perp})(v_z^*),$$

where $z \in Q(\mathscr{R})$ and $v \in \mathfrak{l}_{-1}$. For each p, $(c_{\mathfrak{g}^\perp})^p$ denotes the $C^{p,1}(\mathfrak{g}^\perp)$ component of $c_{\mathfrak{g}^\perp}$, that is,

$$(c_{\mathfrak{g}^\perp})^p(z)(v) = (\chi_{\mathfrak{g}^\perp})_{p-1}(v_z^*).$$

By (3) of Proposition 5. 1. 1, we have $(c_{\mathfrak{g}^\perp})^p = 0$ for $p \leqq 0$.

A G' reduction $Q(\mathscr{R})$ of $P(\mathscr{R})$ is said to be normal if $\partial^*(c_{\mathfrak{g}^\perp})(z) = 0$ for all $z \in Q(\mathscr{R})$. We are now in a position to state the main theorem.

THEOREM 5. 1. 2. (1) *For every integrable linear differential equation* $\mathscr{R} = (M, E, R^n)$ *of type* $(\mathfrak{l}, \rho)$, *there exists a unique normal reduction* $Q(\mathscr{R}), \chi)$ *of* $(P(\mathscr{R}), \omega)$.

(2) *Let* $\mathscr{R} = (M, E, R^n)$ (*resp.* $\mathscr{R}' = (M', E', R^{n'})$) *be an integrable linear differential equation of type* $(\mathfrak{l}, \rho)$ *and let* $(Q(\mathscr{R}), \chi)$ (*resp.* $(Q(\mathscr{R}'), \chi')$) *be the corresponding normal* G' *reduction of* $(P(\mathscr{R}), \omega)$ (*resp. of* $(P(\mathscr{R}'), \omega')$). *For every isomorphism* ϕ *of* $\mathscr{R}$ *onto* $\mathscr{R}'$, *the corresponding bundle isomorphism* $P(\phi) : P(\mathscr{R}) \longrightarrow P(\mathscr{R}')$ *maps* $Q(\mathscr{R})$ *onto* $Q(\mathscr{R}')$, *and the restriction* $Q(\phi)$ *of* $P(\phi)$ *to* $Q(\mathscr{R})$ *is a bundle isomorphism of* $Q(\mathscr{R})$ *onto* $Q(\mathscr{R}')$ *satisfying* $Q(\phi)^*\chi' = \chi$. *Conversely if* $Q(\phi)$ *is a bundle isomorphism of* $Q(\mathscr{R})$ *onto* $Q(\mathscr{R}')$ *such that* $Q(\phi)^*\chi' = \chi$, *then there exists a unique isomorphism* ϕ *of* $\mathscr{R}$ *onto* $\mathscr{R}'$ *which induces the given* $Q(\phi)$.

This theorem will be proved in 5. 4～5. 7.

5. 2. Fundamental system of invariants. Let $Q(\mathscr{R})$ be the normal G' reduction. Let $Hc_{\mathfrak{g}^\perp}$ be the harmonic part of $c_{\mathfrak{g}^\perp}$. The next proposition follows from Lemma 4. 7. 1 and Proposition 5. 1. 1.

PROPOSITION 5. 2. 1. $Hc_{\mathfrak{g}^\perp}(z \cdot a_0 exp(X)) = a_0^{-1} Hc_{\mathfrak{g}^\perp}(z)$ *for all* $z \in Q(\mathscr{R})$, $a_0 \in G_0$ *and* $X \in \mathfrak{l}_1$.

The main purpose of this paragraph is to show that the harmonic part $Hc_{\mathfrak{g}^\perp}$ of $c_{\mathfrak{g}^\perp}$ gives a fundamental system of invariants of the differential equation $\mathscr{R}$.

For each integer p, we define a $C^{p-1,2}(\mathfrak{g}^\perp)$ valued function b^{p-1} on $Q(\mathscr{R})$ by

$$b^{p-1}(z)(v_1, v_2) = -(v_1{}^*(c_{\mathfrak{g}^\perp})^{p-1})(z)(v_2) + (v_2{}^*(c_{\mathfrak{g}^\perp})^{p-1})(z)(v_1) - \sum_{q=1}^{p-1}[(c_{\mathfrak{g}^\perp})^q(z)(v_1), (c_{\mathfrak{g}^\perp})^{p-q}(z)(v_2)]_{\mathfrak{g}^\perp},$$

where $v_1, v_2 \in \mathfrak{l}_{-1}$ and $[(c_{\mathfrak{g}^\perp})^q(z)(v_1), (c_{\mathfrak{g}^\perp})^{p-q}(z)(v_2)]_{\mathfrak{g}^\perp}$ denotes the $\mathfrak{g}^\perp$ component of $[(c_{\mathfrak{g}^\perp})^q(z)(v_1), (c_{\mathfrak{g}^\perp})^{p-q}(z)(v_2)]$. One should note that $b^0=0$.

Now we recall that Δ: Im $\partial^* \longrightarrow$ Im ∂^* is a linear isomorphism and that $(c_{\mathfrak{g}^\perp})^p - H(c_{\mathfrak{g}^\perp})^p \in \mathrm{Im}\ \partial^*$.

THEOREM 5.2.2. *$(c_{\mathfrak{g}^\perp})^p = H(c_{\mathfrak{g}^\perp})^p + \Delta^{-1}\partial^* b^{p-1}$ for every p. In particular $(c_{\mathfrak{g}^\perp})^1$ is harmonic.*

PROOF. By (3) of Proposition 3.2.2, we have $d\chi + \frac{1}{2}\chi \wedge \chi = 0$. Taking the $\mathfrak{g}$ component and the $\mathfrak{g}^\perp$ component of this equality, we have

$$d\chi_{\mathfrak{g}} + \frac{1}{2}\chi_{\mathfrak{g}} \wedge \chi_{\mathfrak{g}} + \frac{1}{2}[\chi_{\mathfrak{g}^\perp} \wedge \chi_{\mathfrak{g}^\perp}]_{\mathfrak{g}} = 0,$$

$$d\chi_{\mathfrak{g}^\perp} + \chi_{\mathfrak{g}} \wedge \chi_{\mathfrak{g}^\perp} + \frac{1}{2}[\chi_{\mathfrak{g}^\perp} \wedge \chi_{\mathfrak{g}^\perp}]_{\mathfrak{g}^\perp} = 0,$$

where $[\chi_{\mathfrak{g}^\perp} \wedge \chi_{\mathfrak{g}^\perp}]_{\mathfrak{g}}$ (resp. $[\chi_{\mathfrak{g}^\perp} \wedge \chi_{\mathfrak{g}^\perp}]_{\mathfrak{g}^\perp}$) stands for the $\mathfrak{g}$ component (resp. $\mathfrak{g}^\perp$ component) of $\chi_{\mathfrak{g}^\perp} \wedge \chi_{\mathfrak{g}^\perp}$.

From these equalities, we have

$$(5.2.1) \quad -(\chi_{\mathfrak{g}})([v_1{}^*, v_2{}^*]) + [c_{\mathfrak{g}^\perp}(v_1), c_{\mathfrak{g}^\perp}(v_2)]_{\mathfrak{g}} = 0,$$

$$(5.2.2) \quad (v_1{}^* c_{\mathfrak{g}^\perp})(v_2) - (v_2{}^* c_{\mathfrak{g}^\perp})(v_1) - (\chi_{\mathfrak{g}^\perp})([v_1{}^*, v_2{}^*]) + [v_1, c_{\mathfrak{g}^\perp}(v_2)] - [v_2, c_{\mathfrak{g}^\perp}(v_1)] + [c_{\mathfrak{g}^\perp}(v_1), c_{\mathfrak{g}^\perp}(v_2)]_{\mathfrak{g}^\perp} = 0,$$

where $v_1, v_2 \in \mathfrak{l}_{-1}$ and $[c_{\mathfrak{g}^\perp}(v_1), c_{\mathfrak{g}^\perp}(v_2)]_{\mathfrak{g}}$ (resp. $[c_{\mathfrak{g}^\perp}(v_1), c_{\mathfrak{g}^\perp}(v_2)]_{\mathfrak{g}^\perp}$) *stands for the $\mathfrak{g}$ component (resp. the $\mathfrak{g}^\perp$ component) of* $[c_{\mathfrak{g}^\perp}(v_1), c_{\mathfrak{g}^\perp}(v_2)]$. Taking the $\mathfrak{g}_{-1}$ component of (5.2.1), we have

$$(\chi_{\mathfrak{g}})_{-1}([v_1{}^*, v_2{}^*]) = 0.$$

This means that $[v_1{}^*, v_2{}^*]$ is vertical and hence

$$(\chi_{\mathfrak{g}^\perp})([v_1{}^*, v_2{}^*]) = 0.$$

Therefore, (5.2.2) yields

$$\partial(c_{\mathfrak{g}^\perp})^p = b^{p-1} \text{ for every } p.$$

Hence we have

$$\Delta((c_{\mathfrak{g}^\perp})^p - H(c_{\mathfrak{g}^\perp})^p) = \Delta(c_{\mathfrak{g}^\perp})^p = \partial^*\partial(c_{\mathfrak{g}^\perp})^p = \partial^*(b^{p-1}).$$

q. e. d.

5.3. Flat differential equations. An integrable linear differential equation $\mathscr{R}=(M, E, R^n)$ of type $(\mathfrak{l}, \rho)$ is said to be flat if $Hc_{\mathfrak{g}^\perp}=0$. In this paragraph, we will show the following

THEOREM 5.3.1. *The model equation* $\mathscr{R}_S=(G/G', E_S, R^n_S)$ *of type* $(\mathfrak{l}, \rho)$ *is flat. Conversely every flat integrable linear differential equation* $\mathscr{R}=(M, E, R^n)$ *of type* $(\mathfrak{l}, \rho)$ *is locally isomorphic to the model equation of type* $(\mathfrak{l}, \rho)$.

PROOF. The first assertion follows from Proposition 3.4.1. Let $\mathscr{R}=(M, E, R^n)$ be a flat linear differential equation of type $(\mathfrak{l}, \rho)$. From Theorem 5.2.2, it follows that $(c_{\mathfrak{g}^\perp})^q=0$ for all q. Hence we have $\chi_{\mathfrak{g}^\perp}=0$ and $\chi=\chi_{\mathfrak{g}}$. Therefore $(Q(\mathscr{R}), \chi)$ is a flat Cartan connection of type G/G'. Consequently, there exists a local bundle isomorphism $Q(\phi)$ of $Q(\mathscr{R}_S)$ onto $Q(\mathscr{R})$ such that $Q(\phi)^*\chi=\chi_S$. By (2) of Theorem 5.1.2, there exists a local isomorphism ϕ of $\mathscr{R}_S$ onto $\mathscr{R}$ which induces $Q(\phi)$. q.e.d.

5.4. The Lie subgroups $G^{(q)}$ of $G^{(0)}$. For each integer $q\geqq 1$, we define a subgroup $G^{(q)}$ of $G^{(0)}$ inductively by

$$G^{(q)}=\{a\in G^{(q-1)}|Ad(a)(\mathfrak{g}_{-1}+\mathfrak{g}^{(q-1)})=\mathfrak{g}_{-1}+\mathfrak{g}^{(q-1)}\},$$

where $\mathfrak{g}^{(q-1)}$ denotes the Lie algebra of $G^{(q-1)}$. Since $Ad(a)\mathfrak{g}^{(q-1)}=\mathfrak{g}^{(q-1)}$ for all $a\in G^{(q-1)}$, we may write

$$G^{(q)}=\{a\in G^{(q-1)}|Ad(a)\mathfrak{g}_{-1}\subset\mathfrak{g}_{-1}+\mathfrak{g}^{(q-1)}\},$$

It is easy to see that $G^{(q)}\supset G'$ for every q.

LEMMA 5.4.1. (1) *The group* $G^{(q)}$ *consists of all elements* $a\in G^{(0)}$ *in the form*:

$$a=a_0exp(X_1)\ldots exp(X_q)exp(X_{q+1})\ldots exp(X_{n-1}),$$

where $a_0\in G_0$, $X_p\in\mathfrak{g}_p$ *for* $p\leqq q$ *and* $X_p\in gl(S)_p$ *for* $p\geqq q+1$. *In particular* $G^{(q)}=G'$ *for* $q\geqq n-1$.

(2) $\mathfrak{g}^{(q)}=(\bigoplus_{p=0}^{q}\mathfrak{g}_p)\oplus(\bigoplus_{p=q+1}^{n-1}gl(S)_p)$. *In particular* $\mathfrak{g}^{(q)}=\mathfrak{g}'$ *for* $q\geqq n-1$.

PROOF. We proceed by induction on q. The case $q=0$ has already shown in Lemma 3.1.1. Assume that $q\geqq 1$ and that every element $a\in G^{(q-1)}$ can be written in the form:

$$a=a_0exp(X_1)\ldots exp(X_{q-1})exp(X_q)\ldots exp(X_{n-1}),$$

where $a_0\in G_0$, $X_p\in\mathfrak{g}_p$ for $p\leqq q-1$ and $X_p\in gl(S)_p$ for $p\geqq q$. We put

$$a' = a_0 exp(X_1) \dots exp(X_{q-1}),$$
$$a'' = exp(X_q) \dots exp(X_{n-1}),$$

Since $a' \in G' \subset G^{(q)}$, $a \in G^{(q)}$ if and only if $a'' \in G^{(q)}$.

On the other hand, for every $v \in \mathfrak{g}_{-1}$, we have

$$Ad(a'')v = v + [X_q, v] \quad (\text{mod. } \bigoplus_{p=q}^{n} gl(S)_p).$$

Hence $a'' \in G^{(q)}$ if and only if $[X_q, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{q-1}$. By (2) of Proposition 2. 2. 2, this is equivalent to the condition $X_q \in \mathfrak{g}_q$. The assertion now follows immediately. q. e. d.

5. 5. The functions c_H. Let H be a connection in the principal bundle $P(\mathscr{R})$, namely, a subbundle of the tangent bundle $T(P(\mathscr{R}))$ such that

$$T(P(\mathscr{R}))_z = H_z + V(P(\mathscr{R}))_z, \ z \in P(\mathscr{R}),$$
$$H_{za} = R_{a*}(H_z), \ z \in P(\mathscr{R}), \ a \in G^{(0)}.$$

Then it follows from (4) of Proposition 3. 2. 2 that $\omega_{-1}: H_z \longrightarrow \mathfrak{l}_{-1} (= \mathfrak{g}_{-1})$ is a linear isomorphism. For each $v \in \mathfrak{l}_{-1}$, we define a vector field $v_H{}^*$ by

$$(v_H{}^*)_z \in H_z \text{ for all } z \in P(\mathscr{R}),$$
$$(\omega_{-1})(v_H{}^*) = v.$$

Now we define a C^1 valued function c_H on $P(\mathscr{R})$ by

$$c_H(z)(v) = \omega(v_H{}^*)_z.$$

We denote by $(c_H)^p$ the $C^{p,1}$ component of c_H.

LEMMA 5. 5. 1. (1) *$c_H(za) = a^{-1} c_H(z)$ for all $z \in P(\mathscr{R})$, $a \in G^{(0)}$.*
(2) *$(c_H)^p = 0$ for $p \leqq -1$.*
$(c_H)^0(z)(v) = v$ for all $z \in P(\mathscr{R})$, $v \in \mathfrak{g}_{-1}$.

PROOF. These assertions follow from Proposition 3. 2. 2 and the definition of c_H. q. e. d.

Let H' be another connection in $P(\mathscr{R})$. For each $v \in \mathfrak{l}_{-1}$, we also define a vector field $v_{H'}{}^*$ as above. Since $v_H{}^* - v_{H'}{}^*$ is a vertical vector field, there exists a unique $\mathrm{Hom}(\mathfrak{l}_{-1}, \mathfrak{g}^{(0)})$ valued function $b_{H,H'}$ on $P(\mathscr{R})$ such that

$$(v_H{}^*)_z - (v_{H'}{}^*)_z = (b_{H,H'}(z)v)^*$$

for all $z \in P(\mathscr{R})$ and $v \in \mathfrak{l}_{-1}$. Clearly we have

$$(5.5.1) \quad c_H - c_{H'} = b_{H,H'}.$$

5. 6. Normal $G^{(q)}$ reduction of $P(\mathscr{R})$. Let $P^{(q)}(\mathscr{R})$ be a $G^{(q)}$ reduction

of $P(\mathscr{R})$. We choose a connection H in $P(\mathscr{R})$ in such a way that
(5.6.1) $H_z \in T(P^{(q)}(\mathscr{R}))_z$ for all $z \in P^{(q)}(\mathscr{R})$.

Let $c_H^{\perp}$ denote the $C^1(\mathfrak{g}^{\perp})$ component of c_H. It should be remarked that for every point $z \in P^{(q)}(\mathscr{R})$ and for every $p \leqq q+1$, $(c_H^{\perp})^p(z)$ does not depend on the choice of the connection H satisfying (5.6.1). Indeed, if H' is another connection in $P(\mathscr{R})$ satisfying (5.6.1), then we have

$$b_{H,H'}(z) \in \mathrm{Hom}(\mathfrak{l}_{-1}, \mathfrak{g}^{(q)}) \quad \text{for all } z \in P^{(q)}(\mathscr{R}).$$

Therefore, by (5.5.1), we have

$$(c_H)^p(z) - (c_{H'})^p(z) = (b_{H,H'})^p(z) \in C^{p,1}(\mathfrak{g})$$

for every $z \in P^{(q)}(\mathscr{R})$ and $p \leqq q+1$.

These being remarked, we say that a $G^{(q)}$ reduction $P^{(q)}(\mathscr{R})$ is normal if $\partial^*(c_H^{\perp})^p = 0$ for all $z \in P^{(q)}(\mathscr{R})$ and all $p \leqq q$.

PROPOSITION 5.6.1. *For every nonnegatve integer q, there exists a unique normal $G^{(q)}$ reduction of $P(\mathscr{R})$.*

PROOF. We proceed by induction on q. The case $q=0$ is trivial. Assume that there exists a unique normal $G^{(q)}$ reduction $P^{(q)}(\mathscr{R})$ of $P(\mathscr{R})$.

Let H be a connection in $P^{(q)}(\mathscr{R})$ satisfying (5.6.1). Then we have $\partial^*(c_H^{\perp})^p = 0$ for all $z \in P^{(q)}(\mathscr{R})$ and $p \leqq q$. Let $P^{(q+1)}(\mathscr{R})$ be the subset of $P^{(q)}(\mathscr{R})$ defined by

$$P^{(q+1)}(\mathscr{R}) = \{z \in P^{(q)}(\mathscr{R}) \mid \partial^*(c_H^{\perp})^{q+1}(z) = 0\}.$$

Note that the definition of $P^{(q+1)}(\mathscr{R})$ does not depend on the choice of the connection H.

We claim that, for every point $x \in M$, the fiber $P^{(q+1)}(\mathscr{R})_x$ of $P^{(q+1)}(\mathscr{R})$ over x is nonempty. For this purpose, we fix a point $z \in P^{(q)}(\mathscr{R})_x$. By (1) of Lemma 5.5.1, we have

$$\begin{aligned}(c_H)^{q+1}(z \cdot exp(Y)) &= (c_H)^{q+1}(z) - [Y, (c_H)^0(z)] \\ &= (c_H)^{q+1}(z) + \partial Y,\end{aligned}$$

where $Y \in gl(S)_{q+1}$. Since $C^1 = \mathrm{Ker}\ \partial^* \oplus \mathrm{Im}\ \partial$, we can take Y in such a way that $\partial^*(c_H)^{q+1}(z \cdot exp(Y)) = 0$. This means that $z \cdot exp(Y) \in P^{(q+1)}(\mathscr{R})$.

Let $z \in P^{(q+1)}(\mathscr{R})$ and $a \in G^{(q)}$. We will show that $za \in P^{(q+1)}(\mathscr{R})$ if and only if $a \in G^{(q+1)}$. To see this, we write $a \in G^{(q)}$ as in Lemma 5.4.1. We put $a' = a_0 exp(X_1) \ldots exp(X_q)$ and $a'' = exp(X_{q+1}) \ldots exp(X_{n-1})$. Note that $a' \in G'$ and hence $a'^{-1}(\bigoplus_{p \geqq q+2} C^{p,1}(\mathfrak{g}^{\perp})) = \bigoplus_{p \geqq q+2} C^{p,1}(\mathfrak{g}^{\perp})$. Therefore we have

$$(5.6.2)\quad (c_H^{\perp})(za')=a'^{-1}(c_H^{\perp})(z) \equiv a'^{-1}\Big(\sum_{p\leq q+1}(c_H^{\perp})^p(z)\Big)\ \Big(\text{mod. } \bigoplus_{p\geq q+2} C^{p,1}(\mathfrak{g}^{\perp})\Big).$$

From Lemma 4.6.2 and the fact that $\sum_{p\leq q+1}(c_H^{\perp})^p(z)\in \text{Ker } \partial^*$, it follows that $a'^{-1}(\sum_{p\leq q+1}(c_H^{\perp})^p(z))\in \text{Ker } \partial^*$. Therefore, by (5.6.2), we have $(c_H^{\perp})^{q+1}(za')\in \text{Ker } \partial^*$ and hence $za'\in P^{(q+1)}(\mathscr{R})$. Thus we may assume that $a'=e$, i.e., $a_0=e$, $X_1=\ldots=X_p=0$. Then we have

$$(c_H^{\perp})^{q+1}(za)=(c_H^{\perp})^{q+1}(z)+[\partial X_{q+1}]^{\perp},$$

where $[\partial X_{q+1}]^{\perp}$ stands for the $C^{q+1,1}(\mathfrak{g}^{\perp})$ component of ∂X_{q+1}. Hence $(c_H^{\perp})^{q+1}(za)\in \text{Ker } \partial^*$ if and only if $[\partial X_{q+1}]^{\perp}\in \text{Ker } \partial^*$. By Lemma 4.6.3, this is equivalent to the condition $X_{q+1}\in \mathfrak{g}_{q+1}$.

Thus we have shown that $P^{(q+1)}(\mathscr{R})$ is a principal $G^{(q+1)}$ bundle over M. By the construction of $P^{(q+1)}(\mathscr{R})$, it is clear that $P^{(q+1)}(\mathscr{R})$ is a normal $G^{(q+1)}$ reduction.

Next we will show the uniqueness of the normal $G^{(q+1)}$ reduction of $P(\mathscr{R})$. Let $P^{(q+1)}(\mathscr{R})'$ be a normal $G^{(q+1)}$ reduction of $P(\mathscr{R})$. We choose a connection H' in $P(\mathscr{R})$ in such a way that

$$H'_z\in T(P^{(q+1)}(\mathscr{R})')_z \text{ for all } z\in P^{(q+1)}(\mathscr{R})'.$$

Let $P^{(q)}(\mathscr{R})'$ be the $G^{(q)}$ extension of $P^{(q+1)}(\mathscr{R})'$. Since every point $z'\in P^{(q)}(\mathscr{R})'$ can be written in the form:

$$z'=z''exp(X),\ z''\in P^{(q+1)}(\mathscr{R})',\ X\in gl(S)_{q+1},$$

we have

$$(c_{H'}^{\perp})^p(z''exp(X))=(c_{H'}^{\perp})^p(z'')\in Ker\ \partial^* \text{ for all } p\leq q.$$

This shows that $P^{(q)}(\mathscr{R})'$ is a normal $G^{(q)}$ reduction of $P(\mathscr{R})$. By induction assumption, we have $P^{(q)}(\mathscr{R})'=P^{(q)}(\mathscr{R})$. From the definition of $P^{(q+1)}(\mathscr{R})$, it follows that $P^{(q+1)}(\mathscr{R})'\subset P^{(q+1)}(\mathscr{R})$. Since both $P^{(q+1)}(\mathscr{R})'$ and $P^{(q+1)}(\mathscr{R})$ are principal $G^{(q+1)}$ bundles, we have $P^{(q+1)}(\mathscr{R})'=P^{(q+1)}(\mathscr{R})$. q.e.d.

5.7. Proof of Theorem 5.1.2. We are now in a position to complete the proof of Theorem 5.1.2. We first remark that $G'=G^{(n-1)}$. The assertion (1) of Theorem 5.1.2 follows from the next lemma.

LEMMA 5.7.1. *A G' reduction $Q(\mathscr{R})$ of $P(\mathscr{R})$ is normal in the sense in* 5.1 *if and only if it is normal in the sense in* 5.6.

PROOF. We choose a connection H in $P(\mathscr{R})$ such that

$$H_z \in T(Q(\mathscr{R}))_z \text{ for all } z \in Q(\mathscr{R}).$$

For each $v \in \mathfrak{l}_{-1}$, let v^* be the vector field on $Q(\mathscr{R})$ defined by $\chi_{\mathfrak{g}}(v^*) = v$ and let $v_H{}^*$ be the vector field on $P(\mathscr{R})$ defined by $\omega_{-1}(v_H{}^*) = v$ and $(v_H{}^*)_z \in H_z$, $z \in P(\mathscr{R})$. Then we see that $v^* - v_H{}^*$ is a vertical vector field. Hence there exists a $C^1(\mathfrak{g})$ valued function b_H such that

$$(v^*)_z - (v_H{}^*)_z = (b_H(z)v)^* \text{ for all } z \in Q(\mathscr{R}) \text{ and } v \in \mathfrak{l}_{-1}.$$

This yields $c - c_H = b_H$. Hence we have $c_{\mathfrak{g}^\perp} = c_H^\perp$. The assertion now follows immediately q. e. d.

Next we will show (2) of Theorem 5. 1. 2. Let ϕ be an isomorphism of $\mathscr{R}$ onto $\mathscr{R}'$. From Proposition 3. 3. 1, it follows that there corresponds the bundle isomorphism $P(\phi): P(\mathscr{R}) \longrightarrow P(\mathscr{R}')$ such that $P(\phi)^* \omega' = \omega$. It is easy to see that $P(\phi)(Q(\mathscr{R}))$ is also a normal G reduction of $P(\mathscr{R}')$. From the uniqueness of the normal G' reduction, we conclude that $Q(\mathscr{R}') = P(\phi)(Q(\mathscr{R}))$. It is then clear that the restriction $Q(\phi)$ of $P(\phi)$ to $Q(\mathscr{R})$ is a bundle isomorphism of $Q(\mathscr{R})$ onto $Q(\mathscr{R}')$ such that $Q(\phi)^* \chi' = \chi$.

Conversely let $Q(\phi): Q(\mathscr{R}) \longrightarrow Q(\mathscr{R}')$ be a bundle isomorphism such that $Q(\phi)^* \chi' = \chi$. Let $P(\phi)$ denote the extension of $Q(\phi)$ as a bundle isomorphism of $P(\mathscr{R})$ onto $P(\mathscr{R}')$. Clearly we have $Q(\phi)^* \chi' = \chi$. By Proposition 3. 3. 1, there exists a unique bundle isomorphism ϕ of $\mathscr{R}$ onto $\mathscr{R}'$ which induces $P(\phi)$. Thus we have proved Theorem 5. 1. 2.

5. 8. Invariants of linear ordinary differential equations. Let $\mathfrak{l}$, ρ and S be as in Example 4. 3. 3. It is well known that $gl(S) = \bigoplus_{p=1}^{n} U_p$, where U_p is an irreducible $\mathfrak{l}$ submodule with dim $U_p = 2p-1$. It is obvious that $\mathfrak{g}^\perp = \bigoplus_{p=3}^{n} U_p$. Put $U_{p,q} = U_p \cap gl(S)_q$. Then we have dim $U_{p,q} = 1$ for $q = -p+1, -p+2, \dots, p-1$ and $U_{p,q} = 0$ for $q \leqq -p$ or $q \geqq p$. It is clear that $\ker \partial^* \cap C^1(\mathfrak{g}^\perp) = \mathscr{H}^1 \cap C^1(\mathfrak{g}^\perp) = \bigoplus_{p=3}^{n} V^* \otimes U_{p,p-1}$.

THEOREM 5. 8. 1. *For a linear ordinary differential equation* $\mathscr{R}$: $\left(\frac{d}{dt}\right)^n u + a_1(t)\left(\frac{d}{dt}\right)^{n-1} u + a_2(t)\left(\frac{d}{dt}\right)^{n-2} u + \dots + a_n(t)u = 0$, *the* $V^* \oplus U_{p,p-1}$-*valued functions* $(c_{\mathfrak{g}^\perp})^p$, $p = 3, \dots, n$ *form the fundamental system of invariants of* $\mathscr{R}$.

Here we remark that $\partial c_{\mathfrak{g}^\perp} = 0$ and hence $c_{\mathfrak{g}^\perp} = H c_{\mathfrak{g}^\perp}$.

EXAMPLE 5. 8. 2. Relation between the invariants $(c_{\mathfrak{g}^\perp})^p$ and the

Laguerre-Forsyth's invariants θ_p. For the sake of the simplicity, we consider the case where $n=6$. Let Y^* be the vector field on $Q(\mathscr{R})$ defined by $\chi_g(Y^*)=Y$, $Y\in sl(2, \boldsymbol{F})$ being defined as in Example 4.3.3. Choosing the basis $s_0, s_1, \ldots, s_5$ of S as in Example 4.3.3, we have

$$\chi(Y^*)_z=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 10c_3(z) & 0 & 0 & 3 & 0 & 0 \\ 10c_4(z) & 6c_3(z) & 0 & 0 & 4 & 0 \\ 5c_5(z) & 4c_4(z) & 3c_3(z) & 0 & 0 & 5 \\ c_6(z) & c_5(z) & c_4(z) & c_3(z) & 0 & 0 \end{bmatrix},$$

for every point $z\in Q(\mathscr{R})$, because of the normal condition.

Let $z(t)$ be an integral curve of Y^*. Since the map $t\in\boldsymbol{F}\longrightarrow x(t)=\pi(z(t))\in M$ is a local diffeomorphism, we may regard t as a local coordinate system of M. For an arbitrary solution σ of $\mathscr{R}$, we define a S valued function $u(t)$ by

$$u(t)=z(t)^{-1}(j^6(\sigma)_{x(t)}).$$

We express $u(t)$ as

$$u(t)=\sum_{i=0}^{5}u_i(t)s_i.$$

It should be remarked that $u_0(t)$ can be considered as the coordinate of $\sigma(x(t))$ in the fiber $E_{x(t)}$. Since $j^6(\sigma)$ is a flat cross section of R^6, it follows that

$$\frac{d}{dt}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}+\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 10c_3(t) & 0 & 0 & 3 & 0 & 0 \\ 10c_4(t) & 6c_3(t) & 0 & 0 & 4 & 0 \\ 5c_5(t) & 4c_4(t) & 3c_3(t) & 0 & 0 & 5 \\ c_6(t) & c_5(t) & c_4(t) & c_3(t) & 0 & 0 \end{bmatrix}\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where we simply write $c_i(t)=c_i(z(t))$. By eliminating the variables $u_1, u_2, \ldots, u_5$ from the above equations, we have a linear ordinary differential equation of order 6

$$u_0^{(6)}+112c_3u_0'''+(168c_3'-216c_4)u_0''+(96c_3''-216c_4'+240c_5)u_0' \\ +(20c_3'''-60c_4''+120c_5'-120c_6+400c_3^2)u_0=0.$$

By comparing the coefficients a_3, a_4, a_5, a_6 of a Laguerre-Forsyth's canonical form of $\mathscr{R}$, we have

$$a_3=112c_3$$
$$a_4=168c_3'-216c_4$$
$$a_5=96c_3''-216c_4'+240c_5$$
$$a_6=20c_3'''-60c_4''+120c_5'-120c_6+400c_3^2$$

From these equalities, we have

$$c_3=\frac{1}{112}a_3=\frac{5}{56}\theta_3$$
$$c_4=-\frac{1}{216}(a_4-\frac{3}{2}a_3')=-\frac{5}{144}\theta_4$$
$$c_5=\frac{1}{240}(a_5-a_4'+\frac{9}{14}a_3'')=\frac{1}{80}\theta_5$$
$$c_6-\frac{10}{3}c_3^2=-\frac{1}{120}(a_6-\frac{1}{2}a_5'+\frac{2}{9}a_4''-\frac{1}{12}a_3''')=-\frac{1}{240}\theta_6.$$

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