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Operator $\Delta-aK$ on surfaces

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§ 1. Introduction

Let M be an oriented 2-dimensional complete non-compact Riemannian manifold. Let denote by $\Delta=\mathrm{trace}\,\nabla\nabla$ and K the laplacian and the Gauss curvature respectively. In this note, we assume that K does not vanish identically, and consider the operator $\Delta-aK$ acting on compactly supported function on M where a is a positive constant.

D. Fischer-Colbrie and R. Schoen [2] noted that the existence of a positive function f on M satisfying $\Delta f-qf=0$ is equivalent to the condition that the first eigenvalue of $\Delta-q$ be positive on each bounded domain in M where q is a function on M. This fact has many interesting applications to stable minimal immersions and some sort of surfaces of constant mean curvature.

They also showed the following fact: For every complete metric on the disc, there exists a number a_0 depending on the metric satisfying $0\leqq a_0<1$ so that for $a\leqq a_0$ there is a positive solution of $\Delta-aK$, and for $a>a_0$ there is no positive solution ([2] COROLLARY 2). They remarked that the value a_0 is $1/4$ for the Poincaré metric on the disc and that possible values of a_0 are not known for metrics of variable curvature.

Though not stated explicitly, it was proved in M. do CARMO and C. K. PENG [1] that $a_0\leqq 1/2$ for every complete metric on the disc. A. V. POGORELOV [4] proved the same result under the assumption $K\leqq 0$. He did not state this explicitly either.

We show in this note that $a_0\leqq 1/4$ for metrics of non-positive curvature.

THEOREM. *Let M be an oriented 2-dimensional complete non-compact Riemannian manifold of non-positive curvature $K\not\equiv 0$. Suppose that a is greater than $1/4$. Then there is no positive solution of $\Delta-aK$, i. e., there exists a function f with compact support which satisfies the inequality*

$$\int_M (|df|^2+aKf^2)*1<0.$$

We use the method of A. V. Pogorelov and choose a slightly different function f from that of [4].

As an application, we show that a theorem of M. J. MICALLEF [3] concerning stable degenerate minimal surfaces in $\boldsymbol{R}^4$ can be improved.

§ 2. Proof of the theorem

By the result of [2] mentioned in § 1, we can assume that M is simply connected. As in [4], we take a polar geodesic coordinate (u, v) for which the line element is $ds^2=du^2+g(u,v)^2dv^2$ where $g(u, v)$ is a positive function.

Let $l(\rho)$ denote the length of the boundary of the geodesic disc of radius ρ centered at the origin. Then one of the following two cases occurs ([4] p. 276).

(1) There exists a constant c with $l(\rho)/\rho \longrightarrow c(\rho \longrightarrow \infty)$.
(2) $l(\rho)/\rho \longrightarrow \infty(\rho \longrightarrow \infty)$.

In the first case, the proof is quite the same as that of [4]. In the second case, we consider a function f depending only on u with $f(0)=1$ and $f(u)=0$ for $u \geqq \rho$. Then we can rewrite the expression

$$\begin{aligned}\int_M(|df|^2+aKf^2)*1&=\int_0^{2\pi}\int_0^{\rho}[(df/du)^2+aKf^2]g\,dudv\\&=\int_0^{2\pi}\int_0^{\rho}[(df/du)^2g-a(\partial^2g/\partial u^2)f^2]\,dudv,\end{aligned}$$

because $K=-(\partial^2g/\partial u^2)/g$.

Integrating the second term by parts twice, and considering the facts $g(0,v)=0$, $\partial g/\partial u(0,v)=1$, $f(\rho)=0$ and $f(0)=1$, we have

$$(*)\quad \begin{aligned}\int_M(|df|^2+aKf^2)*1=2a\pi&-2a\int_0^{2\pi}\int_0^{\rho}f(d^2f/du^2)g\,dudv\\&+(1-2a)\int_0^{2\pi}\int_0^{\rho}(df/du)^2g\,dudv.\end{aligned}$$

Now we define a family of functions $f_{n,\rho}$ as follows:

$$f_{n,\rho}(u)=\begin{cases}(1-u/\rho)^n & (0\leqq u\leqq\rho)\\ 0 & (\rho\leqq u).\end{cases}$$

Then we have

$$\begin{aligned}&(df_{n,\rho}/du)^2=(n^2/\rho^2)(1-u/\rho)^{2n-2},\\&f_{n,\rho}(d^2f_{n,\rho}/du^2)=[n(n-1)/\rho^2](1-u/\rho)^{2n-2}.\end{aligned}$$

Hence the right hand side of $(*)$ is

$$2a\pi+(n/\rho^2)[2a+(1-4a)n]\int_0^{2\pi}\int_0^{\rho}(1-u/\rho)^{2n-2}g\,dudv.$$

Since $a>1/4$, we can choose a sufficiently large number n so that $2a+(1-4a)n<0$. To prove that the right hand side of $(*)$ is negative for some $f_{n,\rho}$, it suffices to show that

$$(1/\rho^2)\int_0^{2\pi}\int_0^{\rho}(1-u/\rho)^{2n-2}g\,dudv\longrightarrow\infty$$

as ρ tends to infinity. This quantity equals to

$$(1/\rho^2)\int_0^{\rho}[(1-v/\rho)^{2n-2}1(u)]du$$
$$=(1/\rho^2)\int_0^{\rho}(1-u/\rho)^{2n-2}u(1(u)/u)du.$$

Since $1(u)/u\longrightarrow\infty(u\longrightarrow\infty)$, for arbitrarily large N, there exists a number t so that $1(u)/u>N$ for every $u>t$. Hence for $\rho>t$, the above quantity is greater than

$$(N/\rho^2)\int_t^{\rho}(1-u/\rho)^{2n-2}u\,du$$
$$=(N/\rho^2)[t\rho(1-t/\rho)^{2n-1}/(2n-1)+\rho/(2n-1)\int_t^{\rho}(1-u/\rho)^{2n-1}du]$$
$$=-[tN(1-t/\rho)^{2n-1}]/[(2n-1)\rho]+[N(1-t/\rho^{2n}]/[2n(2n-1)].$$

When ρ tends to infinity, the first and the second terms tend to 0 and $N/2n(2n-1)$ respectively. This shows that the right hand side of $(*)$ is negative for some $f_{n,\rho}$. Approximating $f_{n,\rho}$ by smooth functions, we can complete the proof of the theorem.

§ 3. Application

M. J. Micallef [3] proved the following theorem.

THEOREM (Micallef). *Let* $F: M\longrightarrow \boldsymbol{R}^4$ *be an isometric stable minaimal immersion of a complete oriented surface M. If the Gauss map of F is at least* $1/3$ *degenerate (i. e., there exists a non-zero fixed vector* $A\in\boldsymbol{C}^4$ *such that* $|A\cdot A|\geqq(1/3)|A|^2$ *and* $A\cdot F_z\equiv 0$*), then the image of F is a plane.*

The proof roughly goes as follows: We take $(1,0)$-part of the normal component of A and denote it by s. Since $A\cdot F_z\equiv 0$ and $|A\cdot A|\geqq(1/3)|A|^2$, normal section s satisfies $D_{\bar z}s\equiv 0$ and is nowhere vanishing, where z is a holomorphic coordinate and D denotes the normal connection. The stability of F implies the following inequality for every real valued function h of compact support.

$$(**)\quad \int_M|dh|^2*1\geqq\int_M h^2[(-K)+q]*1,$$

where

$$q=[|(F_{zz})^{1,0}|^2/|F_z|^4|s|^2][|A\cdot A|-(1/2)|A|^2+(1/2)|s|^2+(1/8)|A\cdot A|^2/|s|^2].$$

Concerning this inequality,see [3] p. 77-p. 78.

If $|A\cdot A|\geqq(1/3)|A|^2$, then we have

$$q\geqq[|(F_{zz})^{1,0}|^2/|F_z|^4|s|^2][-(1/6)|A|^2+(1/2)|s|^2+(1/8)|A\cdot A|^2/|s|^2]$$
$$=(1/2)[|(F_{zz})^{1,0}|2/|F_z|^4|s|^4][|s|^2-(1/6)|A|^2]^2\geqq 0.$$

Hence some argument using the result of [2] shows that K vanishes identically.

We can weaken the assumption $|A\cdot A|\geqq(1/3)|A|^2$ by our previous result. The inequality (**) is written as follows:

$$\int_M |dh|^2 * 1\geqq\int_M h^2[(1/4+\varepsilon)(-K)+q'] * 1,$$

where

$$q'=q+(3/4-\varepsilon)(-K)$$
$$\geqq[|F_{zz})^{1,0}|^2/|F_z|^4|s|^2][|A\cdot A|-(1/2)|A|^2+(5/4-\varepsilon)|s|^2+(1/8)|A\cdot A|^2/|s|^2]$$

for any positive number ε. Here we used the relation

$$-K=|(F_{zz})^N|^2/|F_z|^4\geqq|(F_{zz})^{1,0}|^2/|F_z|^4.$$

If $|A\cdot A|\geqq\{(4-\sqrt{10})/3\}|A|^2$, then q' is non-negative. Hence the same argument as [3] and our theorem give the conclusion $K\equiv 0$.

References

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