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On basic properties of convex functions and convex integrands

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Introduction.

This paper is devoted to proving some fundamental results and generalized formulas in convex analysis. In Chapter 1, we will consider some properties of pointwise convergent sequences of convex functions from the viewpoint of convergences of infima. In [3], Heinz könig proved that inf $f_n \longrightarrow \inf f$ as $n \longrightarrow \infty$ with some conditions where the sequence $\{f_n\}$ converges decreasingly to f, and each f_n and f belong to a certain class of convex functions. In addition, the τ-convergence of convex functions was considered in [6], [7], and [13], and the equivalence between the τ-convergence of $\{f_n\}$ and that of $\{f_n{}^*\}$ was derived with some hypotheses. In this paper, we deal with only the pointwise convergence, and prove some results about the convergence of inf f_n and that of $\{f_n{}^*\}$ with natural hypotheses.

In Chapter 2, we deal with a type of convex integrands, and derive some formulas for them. In our preceding paper [4], we considered convex operators $\tilde{f} : \boldsymbol{R} \supset D(\tilde{f}) \longrightarrow S(\Omega)$ where $\boldsymbol{R}$ is the space of real numbers and $S(\Omega)$ is the linear space of finite valued measurable functions on a finite measure space Ω. We proved that $\tilde{f}$ is represented by a convex integrand $f(\cdot, \cdot) : \boldsymbol{R} \times \Omega \longrightarrow \boldsymbol{R} \cup \{+\infty\}$ such that $(\tilde{f}(a))(\cdot) = f(a, \cdot)$ in $L^1(\Omega)$ for every $a \in \boldsymbol{R}$. Moreover, we proved that $(\tilde{f}^*(\xi))(\cdot) = f^*(\xi, \cdot)$ in $L^1(\Omega)$ for every $\xi \in \boldsymbol{R}$ where $\tilde{f}^*$ in the conjugate operator of $\tilde{f}$ and each $f^*(\cdot, t)$ is the conjugate function of $f(\cdot, t)$. Now, we are interested in further properties of $\tilde{f}$ when the range of $\tilde{f}$ is contained in $L^1(\Omega)$. From this viewpoint, we will define a class of convex integrands which represent some convex operators from convex subsets of $\boldsymbol{R}^d$ to $L^1(\Omega)$. The aim of this chapter is to prove some fundamental formulas which are valid in such a class of convex integrands. Some of our results are considered to be extensions of the following well-known formulas:

$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$ ([1], Theorem I-28)

$(f_1 \nabla f_2)^* = f_1{}^* + f_2{}^*$ ([1], Proposition I-19) where f_1 and f_2 are convex functions, and $f_1 \nabla f_2$ is the infimum convolution of f_1 and f_2. General studies of convex integrands are given in [1], [8], [11], and some other articles. We will use some of them to apply the theory of normality of convex integrands.

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Chapter 1. Pointwise convergent sequences of convex functions.

Let $f,\ f_n : X \longrightarrow \boldsymbol{R} \cup \{+\infty\}$ be convex functions where X is a real vector space. The sequence $\{f_n\}$ is said to *converge pointwise* to f on X, if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$. We denote the effective domain of f by $D(f)$, i. e.,

$$D(f)=\{x \in X \mid f(x) \text{ is finite}\}.$$

In addition, we adopt the following notations in this paper. For a subset C of a topological space, we write $\mathring{C}$ for the set of all interior points of C. For a sequence $\{a_n\} \subset \boldsymbol{R}$, we denote the upper limit and the lower limit of $\{a_n\}$ by $\overline{\lim}_{n\to\infty} a_n$ and $\underline{\lim}_{n\to\infty} a_n$ respectively, i. e., $\overline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} \sup_{n \leqq i} a_i$ and $\underline{\lim}_{n\to\infty} a_n = \lim_{n\to\infty} \inf_{n \leqq i} a_i$.

§ 1. 1

THEOREM 1. 1. *Let X be a real Banach space, and let $f, f_n : X \longrightarrow \boldsymbol{R} \cup \{+\infty\}$ be lower-semicontinuous convex functions such that $\{f_n\}$ converges pointwise to f on X. Suppose that $D(f)$ has nonempty interior and let $K \subset D^\circ(f)$ be a compact set. Then f_n converges uniformly to f on K.*

RERARK. Through this chapter, we do not assume any relation between $D(f)$ and each $D(f_n)$.

PROOF. For each $n=1, 2, \cdots$, we put

$$F_n(x)=\sup_{n \leqq i} f_i(x).$$

We will prove that F_n is locally bounded on K for some n. We define the level sets of f_i as follows:

$$L_m(f_i)=\{x \in D(f_i) \mid f_i(x) \leqq m\},$$

where $m=1, 2, \cdots,\ i=1, 2, \cdots$.

Since each f_i is lower-semicontinuous, $L_m(f_i)$ is a closed convex set which is possibly empty. Let x be an interior point of $D(f)$, and let V be a closed convex symmetric neighborhood of x such that $V \subset D(f)$.

We consider the sets $A_{n,m}$ defined by:

$$A_{n,m} = \bigcap_{n \leqq i}^{\infty} (L_m(f_i) \cap V)$$

where $n=1, 2, \cdots, m=1, 2, \cdots$.

Each $A_{n,m}$ is a closed covex set, and since $\{f_i(x)\}$ converges to $f(x)$, $A_{n,m}$ is nonempty for sufficiently large n and m. Moreover, $A_{n,m} \subset A_{n',m'}$ holds whenever $n \leqq n'$ and $m \leqq m'$. It is easy to see that

$$D(F_n) \cap V = \bigcup_{m=1}^{\infty} A_{n,m}$$

$$\bigcup_{n=1}^{\infty} (D(F_n) \cap V) = V.$$

Hence We have

$$V = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} A_{n,m}.$$

By the Baire's theorem, there exist indices n_0 and m_0 such that A_{n_0,m_0} has an interior point u. If we put

$$v = 2x - u,$$

then the neighborhood V includes v, because V is symmetric. Hence there exist indices n_1 and m_1 such that $A_{n_1,m_1} \ni v$. Thus we have

$$A_{n_2,m_2} \ni v, \quad \mathring{A}_{n_2,m_2} \ni u$$

where $n_2 = \max\{n_0, n_1\}$, $m_2 = \max\{m_0, m_1\}$.

Since A_{n_2,m_2} is convex,

$$x = \frac{1}{2}(u+v) \in \mathring{A}_{n_2,m_2},$$

and this implies that there exists a neighborhood U_x of x such that $f_i(y) \leqq m_2$ holds for any $i \geqq n_2$ and $y \in U_x$. In other words, F_n is bounded on U_x whenever $n \geqq n_2$. Now we assume that $\{f_n\}$ does not converge uniformly to f on K. Then there exist $\varepsilon > 0$ and a sequence $\{x_n\} \subset K$ such that

$$|f_n(x_n) - f(x_n)| \geqq \varepsilon$$

for $n=1, 2, \cdots$. By taking a subsequence, we can assume that $\{x_n\}$ tends to a limit point: $x_0 \in K$. We note that $x_0 \in D^\circ(f)$, and that f is continuous at x_0. (cf. [10] p. 31) Hence there exists an index n' such that

$$|f(x_0)-f(x_n)|\leqq\frac{\varepsilon}{3}$$

$$|f_n(x_0)-f(x_0)|\leqq\frac{\varepsilon}{3},$$

for $n\geqq n'$. Hence for any $n\geqq n'$, we have

$$\begin{aligned}|f_n(x_n)-f_n(x_0)|&\geqq|f_n(x_n)-f(x_n)|-|f_n(x_0)-f(x_n)|\\&\geqq\varepsilon-(|f(x_0)-f(x_n)|+|f_n(x_0)-f(x_0)|)\\&\geqq\varepsilon-\left(\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\right)=\frac{\varepsilon}{3}.\end{aligned}$$

Using the convexity of each f_n, we can easily see that this inequality implies the unboundedness of F_n on any neighborhood of x_0. This contradicts the previous argument, and hence the theorem is proved.

§ 1.2

In Theorem 1.1, the assumption that K is contained in the interior of $D(f)$ is essential. The following theorem gives a similar result without this assumption in the case where X is a finite dimensional Euclidean space $\boldsymbol{R}^d$. In the proof, we will use the fact that every l.s.c. (lower-semicontinuous) convex function f defined on $\boldsymbol{R}^d$ is continuous on $D(f)$ if $d=1$. At the end of this section, we will give an example which shows that this statement is no longer valid if $d\geqq2$.

THEOREM 1.2. *Let f_n, $f:\boldsymbol{R}^d\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ be convex functions such that $\{f_n\}$ converges pointwise to f on $\boldsymbol{R}^d$. If $K\subset D(f)$ is a compact convex set, then*

$$\inf_{x\in K}f_n(x)\longrightarrow\inf_{x\in K}f(x),$$

as $n\longrightarrow\infty$.

REMARK. If f and $\{f_n\}$ are l.s.c. and $D(f)$ has nonempty interior, the conclusion of Theorem 1.2 holds true for any compact K that is not necessarily convex. In fact, $D^\circ(f)\neq\phi$ implies $f_n\stackrel{\tau}{\longrightarrow}f$ (τ-convergence) (see [13], Cor. 2C), and the τ-convergence implies the equi-lower semicontinuity of $\{f_n\}$ ([13], Lemma 3), from which $\inf_K f_n\longrightarrow\inf_K f$ follows immediately.

PROOF. By taking the restriction of f and f_n to the affine hull of K, we can assume that K has nonempty interior. It is easy to see that

$$\overline{\lim_{n\to\infty}}(\inf_{x\in K}f_n(x))\leqq\inf_{x\in K}f(x).$$

Hense if the conclusion is not true, there exist a sequence $\{x_n\}\subset K$ and $\varepsilon>0$ such that

$$f_n(x_n)\leqq \inf_{x\in K} f(x)-\varepsilon.$$

Since K is compact, we can assume that $[x_n\}$ tends to a point $x_0\in K$. Theorem 1.1 asserts that x_0 does not belong to $\mathring{K}\subset D^\circ(f)$. We take intervals $I_1, I_2, \cdots, I_d\subset K$ such that each of them starts from x_0 and that the convex hull of them, co $\{I_1, \cdots, I_d\}$, contains an interior point of K. Let $\bar{f}$ be the closure of f which is defined by the relation $\text{epi}(\bar{f})=\overline{\text{epi}(f)}$ where $\text{epi}(f)=\{(x, a)\in \boldsymbol{R}^{d+1}|f(x)\leqq a\}$ and $\overline{\text{epi}(f)}$ is its closure. ([1], p. 2) Since $\bar{f}$ is continuous on each interval I_i $(i=1, 2, \cdots, d)$ and l. s. c. on K, there exists $r>0$ satisfying the following (a) and (b).

(a) $$\bar{f}(x)\leqq \bar{f}(x_0)+\frac{\varepsilon}{4}$$

for all $x\in I_i(r)=\{x\in I_i|\ \|x-x_0\|\leqq r\}$ $(i=1, 2, \cdots, d)$,

(b) $$\bar{f}(x)\geqq \bar{f}(x_0)-\frac{\varepsilon}{4}$$

for all $x\in L(r)=\text{co}\{I_1(r), \cdots, I_d(r)\}$. By the convexity of f, (a) implies that

$$\bar{f}(x)\leqq \bar{f}(x_0)+\frac{\varepsilon}{4}$$

for all $x\in L(r)$. We take two points $y, z\in L^\circ(r)$ such that $z=\frac{1}{2}(x_0+y)$. For simplicity, we assume that $z=0$ and $y=-x_0$ without losing generality. Since $\bar{f}=f$ on $L^\circ(r)$, it follows from (a) and (b) that

$$\begin{aligned}
f_n(-x_n)&\geqq -f_n(x_n)+2f_n(0)\\
&\geqq \varepsilon-\inf_{x\in K} f(x)+2f_n(0)\\
&\longrightarrow \varepsilon-\inf_{x\in K} f(x)+2f(0)\\
&\geqq \varepsilon-\bar{f}(x_0)+2\left(\bar{f}(x_0)-\frac{\varepsilon}{4}\right)\\
&\geqq \bar{f}(x_0)+\frac{\varepsilon}{2}\\
&\geqq \sup_{x\in L(r)} \bar{f}(x)-\frac{\varepsilon}{4}+\frac{\varepsilon}{2}.
\end{aligned}$$

Consequently,

$$\varlimsup_{n\to\infty} f_n(-x)_n \geqq \sup_{x\in L(r)} \bar{f}(x)+\frac{\varepsilon}{4}.$$

Since $-x_n \longrightarrow -x_0 \in L^\circ(r)$ and $\bar{f}=f$ on $L^\circ(r)$, this implies that $\{f_n\}$ does not converge uniformly to f on any neighborhood of $-x_0$. This contradicts the assertion of Theorem 1.1.

EXAMPLE 1. The following example shows that $\{f_n\}$ does not always converge uniformly to f on K under the conditions in Theorem 1.2. For every $t>0$, we define convex functions f and $f_t : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}\cup\{+\infty\}$ as follows:

$$\begin{aligned}
&D(f)=D(f_t)=\{(x, y)\in \boldsymbol{R}^2 | y \geqq x^2\}\,; \\
&f(x, y)=\frac{x^2}{y} && (\text{if } y\neq 0)\,; \\
&f(0, 0)=0\,; \\
&f_t(x, y)=\frac{(|x|-t)^2}{y-2t|x|+t^2} && (\text{if } |x|>t)\,; \\
&f_t(t, t^2)=0\,; \\
&f_t(x, y)=0 && (\text{if } |x|\leqq t).
\end{aligned}$$

One can easily check that f and f_t are midpoint convex and continuous on $D^\circ(f)$. Hence they are convex functions. Although f and f_t are l. s. c. on $D(f)$, f is not continuous at $(0, 0)$. In fact, $f(x, y)=1$ whenever $y=x^2$ and $y\neq 0$. It is easy to see that $\{f_t\}$ converges pointwise to f as $t\longrightarrow 0$. However,

$$f(t, t^2)-f_t(t, t^2)=1-0=1$$

holds for every $t>0$, and this implies that the convergence of $\{f_t\}$ is not uniform on any neighborhood of $(0, 0)$. Next we define a compact set $K=\{(x, y) | y=x^2, 0\leqq x\leqq 1\}$, and replace the values of $f(0, 0)$ and $f_t(0, 0)$ by 1. Then we have

$$\inf_{(x,y)\in K} f_t(x, y)=0\neq \inf_{(x,y)\in K} f(x, y)$$

for every $t>0$. This fact shows that the conclusion of Theorem 1.2 is not valid if we do not assume the convexity of K.

§ 1.3.

Let $f : \boldsymbol{R}^d \longrightarrow \boldsymbol{R}\cup\{+\infty\}$ be a proper convex function, i. e, $D(f)\neq\phi$. For $\xi\in\boldsymbol{R}^d$, we define the *conjugate function* f^* of f as follows:

$$f^*(\xi)=\sup_{x\in D(f)} (\langle \xi, x\rangle - f(x)),$$

$$D(f^*)=\{\xi\in\boldsymbol{R}^d|\sup_{x\in D(f)}(\langle\xi, x\rangle-f(x))<+\infty\}.$$

In [5], U. Mosco proved the equivalence between a type of convergence of $\{f_n\}$ and that of $\{f_n^*\}$. However, pointwise convergence of $\{f_n\}$ does not imply the convergence of $\{f_n^*\}$ in general. The following theorem shows that this implication is partly true in finite dimensional cases.

THEOREM 1.3. *Let $f,\ f_n:\boldsymbol{R}^d\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ be convex functions such that $\{f_n\}$ converges pointwise to f on $\boldsymbol{R}^d$. If $D(f)$ has nonempty interior, then*

$$f_n^*(\xi)\longrightarrow f^*(\xi)$$

holds whenever $\xi\in D^\circ(f^*)$, or $\xi\notin D(f^*)$.

LEMMA 1.4. *Let $f:\boldsymbol{R}^d\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ be a l. s. c. proper convex function, and define a set :*

$$K_\delta=\{x\in D(f)|f(x)-\langle\xi, x\rangle\leqq-f^*(\xi)+\delta\}$$

for a positive number δ. If $\xi\in D^\circ(f^*)$, then K_δ is a nonempty compact convex set.

PROOF. Since f is a lower-semicontinuous convex function, K_δ is a closed convex set. It follows from the definition of f^* that K_δ is not empty for any $\delta>0$. Therefore the only thing we nust prove is the boundedness of K_δ. Suppose that K_δ is unbounded for some $\delta>0$. Then there exists $\eta\in\boldsymbol{R}^d$ such that $\sigma(K_\delta;\eta)=\infty$ where $\sigma(K_\delta;\cdot)$ is the *supporting functional* of K_δ defined by

$$\sigma(K_\delta;\eta)=\sup\{\langle\eta, x\rangle|x\in K_\delta\}.$$

Hence for all $r>0$,

$$\begin{aligned}f^*(\xi+r\cdot\eta)&=\sup_{x\in D(f)}\{\langle\xi+r\cdot\eta, x\rangle-f(x)\}\\&\geqq\sup_{x\in K_\delta}\{\langle\xi, x\rangle-f(x)+r\langle\eta, x\rangle\}\\&\geqq\sup_{x\in K_\delta}\{f^*(\xi)-\delta+r\langle\eta, x\rangle\}=\infty.\end{aligned}$$

This implies that ξ does not belong to the interior of $D(f^*)$, and the lemma is proved.

PROOF of THEOREM 1.3. In case when $f^*(\xi)=\infty$, it is easy to see that $f_n^*(\xi)\longrightarrow\infty$. Therefore we consider only the case when ξ is an interior point of $D(f^*)$. By taking the convex function $g(x)=f(x)-\langle\xi, x\rangle$, we can assume that $\xi=0$ without losing generality. In other words, it suffices to show that

$$\inf_{x\in D(f_n)} f_n(x) \longrightarrow \inf_{x\in D(f)} f(x)$$

under the condition that $D^\circ(f)\neq\phi$ and that $0\in D^\circ(f^*)$. For simplicity, we denote $\inf_{x\in D(f_n)} f_n(x)$ and $\inf_{x\in D(f)} f(x)$ by $\inf f_n$ and $\inf f$ respectively. It is easy to see that

$$\overline{\lim_{n\to\infty}}\{\inf f_n\}\leqq\inf f.$$

Hence, if $\inf f_n$ does not converge to $\inf f$, there exist $\varepsilon>0$ and a sequence $\{x_n\}\subset \boldsymbol{R}^d$ such that

$$f_n(x_n)<\inf f-\varepsilon$$

for $n=1, 2, \cdots$. By virtue of Theorem 1.1, we can see that the sequence $\{x_n\}$ does not have a cluster point in the interior of $D(f)$. Hence the following cases are possible.

Case 1: $\{x_n\}$ has a cluster point x_0 which belongs to the boundary of $D(f)$.

Case 2: $\{x_n\}$ has a cluster point x_0 which is an exterior point of $D(f)$.

Case 3: $\{x_n\}$ has no cluster point.

We will derive a contradiction to the assertion of Theorem 1.1 in each case. Let $u\in D^\circ(f)$ be such that

$$f(u)\leqq\inf f+\frac{\varepsilon}{2}.$$

The assumption that $D(f)$ has nonempty interior guarantees the existence of such u in the interior of $D(f)$.

In case 1, we can assume that $\{x_n\}$ converges to a point x_0 of the boundary of $D(f)$. It follows that

$$\begin{aligned} f_n\left(\frac{1}{2}(u+x_n)\right) &\leqq\frac{1}{2}(f_n(u)+f_n(x_n)) \\ &<\frac{1}{2}(f_n(u)+\inf f-\varepsilon) \\ &\longrightarrow\frac{1}{2}(f(u)+\inf f-\varepsilon) \\ &\leqq\frac{1}{2}\left(\inf f+\frac{\varepsilon}{2}+\inf f-\varepsilon\right) \\ &=\inf f-\frac{\varepsilon}{4}. \end{aligned}$$

Hence we have

$$\overline{\lim_{n\to\infty}} f_n\left(\frac{1}{2}(u+x_n)\right) \leqq \inf f - \frac{\varepsilon}{4}$$
$$\leqq f\left(\frac{1}{2}(u+x_0)\right) - \frac{\varepsilon}{4}.$$

Since $\frac{1}{2}(u+x_n) \longrightarrow \frac{1}{2}(u+x_0) \in D^\circ(f)$ and f is continuous at $\frac{1}{2}(u+x_0)$, this implies that $\{f_n\}$ does not converge uniformly to f on any neighborhood of $\frac{1}{2}(u+x_0)$. This contradicts the assertion of Theorem 1. 1, and we obtain that the case 1 is impossible.

In case 2, we similarly assume that $\{x_n\}$ converges to an exterior point x_0 of $D(f)$. Then there exists $0<\lambda<1$ such that $y_0=\lambda\cdot u+(1-\lambda)x_0$ is also an exterior point of $D(f)$. Let $\{u_n\}$ be a sequence such that

$$y_0=\lambda\cdot u_n+(1-\lambda)x_n$$

for $n=1, 2, \cdots$. Then $\{u_n\}$ tends to u and

$$f_n(y_0) \leqq \lambda\cdot f_n(u_n)+(1-\lambda)\cdot f_n(x_n)$$
$$< \lambda\cdot f_n(u_n)+(1-\lambda)(\inf f-\varepsilon)$$

Since $f_n(y_0) \longrightarrow \infty$, this inequality shows that $f_n(u_n) \longrightarrow \infty$. Thus we obtain the same contradiction as in the case 1.

In case 3, we will apply Lemma 1. 4 to the closure $\bar{f}$ of f. Since we are assuming $\xi=0$, K_δ in Lemma 1. 4 is the set of all $x \in D(f)$ such that $\bar{f}(x) \leqq \inf \bar{f}+\delta=\inf f+\delta$. Therefore, from Lemma 1. 4, there exists $r>0$ such that

$$\bar{f}(x) \geqq \inf f+2\varepsilon$$

holds whenever $\|x-u\| \geqq r$. Since the sequence $\{x_n\}$ is unbounded, we can assume that $\|u-x_n\| \longrightarrow \infty$. Hence each interval $[u, x_n]$ includes a point y_n such that $\|u-y_n\|=r$, where

$$[u, x_n]=\{\lambda\cdot u+(1-\lambda)\cdot x_n | 0 \leqq \lambda \leqq 1\}.$$

From the convexity of each f_n, it follows that

$$f_n(y_n) \leqq \text{Max } \{f_n(u), f_n(x_n)\}$$
$$=f_n(u)$$
$$\longrightarrow f(u)$$
$$\leqq \inf f+\frac{\varepsilon}{2}. \qquad (1, 1)$$

By taking a subsequence of the bounded sequence $\{y_n\}$, we can assume that $\{y_n\}$ tends to a point y_0. Since $\|u-y_0\|=r$, it follows that

$$\bar{f}(y_0)\geqq\inf f+2\varepsilon. \tag{1, 2}$$

If y_0 belongs to the interior of $D(f)$, then the estimations (1, 1) and (1, 2) yield a contradiction to Theorem 1. 1. Moreover, if y_0 is an exterior point of $D(f)$, we can also obtain a contradiction in the same way as in the case 2. Hence we suppose that y_0 belongs to the boundary of $D(f)$. From the lower-semicontinuity of $\bar{f}$ and (1, 2), we can choose $v=\lambda\cdot u+(1-\lambda)\cdot y_0$ such that $0<\lambda<1$, and

$$\bar{f}(v)\geqq\inf f+\varepsilon. \tag{1, 3}$$

On the other hand,

$$\begin{aligned} f_n(\lambda\cdot u+(1-\lambda)\cdot y_n) &\leqq\lambda\cdot f_n(u)+(1-\lambda)\cdot f_n(y_n) \\ &\leqq\lambda\cdot f_n(u)+(1-\lambda)\cdot f_n(u) \\ &=f_n(u) \\ &\longrightarrow f(u) \\ &\leqq\inf f+\frac{\varepsilon}{2}. \end{aligned} \tag{1, 4}$$

Since $\bar{f}(v)=f(v)$, it follows from (1, 3) and (1, 4) that

$$\varliminf_{n\to\infty} f_n(\lambda\cdot u+(1-\lambda)\cdot y_n)\leqq f(v)-\frac{\varepsilon}{2}.$$

Since $\lambda\cdot u+(1-\lambda)\cdot y_n\longrightarrow v$ and v is an interior point of $D(f)$, this contradicts Theorem 1. 1. Consequently, the case 3 is impossible, and the theorem has been proved.

Chapter 2. Fundamental properties of a class of convex integrands.

DEFINITIONS. Let (Ω, μ) be a probability space. A function $f:\boldsymbol{R}^d\times\Omega\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ is called a *convex integrand* if $f(\cdot, t)$ is a proper convex function for every $t\in\Omega$ and if further $f(a, \cdot)$ is measurable for every $a\in\boldsymbol{R}^d$. We will consider the integral of the form

$$F(a)=\int_\Omega f(a, t)d\mu(t)$$

for a convex integrand $f(\cdot, \cdot)$. The function $F(\cdot)$ is obviously a convex function. A convex integrand $f(\cdot, \cdot)$ is said to represent a convex operator $\tilde{f}:\boldsymbol{R}^d\supset D(\tilde{f})\longrightarrow L^1(\Omega)$ if $(\tilde{f}(a))(\cdot)=f(a, \cdot)$ in $L^1(\Omega)$ for every $a\in D(\tilde{f})$. For such a convex integrand $f(\cdot, \cdot)$, it is natural to assume the following condition (A′).

(A′) $D(f(\cdot, t))=D(F)\neq\phi$

for almost every $t\in\Omega$, where $D(f(\cdot, t))$ is the effective domain of $f(\cdot, t)$. The condition (A′) implies that the function $f(a, \cdot)$ is identically $+\infty$ on Ω (in the a. e. sense), whenever $a\notin D(F)$. In this paper, we adopt a slightly weaker assumption for convex integrands as follows. We will say that a convex integrand $f(\cdot, \cdot)$ satisfies the condition (A) if

(A) $\overline{D(f(\cdot, t))}=\overline{D(F)}\neq\phi$

for almost every $t\in\Omega$, where $\overline{D(f(\cdot, t))}$ and $\overline{D(F)}$ are the closure of $D(f(\cdot, t))$ and $D(F)$ respectively. For a convex subset $C\subset\boldsymbol{R}^d$, $ri\ C$ denotes the relative interior of C which is the interior of C with respect to the relative topology of the affine hull of C. It is easy to see that a convex integrand $f(\cdot, \cdot)$ satisfies the condition (A) if and only if $ri\ D(f(\cdot, t))=ri\ D(F)\neq\phi$. Moreover, F cannot take $-\infty$ anywhere under the condition (A).

§ 2. 1

In this section, we will give a proof of a measurable selection theorem for the subdifferentials of convex integrands. For a multifunction T: $\Omega\longrightarrow 2^{\boldsymbol{R}^d}$, $\int_\Omega T(t)\,d\mu(t)$ denotes the set of all the integrals of summable selectors of T, i. e.,

$$\int_\Omega T(t)\,d\mu(t)=\left\{\int_\Omega \zeta(t)\,d\mu(t) \,\middle|\, \zeta:\Omega\longrightarrow\boldsymbol{R}^d \text{ is summable and } \zeta(t)\in T(t) \text{ for almost every } t\in\Omega\right\}.$$

THEOREM 2. 1. *Let $f(\cdot, \cdot)$ be a convex integrand with the condition* (A). *For every $a\in D(F)$ with $\partial F(a)\neq\phi$, we have*

$$\partial F(a)=\int_\Omega \partial f(a, t)\,d\mu(t)$$

where $\partial f(a, t)$ is the subdifferential of $f(\cdot, t)$ at a.

REMARK. If we remove the condition (A), this formula is not valid. We will show a counterexample at the end of this chapter.

We will prepare some lemmas and notations for our proof of this theorem. For $x\in\boldsymbol{R}^d$ with $x\neq 0$, we denote the orthogonal space of x by N_x, i. e.,

$$N_x=\{y\in \boldsymbol{R}^d|\langle x, y\rangle=0\}.$$

N_x can be identified with $\boldsymbol{R}^{d-1}$. Moreover, we represent every $z\in\boldsymbol{R}^d$ by the form

$$z=(z_1, z_2)_x$$

where $z_1\in\boldsymbol{R}$, $z_2\in N_x\simeq\boldsymbol{R}^{d-1}$, and $z=\frac{z_1}{\|x\|}x+z_2$. We can regard z_2 as an element of $\boldsymbol{R}^{d-1}$. Now a function $G:\boldsymbol{R}^d\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ is said to be a *sublinear function* if

$$G(x+y)\leqq G(x)+G(y)$$
$$G(\lambda x)=\lambda G(x)$$

for every $x, y\in\boldsymbol{R}^d$ and $\lambda>0$.

LEMMA 2.2. *Let $G:\boldsymbol{R}^d\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ be a sublinear function, and let $x\neq 0$ be a fixed element of $\boldsymbol{R}^d$. We define $\bar{G}:\boldsymbol{R}^{d-1}\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ by $\bar{G}(y)=G(\|x\|, y)_x:=G((\|x\|, y)_x)$. Then*

$$\left(\frac{G(x)}{\|x\|}, \partial\bar{G}(0)\right)_x=\partial G(x)\subset\partial G(0).$$

PROOF. Let η be an arbitrary element of $\partial\bar{G}(0)$. For $z=(z_1, z_2)_x\in\boldsymbol{R}^d$, we put

$$\begin{aligned}G_1(z_1, z_2)_x&=G(z)-G(x)-\left\langle\left(\frac{G(x)}{\|x\|}, \eta\right)_x, z-x\right\rangle\\&=G(z_1, z_2)_x-G(\|x\|, 0)_x-\left\langle\left(\frac{G(x)}{\|x\|}, \eta\right)_x, (z_1-\|x\|, z_2)_x\right\rangle\\&=F(z_1, z_2)_x-\frac{G(x)}{\|x\|}z_1-\langle\eta, z_2\rangle.\end{aligned}$$

It suffices to show that $G_1(z_1, z_2)_x\geqq 0$ for any $z\in\boldsymbol{R}^d$. G_1 is clearly a sublinear function and

$$G_1(x)=G_1(\|x\|, 0)_x=0.$$

Moreover, for every $z_2\in\boldsymbol{R}^{d-1}$,

$$\begin{aligned}G_1(\|x\|, z_2)_x&=G(\|x\|, z_2)_x-G(\|x\|, 0)_x-\langle\eta, z_2\rangle\\&=\bar{G}(z_2)-\bar{G}(0)-\langle\eta, z_2-0\rangle\\&\geqq 0.\end{aligned}$$

Hence, by the following lemma, we obtain $G_1(z_1, z_2)_x\geqq 0$, and this completes the proof.

LEMMA 2.3. *Let* $G:\boldsymbol{R}^d\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ *be a sublinear function, and let* $x\neq 0$, N_x *be as above. If* $G(x)=0$, *and* $G(x+u)\geqq 0$ *for every* $u\in N_x$, *then* $G(y)\geqq 0$ *for every* $y\in\boldsymbol{R}^d$.

PROOF. We put $y=\lambda\cdot x+u$, where $\lambda\in\boldsymbol{R}$ and $u\in N_x$. If $\lambda>0$, then

$$G(y)=\lambda\cdot G\left(\frac{1}{\lambda}y\right)=\lambda\cdot G\left(x+\frac{1}{\lambda}u\right)\geqq 0.$$

If $\lambda\leqq 0$, then

$$\begin{aligned}G(y)&=G(\lambda\cdot x+u)\\ &\geqq 2G(x+u)-G((2-\lambda)x+u)\\ &=2G(x+u)-(2-\lambda)G\left(x+\frac{1}{2-\lambda}u\right)\\ &\geqq 2G(x+u)-(2-\lambda)\left(\frac{1-\lambda}{2-\lambda}G(x)+\frac{1}{2-\lambda}G(x+u)\right)\\ &=2G(x+u)-(1-\lambda)G(x)-G(x+u)\\ &=G(x+u)\geqq 0.\end{aligned}$$

Thus the lemma is proved.

We will prepare two more lemmas which are useful in general. The following one gives the definition of recession functions of l. s. c. convex functions.

LEMMA 2.4. *Let* $f:\boldsymbol{R}^d\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ *be a l. s. c. convex function. For* $x_0\in D(f)$, *wefine*

$$f_\infty(x_0\,;x)=\sup_{h>0}\frac{1}{h}(f(x_0+h\cdot x)-f(x_0))$$

Then $f_\infty(x_0\,;\cdot)$ *is a l. s. c. sublinear function, and this does not depend on the choice of* $x_0\in D(f)$, *i. e.,*

$$f_\infty(x_0\,;x)=f_\infty(x_1\,;x)$$

for every x_0, $x_1\in D(f)$, *and* $x\in\boldsymbol{R}^d$.

PROOF. It is easy to see that $f_\infty(x_0\,;\cdot)$ is l. s. c. and sublinear. We will prove the last statenent. By the monotonicity of $\frac{1}{h}(f(x_0+h\cdot x)-f(x_0))$ in h, it follows that

$$f_\infty(x_0\,;x)=\lim_{n\to\infty}\frac{1}{h}(f(x_0+h\cdot x)-f(x_0))$$

For every x_0, $x_1\in D(f)$, and $h>0$, it follows from the lower-semicontinuity of f that, for every $x \in \boldsymbol{R}^d$,

$$\begin{aligned}
\frac{1}{h}(f(x_0+h\cdot x)-f(x_0)) &\leqq \lim_{\lambda\to 0}\frac{1}{h}(f((1-\lambda)x_0+\lambda x_1+h\cdot x)-f(x_0)) \\
&=\lim_{\lambda\to 0}\frac{1}{h}(f((1-\lambda)x_0+\lambda(x_1+\frac{h}{\lambda}x))-f(x_0)) \\
&\leqq \lim_{\lambda\to 0}\frac{1}{h}\{(1-\lambda)f(x_0)+\lambda\cdot f\Big(x_1+\frac{h}{\lambda}x\Big)-f(x_0)\} \\
&=\frac{1}{h}\lim_{\lambda\to 0}\lambda\cdot f\Big(x_1+\frac{h}{\lambda}x\Big) \\
&=\frac{1}{h}\lim_{\lambda\to 0}\lambda\{f\Big(x_1+\frac{h}{\lambda}x\Big)-f(x_1) \\
&=\frac{1}{h}f_\infty(x_1\,;h\cdot x) \\
&=f_\infty(x_1\,;x).
\end{aligned}$$

This implies that $f_\infty(x_0\,;x)\leqq f_\infty(x_1\,;x)$ for every $x\in\boldsymbol{R}^d$. Similarly, we can get that $f_\infty(x_1\,;x)\leqq f_\infty(x_0\,;x)$ for every $x\in\boldsymbol{R}^d$, and this completes the proof.

The function $f_\infty(x_0\,;x)=f_\infty(x)$ in called the *recession function* of f. We will use this in proving the following lemma. The well known formula: $\sigma(\partial f(a)\,;x)=f'(a\,;x)$ is not always true for a boundary point a of $D(f)$, where $\sigma(A\,;x)$ denotes the supporting functional defined by

$$\sigma(A\,;x)=\sup_{\xi\in A}\langle\xi, x\rangle\,;$$

and

$$f'(a\,;x)=\lim_{h\to 0}\frac{1}{h}(f(a+h\cdot x)-f(a)).$$

The following lemma shows that the formula is true whenever $f'(a\,;\cdot)$ is l. s. c. (To see this, let $G(x)=f'(a\,;x)$ and use the fact that $\partial f'(a\,;\cdot)(0)=\partial f(a)$.)

LEMMA 2.5. *Let* $G:\boldsymbol{R}^d\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ *be a l. s. c. sublinear function. Then, for every* $x\in\boldsymbol{R}^d$,

$$\sigma(\partial G(0)\,;x)=G(x).$$

PROOF. If x is an exterior point of $D(G)$, then there exists $\eta\in\boldsymbol{R}^d$ such that $\langle\eta, x\rangle>0$ and that $\langle\eta, y\rangle\leqq 0$ for any $y\in D(G)$ (since $D(G)$ is a closed cone). Now choose $\xi\in\partial G(0)$ arbitrarily. Then $\xi+\lambda\cdot\eta\in\partial G(0)$ for any $\lambda>0$, hence $\sigma(\partial G(0)\,;x)=\infty=G(x)$. Next we suppose that $x\in\overline{D(G)}$. Let x_0 be a relative interior point of $D(G)$, and let $x_n=x_0+n\cdot x$ for $n=1, 2, \cdots$. Since $D(G)$ is a convex cone, every x_n belongs to the relative interior of $D(G)$, and $\partial G(x_n)\neq\phi$. If $\xi_n\in\partial G(x_n)$, then

$$\langle \xi_n, n\cdot x\rangle \geqq G(x_n)-G(x_0).$$

Hence, we have

$$\langle \xi_n, x\rangle \geqq \frac{1}{n}(G(x_n)-G(x_0)) \longrightarrow G_\infty(x) \text{ as } n\longrightarrow\infty.$$

Since G is sublinear, $\partial G(0)\supset\partial G(x_n)$, and $G_\infty(x)=G_\infty(0\,;x)=G(x)$ for every $x\in\boldsymbol{R}^d$. Therefore,

$$\sigma(\partial G(0)\,;x)\geqq\sigma(\partial G(x_n)\,;x)\geqq\langle\xi_n, x\rangle$$

for every n, and hence, $\sigma(\partial G(0)\,;x)\geqq G(x)$ for every $x\in\boldsymbol{R}^d$. The converse inequality is obvious, and the lemma is proved.

PROOF of THEOREM 2.1. It is easy so see the inclusion $\partial F(a)\supset \int_\Omega \partial f(a, t)\, d\eta(t)$, and we will prove the converse inclusion. In one dimensional case, i.e., $d=1$, $\partial F(a)$ is a closed interval $[\alpha, \beta]\subset\boldsymbol{R}$, and α and β are given by

$$\alpha=F'(a\,;-1)=\lim_{h\to0}\frac{1}{h}(F(a)-F(a-h))\,;$$

$$\beta=F'(a\,;1)=\lim_{h\to0}\frac{1}{h}(F(a+h)-F(a)).$$

Similarly, $\partial f(a, t)=[\alpha(t), \beta(t)]$ is given by

$$\alpha(t)=\lim_{h\to0}\frac{1}{h}(f(a, t)-f(a-h, t))\,;$$

$$\beta(t)=\lim_{h\to0}\frac{1}{h}(f(a+h, t)-f(a, t)).$$

Since these are monotone with respect to h, $\alpha(\cdot)$ and $\beta(\cdot)$ are measurable functions and

$$\begin{aligned}\int_\Omega \alpha(t)\, d\mu(t)&=\int_\Omega \lim_{h\to0}\frac{1}{h}(f(a, t)-f(a-h, t))\\&=\lim_{h\to0}\int_\Omega\frac{1}{h}(f(a, t)-f(a-h, t))\\&=\lim_{h\to0}\frac{1}{h}(F(a)-F(a-h))\\&=\alpha\,;\end{aligned}$$

$$\int_\Omega \beta(t)\, d\mu(t)=\beta.$$

We note that the hypothesis: $\overline{D(F)}=\overline{D(f(\cdot, t))}$ is used here in order to apply the monotone convergence theorem. (Refer to Example 2 at the end of this chapter.) It is easy to see that $\int_\Omega \partial f(a, t)\, d\mu(t)$ is a convex set. Thus we obtain

$$[\alpha, \beta]=\int_\Omega [\alpha(t), \beta(t)]\, d\mu(t)$$
$$=\int_\Omega \partial f(a, t)\, d\mu(t).$$

The proof for higher dimensional cases will be done by induction. We will verify a fundamental fact concerning the convex integrands before continuing the proof.

Let $f: \boldsymbol{R}^d \longrightarrow \boldsymbol{R} \cup \{+\infty\}$ be a proper convex function. We recall that $\bar{f}$ (the closure of f) is defined by the relation:

$$\operatorname{epi}(\bar{f})=\overline{\operatorname{epi}(f)}$$

where $\operatorname{epi}(f)$ is the epigraph of f, and $\overline{\operatorname{epi}(f)}$ is its closure. We know that $\bar{f}$ is l. s. c, and $\bar{f}=f^{**}$. Moreover, $\bar{f}=f$ on the relative interior of $D(f)$. By the definition of $\bar{f}$, one can easily verify that

$$\partial \bar{f}(a)=\partial f(a)$$

for every $a \in D(f)$ such that $\partial f(a) \neq \phi$.

PROPOSITION 2.6. *Suppose that* $f(\cdot, \cdot)$ *satisfies the condition* (A), *then*

(i) *if* $f(\cdot, t)$ *is l. s. c. for almost every* $t \in \Omega$, *then so is* $F(\cdot)$;
(ii) $\bar{F}(x)=\int_\Omega \bar{f}(x, t)\, d\mu(t)$ *holds for every* $x \in \boldsymbol{R}^d$.

PROOF. (i). We denote the restriction of F to a linear line $l \subset \boldsymbol{R}^d$ by F_l. If F_l is l. s. c. for every linear line $l \subset \boldsymbol{R}^d$, then $\{x \in \boldsymbol{R}^d | F(x) \leqq \alpha\}$ is closed for every $\alpha \in \boldsymbol{R}$, and this implies that F is l. s. c. Hence (i) can be reduced to one dimensional case by considering F_l and $f_l(\cdot, t)$. Let a be such that $\partial F(a) \neq \phi$. Since Therem 2.1 has been proved in the one dimensional case, we can take a summable function $\zeta: \Omega \longrightarrow \boldsymbol{R}^d$ such that

$$\int_\Omega \zeta(t)\, d\mu(t)=\xi \in \partial F(a),$$

and that $\zeta(t) \in \partial f(a, t)$. We put

$$H(x)=F(x)-\langle \xi, x\rangle,$$
$$h(x, t)=f(x, t)-\langle \zeta(t), x\rangle.$$

Then it suffices to show that H is l. s. c. Since

$$\partial h(a, t)=\partial f(a, t)-\zeta(t)\ni 0,$$

we have $h(x, t)\geqq h(a, t)$ for every $x\in \boldsymbol{R}$ and $t\in\Omega$. Note also that $h(a, \cdot)$ is summable. Let x_0 be an arbitrary point of $\boldsymbol{R}$, and let $\{x_n\}$ be a sequence converging to x_0. Since each $h(\cdot, t)$ is l. s. c.,

$$h(x, t)\leqq \varliminf_{n\to\infty} h(x_n, t).$$

According to the Fatou's Lemma,

$$\int_\Omega \varliminf_{n\to\infty} h(x_n, t)\mathrm{d}\mu(t)\leqq \varliminf_{n\to\infty}\int_\Omega h(x_n, t)d\mu(t).$$

Hence, we get

$$H(x)\leqq \varliminf_{n\to\infty} H(x_n),$$

and this implies the lower-semicontinuity of H.

(ii). In (ii), we do not assume that $f(\cdot, t)$ is l. s. c. for almost every $t\in\Omega$. However, if x belongs to the relative interior of $D(F)$, $\bar{f}(x, t)=f(x, t)$ holds for every $t\in\Omega$. Hence

$$F(x)=\int_\Omega \bar{f}(x, t)d\mu(t)$$

for every relative interior point x. Thus, (ii) follows from (i).

PROOF of THEOREM 2. 1, for d dimensional case. We suppose that this theorem is valid in $d-1$ dimensional case, and consider the d dimensional case. For the first step, we will prove that

$$\xi\in\int_\Omega \partial f(a, t)d\mu(t)$$

when ξ belongs to the boundary of $\partial F(a)$. Since $\partial F(a)$ is closed, $\xi\in \partial F(a)$. By taking the convex functions:

$$H(y)=F(y)-\langle\xi, y\rangle\,;$$
$$h(y, t)=f(y, t)-\langle\xi, y\rangle,$$

we can assume that $\xi=0$ without losing generality. Let x be a nonzero point in $\boldsymbol{R}^d$ such that

$$\sigma(\partial F(a)\,; x)=\langle\xi, x\rangle=0. \tag{2.1}$$

By Lemma 2. 5,

$$\begin{aligned}\sigma(\partial F(a)\,;x)&=\sigma(\partial F'(a\,;\cdot)(0)\,;x)\\&=\sigma(\partial\overline{F'(a\,;\cdot)}(0)\,;x)\\&=\overline{F'(a\,;x)}\end{aligned}\tag{2, 2}$$

where $\partial F'(a\,;\cdot)(0)$ is the subdifferential of $F'(a\,;\cdot)$ at 0, and $\overline{F'(a\,;\cdot)}$ is the closure of $F'(a\,;\cdot)$. Let N_x and $(\cdot,\cdot)_x$ be the notations defined in Lemma 2. 2, and define $F_1: N_x \longrightarrow \boldsymbol{R}\cup\{+\infty\}$ by

$$F_1(y)=\overline{F'(a\,;x+y)}$$

where $y\in N_x$. From (2, 1) and (2, 2), we have

$$F_1(0)=\overline{F'(a\,;x)}=0,$$

and it follows from $\partial F(a)\ni 0$ that

$$F_1(y)=\overline{F'(a\,;x+y)}\geqq 0.$$

This implies that $\partial F_1(0)\ni 0$, and by Lemma 2. 2,

$$(0,\,\partial F_1(0))_x\ni\xi=(0,\,0)_x.$$

Similarly, we define

$$f_1(y,\,t)=\overline{f_t'(a\,;x+y)}$$

where $f_t'(a\,;\cdot)$ is the directional derivative of $f(\cdot,\,t)$ at a, and $\overline{f_t'(a\,;\cdot)}$ is its closure. It is easy to see that a point $z\in\boldsymbol{R}^d$ belongs to the relative interior of $D(F'(a\,;\cdot))$ if and only if there exists $h>0$ such that $a+h\cdot z\in ri\,D(F)$. Hence, our assumption $ri\,D(F)=ri\,D(f(\cdot,\,t))$ implies that $ri\,D(F'(a\,;\cdot))=ri\,D(f_t'(a\,;\cdot))$. Moreover, for every $z\in ri\,D(F'(a\,;\cdot))$, we can apply the monotone convergence theorem to the following integral.

$$\begin{aligned}\int_\Omega f_t'(a\,;z)d\mu(t)&=\int_\Omega\lim_{h\to 0}\frac{1}{h}(f(a+h\cdot z,\,t)-f(a,\,t))d\mu(t)\\&=\lim_{h\to 0}\int_\Omega\frac{1}{h}(f(a+h\cdot z,\,t)-f(a,\,t))d\mu(t)\\&=\lim_{h\to 0}\frac{1}{h}(F(a+h\cdot z)-F(a))\\&=F'(a\,;z).\end{aligned}$$

Similarly, the same relation holds on the exterior of $D(F'(a\,;\cdot))$. Hence,

$$\overline{\int_\Omega f_t'(a\,;z)d\mu(t)}=\overline{F'(a\,;z)}$$

holds for every $z\in\boldsymbol{R}^d$. By Proposition 2. 6, we have

$$\int_\Omega \overline{f_t'(a\,;z)}\,d\mu(t)=\overline{\int_\Omega f_t'(a\,;z)d\mu(t)}$$
$$=\overline{F'(a\,;z)}$$

for every $z\in\boldsymbol{R}^d$. Therefore,

$$\int_\Omega f_1(y,t)d\mu(t)=\int_\Omega \overline{f_t'(a\,;x+y)}\,d\mu(t)$$
$$=\overline{F'(a\,;x+y)}$$
$$=F_1(y)$$

holds for every $y\in N_x$. Moreover,

$$ri\,D(\overline{F'(a\,;\cdot)})=ri\,D(F'(a\,;\cdot))$$
$$=ri\,D(f_t'(a\,;\cdot))$$
$$=ri\,D(\overline{f_t'(a\,;\cdot)}).$$

Hence we can easily see that $ri\,D(F_1)=ri\,D(f_1(\cdot,t))$, and thus the convex integrand $f_1(\cdot,\cdot)$ satisfies the conditions of this theorem in $d-1$ dimensional case. Hence, we have

$$\partial F_1(0)=\int_\Omega \partial f_1(0,t)\,d\mu(t).$$

Since $\int_\Omega \overline{f_t'(a\,;x)}\,d\mu(t)=\overline{F'(a\,;x)}=0$, it follows from Lemma 2.2 that

$$\xi=(0,0)_x\in(0,\partial F_1(0))$$
$$=\int_\Omega\left(\frac{\overline{f_t'(a\,;x)}}{\|x\|},\ \partial f_1(0,t)\right)_x d\mu(t)$$
$$\subset\int_\Omega \partial\overline{f_t'(a\,;\cdot)}(0)\,d\mu(t)$$
$$=\int_\Omega \partial f_t'(a\,;\cdot)(0)\,d\mu(t)$$
$$=\int_\Omega \partial f(a,t)\,d\mu(t).$$

For the final step of the proof, we will consider the interior points of $\partial F(a)$. Let η be an arbitrary interior point of $\partial F(a)$, and let l be the half line which starts from $\xi=0$, and includes η, i. e.,

$$l=\{\xi+\lambda(\eta-\xi)|\lambda\geqq 0\}=\{\lambda\cdot\eta|\lambda\geqq 0\}.$$

If $\partial F(a)\cap l$ is a bounded interval $[\zeta,0]$ for a boundary point ζ of $\partial F(a)$, then $\eta\in[\zeta,0]\subset\int_\Omega \partial f(a,t)\,d\mu(t)$, because ζ belongs to $\int_\Omega \partial f(a,t)\,d\mu(t)$, and the set $\int_\Omega \partial f(a,t)\,d\mu(t)$ is convex. Thus it remains to prove that $\eta\in$

$\int_\Omega \partial f(a, t)\, d\mu(t)$ when $\partial F(a) \cap l = l$. Since $\partial F(a) \supset l$, it follows that

$$\langle \lambda \cdot \eta, b-a \rangle \leqq F(b) - F(a)$$

holds for any $\lambda \geqq 0$ and $b \in \boldsymbol{R}^d$, and hence,

$$\langle \eta, b-a \rangle \leqq 0 \tag{2, 3}$$

holds whenever $b \in D(F)$. Let $\xi(t)$ be a summable selector such that

$$\xi(t) \in \partial f(a, t)$$

and that $\int_\Omega \xi(t)\, d\mu(t) = 0$. Then it follows that

$$\langle \xi(t) + \eta, b-a \rangle - (f(b, t) - f(a, t)) \leqq \langle \eta, b-a \rangle$$

for every $t \in \Omega$. By (2, 3), we can easily see that $\langle \eta, b-a \rangle \leqq 0$ for $b \in D(f(\cdot, t))$. Hence, we have

$$\langle \xi(t) + \eta, b-a \rangle - (f(b, t) - f(a, t)) \leqq 0$$

for every $t \in \Omega$ and $b \in D(f(\cdot, t))$. This implies that

$$\xi(t) + \eta \in \partial f(a, t)$$

for every $t \in \Omega$. Moreover,

$$\int_\Omega (\xi(t) + \eta)\, d\mu(t) = \eta,$$

and this completes the proof of Theorem 2. 1.

§ 2. 2

The aim of this section is to establish a relation between the conjugate function F^* of F and the convex integrand $f^*(\cdot, \cdot)$, where $f^*(\cdot, t)$ is the conjugate function of $f(\cdot, t)$ for each $t \in \Omega$. The recession function which is defined in § 2. 1 is useful for this problem. The following proposition gives a fundamental property of recession functions.

PROPOSITION 2. 7. *Let $f : \boldsymbol{R}^d \longrightarrow \boldsymbol{R} \cup \{+\infty\}$ be a l. s. c. proper convex function. Then*

$$\partial f_\infty(0) = \overline{D(f^*)}$$

holds, where $\overline{D(f^*)}$ is the closure of the effective domain of f^*.

PROOF. For every $\xi \in D(f^*)$, the convex function $g(x) = f(x) - \langle \xi, x \rangle$ is bounded below. Therefore,

$$g_\infty(x) \geqq 0$$

for every $x \in \boldsymbol{R}^d$, and this implies that

$$f_\infty(x) \geqq \langle \xi, x \rangle$$

for every $x \in \boldsymbol{R}^d$. Hence, ξ belongs to $\partial f_\infty(0)$, and we obtain $D(f^*) \subset \partial f_\infty(0)$. Since $\partial f_\infty(0)$ is closed, this implies that $\overline{D(f^*)} \subset \partial f_\infty(0)$. Thus it remains to prove that

$$ri\ \partial f_\infty(0) \subset D(f^*).$$

Let ζ be an arbitrary point of ri $\partial f_\infty(0)$, and let $h(\cdot)$ be a convex function defined by

$$h(x) = f(x) - \langle \zeta, x \rangle.$$

Then we can easily see that

$$\partial h_\infty(0) = \partial f_\infty(0) - \zeta\,;$$
$$D(h^*) = D(f^*) - \zeta.$$

Hence, we can assume that $\zeta = 0$ without losing generality. Let A be the affine hull of $\partial f_\infty(0)$. Since $A \ni 0$, A is a subspace of $\boldsymbol{R}^d$. By $A^\perp$, we denote the orthogonal space of A, i. e., $A^\perp = \{y \in \boldsymbol{R}^d | \langle y, x \rangle = 0$ for every $x \in A\}$. Since $\partial f_\infty(0) \subset A$, we have

$$\sigma(\partial f_\infty(0)\,;\, y) = 0$$

for every $y \in A^\perp$. By the lower-semicontinuity of f and by Lemma 2. 5,

$$\sigma(\partial f_\infty(0)\,;\,\cdot) = f_\infty(\cdot)$$

on $\boldsymbol{R}^d$. Hence we get $f_\infty(y) = 0$ for every $y \in A^\perp$, and this implies that

$$f(x) = f(x + y)$$

for every $x \in A$, and $y \in A^\perp$. On the other hand,

$$f_\infty(x) = \sigma(\partial f_\infty(0)\,;\, x) > 0$$

for every $x \in A$ with $x \neq 0$, because 0 is an interior point of $\partial f_\infty(0)$ with respect to the relative topology of A. Hence, we can easily see that the set $A \cap L_\alpha(f) = \{x \in A | f(x) \leqq \alpha\}$ is bounded for every $\alpha \in \boldsymbol{R}$. This implies that f is bounded below on A. Moreover, by the previous argument,

$$\inf_{x \in \boldsymbol{R}^d} f(x) = \inf_{x \in A} f(x) > -\infty.$$

Thus we obtain $0 \in D(f^*)$, and the proposition is proved.

PROPOSITION 2. 8. *Under the hypotheses in* § 2. 1, *the function* $\sigma(D(f^*(\cdot, t)); x)$ *is measurable with respect to* $t\in\Omega$, *for every* $x\in\boldsymbol{R}^d$, *and*

$$\int_{\Omega}\sigma(D(f^*(\cdot, t)); x)d\mu(t)=\sigma(D(F^*); x).$$

PROOF. We know that $D(F^*)=D(\bar{F}^*)$ and $D(f^*(\cdot, t))=D(\bar{f}^*(\cdot, t))$ hold in general. Therefore, by proposition 2. 6, we can assume that $F(\cdot)$ and $f(\cdot, t)$ are l. s. c, without losing generality. From Lemma 2. 5 and Proposition 2. 7, we have

$$\sigma(D(f^*(\cdot, t)); x)=f_{\infty}(x, t);$$
$$\sigma(D(F^*); x)=F_{\infty}(x).$$

Hence $\sigma(D(f^*(\cdot, t)); x)$ is measurable for every $x\in\boldsymbol{R}^d$. By the monotone convergence theorem,

$$\begin{aligned}\int_{\Omega}\sigma(D(f^*(\cdot, t); x)d\mu(t)&=\int_{\Omega}f_{\infty}(x, t)d\mu(t)\\&=\int_{\Omega}\lim_{h\to\infty}\frac{1}{h}(f(x_0+h\cdot x, t)-f(x_0, t))d\mu(t)\\&=\lim_{h\to\infty}\int_{\Omega}\frac{1}{h}(f(x_0+h\cdot x, t)-f(x_0, t))d\mu(t)\\&=\lim_{h\to\infty}\frac{1}{h}(F(x_0+h\cdot x)-F(x_0))\\&=F_{\infty}(x)\\&=\sigma(D(F^*); x).\end{aligned}$$

Thus the proposition is proved.

LEMMA 2. 9. *Let* ξ *be a boundary point of* $D(F^*)$, *and let* $x\in\boldsymbol{R}^d$ *be such that* $x\neq 0$ *and that*

$$\langle\xi, x\rangle=\sigma(D(F^*); x).$$

Then, for every $x_0\in D(F)$, *the convex function*

$$F(x_0+\lambda\cdot x)-\langle\xi, x_0+\lambda\cdot x\rangle$$

is non increasing with respect to $\lambda\in\boldsymbol{R}$.

PROOF. From Lemma 2. 5 and Proposition 2. 7, we have

$$F_{\infty}(x)=\sigma(D(F^*); x)=\langle\xi, x\rangle.$$

We take a convex function F_1 defined by

$$F_1(\cdot)=F(\cdot)-\langle\xi, \cdot\rangle,$$

then

$$(F_1)_\infty(x)=F_\infty(x)-\langle\xi, x\rangle=0.$$

Hence $F_1(x_0+\lambda\cdot x)$ is non increasing with respect to λ, and the lemma is proved.

We will consider the function of the form $f^*(\zeta(t), t)$ where $\zeta:\Omega\longrightarrow\boldsymbol{R}^d$ is a measurable function and $f(\cdot,\cdot)$ is a convex integrand with the condition (A). The theory of normal convex integrand is applicable to prove the measurability of $f^*(\zeta(t), t)$. A convex integrand $f(\cdot,\cdot)$ is said to be *normal* if $f(\cdot, t)$ is proper l. s. c. for each t, and if further there exists a countable collection Z of measurable functions ζ from Ω to $\boldsymbol{R}^d$ having the following properties:

(a) for each $\zeta\in Z$, $f(\zeta(t), t)$ is measurable in t;
(b) for each $t\in\Omega$, $\{\zeta(t)|\zeta\in Z\}\cap D(f(\cdot, t))$ is dense in $D(f(\cdot, t))$.

The normality of a convex integrand $f(\cdot,\cdot)$ guarantees the measurability of $f(\zeta(t), t)$ in t for every measurable function ζ from Ω to $\boldsymbol{R}^d$. Moreover, if $f(\cdot,\cdot)$ is normal, then $f^*(\cdot,\cdot)$ is also normal. ([11], Corollary 2B and Proposition 2S)

LEMMA 2.10. *Let $f(\cdot,\cdot)$ be a convex integrand with the condition* (A). *Then for every measurable function ζ from Ω to $\boldsymbol{R}^d$, $f^*(\zeta(t), t)$ is measurable in t.*

PROOF. We take the closures $\bar{f}(\cdot, t)$ and $\bar{F}$ of each $f(\cdot, t)$ and F. Then the convex integrand $\bar{f}(\cdot,\cdot)$ also satisfies the condition (A), i. e.,

$$\overline{D(\bar{f}(\cdot, t))}=\overline{D(\bar{F})}\neq\phi \text{ for almost every } t\in\Omega.$$

Let $\{z_n\}_{n=1}^\infty$ be a countable dense subset of $D(\bar{F})$, and for each n, let $\zeta_n:\Omega\longrightarrow\boldsymbol{R}^d$ be a constant function with the value z_n. Then the collection $\{\zeta_n\}_{n=1}^\infty$ satisfies (a) and (b) in the definition of normality. Moreover the lower-semicontinuity of $\bar{f}(\cdot, t)$ is automatic. Hence the convex integrand $\bar{f}(\cdot,\cdot)$ is normal, and we have that $(\bar{f})^*(\cdot,\cdot)=f^*(\cdot,\cdot)$ is normal. Thus we obtain the measurability of $f^*(\zeta(t), t)$.

We are ready to prove the following theorem wich represents the conjugate function F^* in terms of the convex integrand $f^*(\cdot,\cdot)$. For every $\xi\in\boldsymbol{R}^d$, we define $S(\xi)$ as follows.

$$S(\xi)=\{\zeta:\Omega\longrightarrow\boldsymbol{R}^d|\zeta \text{ is summable, and } \int_\Omega\zeta(t)d\mu(t)=\xi\}$$

THEOREM 2.11. *Let $f(\cdot,\cdot)$ be a convex integrand with the condition* (A). *Then*

$$F^*(\xi)=\min\{\int_\Omega f^*(\zeta(t),\ t)d\mu(t)|\zeta\in S(\xi)\}$$

holds for every $\xi\in\boldsymbol{R}^d$.

REMARK. The right side of this formula is called the continuous infimal convolution of $f^*(\cdot, t)$. In [2], this formula is shown with the condition that $f(x, \cdot)$ is summable for every $x\in\boldsymbol{R}^d$. Our theorem shows that the condition is not essential. However, this formula is no longer valid, if we remove the condition (A). (See Example 2.)

PROOF. For every $\zeta(\cdot)\in S(\xi)$, we have

$$\begin{aligned} F^*(\xi) &= \sup_{x\in D(F)}(\langle\xi, x\rangle-F(x)) \\ &= \sup_{x\in D(F)}\int_\Omega(\langle\zeta(t), x\rangle-f(x, t))d\mu(t) \\ &\leqq \int_\Omega\sup_{x\in D(F)}(\langle\zeta(t), x\rangle-f(x, t))d\mu(t) \\ &= \int_\Omega f^*(\zeta(t),\ t)d\mu(t). \end{aligned} \tag{2, 4}$$

Hence it suffices to find a summable function $\zeta\in S(\xi)$ such that

$$F^*(\xi)=\int_\Omega f^*(\zeta(t),\ t)d\mu(t).$$

By Proposition 2.6 (ii), we can assume that F and each $f(\cdot, \cdot)$ are l.s.c. without losing generality. The following three cases are possible, and our proof will be done in each case.

Case 1: $\xi\in D^\circ(F^*)$.
Case 2: $\xi\notin D(F^*)$.
Case 3: $\xi\in D(F^*)\backslash D^\circ(F^*)$.

Case 1. Applying Lemma 1.4, we can easily see that the l.s.c. convex function $F(\cdot)-\langle\xi, \cdot\rangle$ attains its minimum $-F^*(\xi)$, i.e., there exists $a\in D(F)$ such that

$$F^*(\xi)=\langle\xi, a\rangle-F(a).$$

In other words, ξ belongs to $\partial F(a)$ for the same a. Hence from Theorem 2.1, there exists a summable function $\zeta\in S(\xi)$ such that

$$\zeta(t)\in\partial f(a, t)$$

for every $t\in\Omega$. Hence it follows that

$$\int_\Omega f^*(\zeta(t), t)d\mu(t) = \int_\Omega \sup_{x\in D(F)} (\langle\zeta(t), x\rangle - f(x, t))d\mu(t)$$
$$= \int_\Omega (\langle\zeta(t), a\rangle - f(a, t))d\mu(t)$$
$$= \langle\xi, a\rangle - F(a)$$
$$= F^*(\xi).$$

Case 2. From (2, 4), $F^*(\xi)=\infty$ implies that

$$\int_\Omega f^*(\zeta(t), t)d\mu(t)=\infty$$

for every $\zeta\in S(\xi)$.

Case 3. We can assume that $\xi=0$ without losing generality. Hence the assertion of this theorem is equivalent to the following statement:

There exists $\zeta\in S(0)$ such that

$$\inf_{z\in \boldsymbol{R}^d} F(z) = \int_\Omega \inf_{z\in \boldsymbol{R}^d} (f(z, t) - \langle\zeta(t), z\rangle)d\mu(t). \tag{2, 5}$$

Let $x\in\boldsymbol{R}^d$ be such that $x\neq 0$ and that

$$\sigma(D(F^*); x) = \langle\xi, x\rangle = 0,$$

and let $N_x=\{z\in\boldsymbol{R}^d | \langle z, x\rangle=0\}$. Then N_x can be identified with $\boldsymbol{R}^{d-1}$. We define $F_x : N_x \longrightarrow \boldsymbol{R}\cup\{\pm\infty\}$ by

$$F_x(y) = \inf_{\lambda\in\boldsymbol{R}} F(y+\lambda\cdot x)$$

for $y\in N_x$. One can easily see that F_x is a convex function. By Lemma 2.9, $F(y+\lambda\cdot x)$ is non increasing with respect to λ. Hence we have

$$F_x(y) = \lim_{\lambda\to\infty} F(y+\lambda\cdot x).$$

Since $F^*(0)<+\infty$ in case 3, F_x cannot take $-\infty$ anywhere. Similarly, we define $f_x : N_x\times\Omega \longrightarrow \boldsymbol{R}\cup\{\pm\infty\}$ by

$$f_x(y, t) = \inf_{\lambda\in\boldsymbol{R}} (f(y+\lambda\cdot x, t) - \Phi(t)\cdot\lambda)$$

where $\Phi(t)=\sigma(D(f^*(\cdot, t)); x)$. Then by Proposition 2.8, $\Phi(\cdot)$ is summable, and

$$\int_\Omega \Phi(t)d\mu(t) = \sigma(D(F^*); x) = 0.$$

Let $\eta(t)\in\overline{D(f^*(\cdot, t))}$ be such that

$$\langle\eta(t), x\rangle=\sigma(D(f^*(\cdot, t)\,; x)=\Phi(t).$$

Then we have

$$f(y+\lambda\cdot x, t)-\Phi(t)\cdot\lambda=f(y+\lambda\cdot x, t)-\langle\eta(t), y+\lambda\cdot x\rangle+\langle\eta(t), y\rangle.$$

Hence by Lemma 2.9, this is non increasing with respect to λ, and we have

$$f_x(y, t)=\lim_{\lambda\to\infty}(f(y+\lambda\cdot x, t)-\Phi(t)\cdot\lambda).$$

Therefore $f_x(y, \cdot)$ is measurable for every $y\in N_x$. If $y\in D(F_x)$, there exists $\lambda_0\in \boldsymbol{R}$ such that $f(y+\lambda_0\cdot x, \cdot)-\Phi(\cdot)\cdot\lambda_0$ is summable. In addition, by the condition (A), $f(y+\lambda\cdot x, t)-\Phi(t)\cdot\lambda=\infty$ for every $\lambda\in\boldsymbol{R}$ and almost every $t\in\Omega$, if y is an exterior point of $D(F_x)$. Hence by the monotone convergence theorem,

$$\begin{aligned}\int_\Omega f_x(y, t)d\mu(t)&=\int_\Omega\lim_{\lambda\to\infty}(f(y+\lambda\cdot x, t)-\Phi(t)\cdot\lambda)d\mu(t)\\&=\lim_{\lambda\to\infty}\int_\Omega(f(y+\lambda\cdot x, t)-\Phi(t)\cdot\lambda)d\mu(t)\\&=F_x(y)\end{aligned}$$

holds if y belongs to $D(F_x)$ or the exterior of $D(F_x)$. Hence by taking the closures of both sides and by Lemma 2. 6, we get

$$\int_\Omega \bar{f}_x(y, t)\,d\mu(t)=\bar{F}_x(y)\ \text{ for every }\ y\in N_x. \tag{2, 6}$$

If $d=1$, $N_x=\{0\}$ and (2, 5) follows from (2, 6). Thus the theorem is proved in one dimensional case. We assume that this theorem is valid in $d-1$ dimensional case where $d\geqq 2$. By the condition (A), one can easily see that $\bar{f}_x(\cdot, t)$ cannot take $-\infty$ anywhere for almost every $t\in\Omega$, and the convex integrand $\bar{f}_x(\cdot, \cdot)$ also satisfies the condition (A). Hence by identifying N_x and $\boldsymbol{R}^{d-1}$, we can apply this theorem in $d-1$ dimensional case to $\bar{f}_x(\cdot, \cdot)$ and $\bar{F}_x$. By (2. 5), there exists a summable function $\tilde{\zeta}:\Omega\longrightarrow N_x\simeq\boldsymbol{R}^{d-1}$ such that $\int_\Omega\tilde{\zeta}(t)d\mu(t)=0$, and that

$$\inf_{y\in N_x}F_x(y)=\int_\Omega\inf_{y\in N_x}(f_x(y, t)-\langle\tilde{\zeta}(t), y\rangle)d\mu(t).$$

Define $\zeta:\Omega\longrightarrow\boldsymbol{R}^d$ by $\zeta(t)=\tilde{\zeta}(t)+\dfrac{\Phi(t)}{\|x\|^2}x$. Then ζ is summable and $\int_\Omega\zeta(t)d\mu(t)=0$. Moreover we have

$$\inf_{z\in D(F)}F(z)=\inf_{y\in N_x}F_x(y)$$

$$=\int_{\Omega}\inf_{y\in N_x}(f_x(y, t)-\langle\tilde{\xi}(t), y\rangle)d\mu(t)$$
$$=\int_{\Omega}\inf_{y\in N_x}\{\inf_{\lambda\in\boldsymbol{R}}(f(y+\lambda\cdot x, t)-\Phi(t)\cdot\lambda)-\langle\tilde{\xi}(t), y\rangle\}d\mu(t)$$
$$=\int_{\Omega}\inf_{y\in N_x}\inf_{\lambda\in\boldsymbol{R}}(f(y+\lambda\cdot x, t)-\langle\tilde{\xi}(t)+\frac{\Phi(t)}{\|x\|^2}x, \lambda\cdot x\rangle$$
$$-\langle\tilde{\xi}(t)+\frac{\Phi(t)}{\|x\|^2}x, y\rangle)d\mu(t)$$
$$=\int_{\Omega}\inf_{y\in N_x}\inf_{\lambda\in\boldsymbol{R}}(f(y+\lambda\cdot x, t)-\langle\zeta(t), y+\lambda\cdot x\rangle)d\mu(t)$$
$$=\int_{\Omega}\inf_{z\in\boldsymbol{R}^d}(f(z, t)-\langle\zeta(t), z\rangle)d\mu(t).$$

Thus we obtain (2, 5) in d dimensional case, and this completes the proof of Theorem 2. 11.

§ 2. 3.

In this section, we will give two theorems which are derived from Theorem 2. 11.

THEOREM 2. 12. *Let $f(\cdot, \cdot)$ be a convex integrand with the condition* (A). *Then we have*

$$D(F^*)=\int_{\Omega}D(f^*(\cdot, t))d\mu(t).$$

PROOF. The inclusion $D(F^*)\supset\int_{\Omega}D(f^*(\cdot, t))d\mu(t)$ is obvious. On the other hand, if $F^*(\xi)<\infty$, then there exists a summable function ζ : $\Omega\longrightarrow\boldsymbol{R}^d$ such that

$$\int_{\Omega}\zeta(t)d\mu(t)=\xi$$

and that $F^*(\xi)=\int_{\Omega}f^*(\zeta(t), t)d\mu(t)<\infty$. Hence $\zeta(t)\in D(f^*(\cdot, t))$ for almost every $t\in\Omega$, and this completes the proof.

For a proper convex function $f:\boldsymbol{R}^d\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ and a positive number ε, the *ε-subdifferential* of f at $x_0\in D(f)$ is defined by

$$\partial_\varepsilon f(x_0)=\{\xi\in\boldsymbol{R}^d|\langle\xi, x_0\rangle-f(x_0)\geqq\langle\xi, x\rangle-f(x)-\varepsilon, \text{ for every } x\in D(f)\}.$$

The following is a well known formula.

$$\partial_\varepsilon(f_1+f_2)(x_0)=\bigcup_{\substack{\varepsilon_1,\varepsilon_2\geqq0\\ \varepsilon_1+\varepsilon_2=\varepsilon}}(\partial_{\varepsilon_1}f_1(x_0)+\partial_{\varepsilon_2}f_2(x_0))$$

where f_1 and f_2 are proper convex functions and $x_0 \in D(f_1) \cap D(f_2)$. Applying Theorem 2.11, we can get the following theorem which is a natural extension of this formula.

THEOREM 2.13. *Let $f(\cdot, \cdot)$ be a convex integrand with the condition (A). Then for $x_0 \in D(F)$, we have*

$$\partial_\varepsilon F(x_0) = \bigcup_{\varepsilon(\cdot) \in S^+(\varepsilon)} \int_\Omega \partial_{\varepsilon(t)} f(x_0, t) d\mu(t)$$

where $\partial_{\varepsilon(t)} f(x_0, t)$ is the $\varepsilon(t)$-subdifferential of $f(\cdot, t)$ at x_0, and $S^+(\varepsilon)$ is the set of summable functions $\varepsilon(\cdot)$ satisfying $\int_\Omega \varepsilon(t) d\mu(t) = \varepsilon$ and $\varepsilon(t) \geqq 0$ for every $t \in \Omega$.

PROOF. If $\xi \in \partial_\varepsilon F(x_0)$, then

$$F^*(\xi) - (\langle \xi, x_0 \rangle - F(x_0)) \leqq \varepsilon,$$

By Theorem 2.11, there exists a summable function $\zeta \in S(\xi)$ such that

$$F^*(\xi) = \int_\Omega f^*(\zeta(t), t) d\mu(t).$$

We put

$$\varepsilon_1(t) = f^*(\zeta(t), t) - (\langle \zeta(t), x_0 \rangle - f(x_0, t)).$$

Then $\zeta(t) \in \partial_{\varepsilon_1(t)} f(x_0, t)$, and

$$\int_\Omega \varepsilon_1(t) d\mu(t) = F^*(\xi) - (\langle \xi, x_0 \rangle - F(x_0)) = \varepsilon - \alpha$$

where $0 \leqq \alpha \leqq \varepsilon$. We define $\varepsilon(t)$ by $\varepsilon(t) = \varepsilon_1(t) + \alpha$ for every $t \in \Omega$. Then we have

$$\zeta(t) \in \partial_{\varepsilon_1(t)} f(x_0, t) \subset \partial_{\varepsilon(t)} f(x_0, t)$$
$$\int_\Omega \varepsilon(t) d\mu(t) = \varepsilon.$$

Hence the inclusion

$$\partial_\varepsilon \mathrm{F}(x_0) \subset \bigcup_{\varepsilon(\cdot) \in S^+(\varepsilon)} \int_\Omega \partial_{\varepsilon(t)} f(x_0, t) d\mu(t)$$

is obtained, and the converse inclusion is obvious.

EXAMPLE 2. We give a simple example which shows that the condition (A) is essential for our theorems. Let Ω be the interval $[0, 1] \subset \boldsymbol{R}$ with Lebesgue measure, and consider the one dimensional case, i.e., $d=1$. We define a convex integrand $f:\boldsymbol{R}\times[0,1]\longrightarrow\boldsymbol{R}\cup\{+\infty\}$ to be

$$f(x,t)=\begin{cases}x & \text{if } x\geqq -t\\ +\infty & \text{if } x<-t.\end{cases}$$

Then

$$F(x)=\int_0^1 f(x,t)dt=\begin{cases}x & \text{if } x\geqq 0\\ +\infty & \text{if } x<0.\end{cases}$$

Since $F(x)\geqq 0$, $0\in\partial F(0)$. However, $\partial f(0,t)=\{1\}$ for every $t\in(0,1]$, and hence

$$0\notin\int_0^1 \partial f(0,t)dt.$$

Therefore the formula in Theorem 2. 1 does not hold in this case. Moreover, we get by a simple computation that

$$f^*(\xi,t)=\begin{cases}t(1-\xi) & \text{if } \xi\leqq 1\\ +\infty & \text{if } \xi>1,\end{cases}$$

$$\mathrm{F}^*(\xi)=\begin{cases}0 & \text{if } \xi\leqq 1\\ +\infty & \text{if } \xi>1,\end{cases}$$

and $F^*(0)=0$ in particular. Since $f^*(\xi,t)$ is always non negative, only the constant function $\zeta(\cdot)$ whose value is 1 can satisfy

$$\int_0^1 f^*(\zeta(t),t)dt=F^*(0)=0.$$

On the other hand, we have

$$\int_0^1 \zeta(t)dt=1.$$

Consequently, there is no summable function $\zeta(t)$ which satisfies the formula in Theorem 2. 11.

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