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Zeros of integrals along trajectories of ergodic nonsingular flows

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§ 1. Introduction

Let $(X, \mathscr{B}, \mu)$ be a probability space. In 1976, Atkinson [1] proved that if T is an ergodic measure preserving automorphism of $(X, \mathscr{B}, \mu)$ then the following conditions are equivalent for f in $L_1(\mu)$:

(a) $\int f\, d\mu=0.$

(b) $\liminf_{n\to\infty}\left|\sum_{j=0}^{n} f(T^j x)\right|=0$ for almost all $x\in X$.

In 1987, Ullman [6] generalized Atkinson's theorem to noninvariant measures. That is, he considered an ergodic, conservative, nonsingular automorphism T of $(X, \mathscr{B}, \mu)$ and proved that the above condition (a) and the following (b′) are equivalent for f in $L_1(\mu)$.

(b′) $\liminf_{n\to\infty}\left|\sum_{j=0}^{n} f(T^j x)\cdot\frac{d\mu\circ T^j}{d\mu}(x)\right|=0$ for almost all $x\in X$.

In this note we will treat an ergodic, conservative, nonsingular flow of $(X, \mathscr{B}, \mu)$ and prove a corresponding continuous time result. The method of proof is different from that of Ullman. See also Schneiberg [5].

§ 2. Preliminaries and the theorem

From now on, let $\{T_t\}=\{T_t : -\infty<t<\infty\}$ be a measurable flow of nonsingular automorphisms of $(X, \mathscr{B}, \mu)$. All sets and functions introduced below are assumed to be measurable; and all relations are assumed to hold modulo sets of measure zero. Since each T_t is nonsingular, the Radon-Nikodym theorem can be applied to define a function $w_t=\frac{d\mu\circ T_t}{d\mu}$ in $L_1(\mu)$ such that

$$(1)\qquad \int_A w_t d\mu=\mu(T_t A) \text{ for all } A\in\mathscr{B},$$

and let us put

(2) $\quad U_tf(x)=f(T_tx)w_t(x)$ for $f\in L_1(\mu)$.

As is easily seen, $\{U_t\}=\{U_t: -\infty<t<\infty\}$ becomes a group of positile linear isometries of $L_1(\mu)$. Further by Krengel [2] (see also Sato [4]), $\text{strong-}\lim_{t\to 0} U_t=I$ (I being the identity operator). The flow $\{T_t\}$ is called *conservative* if each T_t is conservative. (Recall that a nonsingular automorphism T is conservative if and only if $A\subset TA$ implies $A=TA$. It is known (cf. e.g. Krengel [3], § 3.1) that T is conservative if and only if $\sum_{n=0}^{\infty}\frac{d\mu\circ T^n}{d\mu}(x)=\infty$ on X.) It is easy to see that $\{T_t\}$ is conservative if and only if

$$\int_0^\infty U_t1(x)dt=\int_0^\infty w_t(x)dt=\infty \text{ for almost all } x\in X.$$

The flow $\{T_t\}$ is called *ergodic* if $A=T_tA$ for all t implies $\mu A=0$ or $\mu(X\setminus A)=0$. We are now in a position to state our result.

THEOREM. *Let $\{T_t\}$ be an ergodic, conservative, measurable flow of nonsingular automorphisms of $(X, \mathscr{B}, \mu)$ with $\mu X=1$. Then the following conditions are equivalent for f in $L_1(\mu)$:*

(I) $\int f\, d\mu=0$.

(II) *To almost every $x\in X$ there corresponds a real sequence s_n (dependent on x), with $s_n\uparrow\infty$, such that $\int_0^{s_n} f(T_tx)w_t(x)\, dt=0$ for all $n\geq 1$.*

PROOF. (I)$\Rightarrow$(II): Let us fix an integer $N\geq 1$, and write

$$A=A_N=\{x\in X: \int_0^s U_tf(x)dt>0 \text{ for all } s\geq N\},$$
$$B=B_N=\{x\in X: \int_0^s U_tf(x)dt<0 \text{ for all } s\geq N\},$$
$$C=C_N=(X\setminus A)\cap(X\setminus B).$$

Let

(3) $$g(x)=g_N(x)=\int_0^N U_tf(x)dt, \text{ and}$$

(4) $$D=\{x\in X: \sum_{j=0}^{n-1} U_N^jg(x)>0 \text{ for all } n\geq 1\}.$$

It follows that $A\subset D$. In order to prove $\mu A=0$, we assume $\mu D>0$. Then, since T_N is conservative by hypothesis, for almost every $x\in D$ we can take an integer $n(x)\geq 1$ such that

$$T_N^{n(x)}x \in D \text{ and } T_N^j x \notin D \text{ for all } 1 \leq j < n(x).$$

Put

$$X(n) = \{x \in D : n(x) = n\} \text{ and } Y(n) = \bigcup_{j=0}^{n-1} T_N^j X(n).$$

Then we see that

$$D = \bigcup_{n=1}^{\infty} X(n), \tag{5}$$

and the set $Y = \bigcup_{n=1}^{\infty} Y(n)$ satisfies $T_N Y = Y$. On the other hand, by the continuous time version of the Chacon-Ornstein ratio ergodic theorem (see e. g. [3], Chapter 3), (I) implies

$$\begin{aligned} 0 = \int f\, d\mu &= \lim_{s \to \infty} \int_0^s U_t f(x) dt \Big/ \int_0^s U_t 1(x) dt \\ &= \lim_{n \to \infty} \sum_{j=0}^{n-1} U_N^j g(x) \Big/ \sum_{j=0}^{n-1} U_N^j h(x) \text{ for almost all } x \in X, \end{aligned}$$

where we let $h(x) = \int_0^N U_t 1(x) dt$. Therefore $T_N Y = Y$ implies $\int_Y g\, d\mu = 0$. But this is a contradiction, because

$$\begin{aligned} \int_Y g\, d\mu = \sum_{n=1}^{\infty} \int g \cdot 1_{Y(n)} d\mu &= \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \int g \cdot 1_{T_N^j X(n)} d\mu \\ &= \sum_{n=1}^{\infty} \int \left(\sum_{j=0}^{n-1} g(T_N^j x) \cdot \frac{d\mu \circ T_N^j}{d\mu}(x) \right) \cdot 1_{X(n)}(x) d\mu(x) \\ &= \sum_{n=1}^{\infty} \int \left(\sum_{j=0}^{n-1} U_N^j g \right) \cdot 1_{X(n)} d\mu > 0 \end{aligned}$$

where the last inequality is due to (4) and (5).

We have proved $\mu A = 0$. Similarly, $\mu B = 0$ follows. Hence for almost all $x \in X$ there exists a real number $s(x) \geq N$ such that $\int_0^{s(x)} U_t f(x) dt = 0$. By this (II) follows immediately.

(II)⇒(I): This implication is a direct consequence of the continuous time version of the Chacon-Ornstein theorem, and we omit the details.

References

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