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Combinatorial analysis of point obstructions to local factorizability in three-folds

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Abstract: The paper introduces formally the concept of local factorizability used in earlier factorizability work, identifies the basic form of obstructions to local factorizability of birational morphisms, and outlines a combinatorial technique for analyzing such obstructions. As an application and illustration, two open cases in the classification of birational morphisms with small canonical divisors are settled.

0. Introduction

If $f: X \to Y$ is a birational morphism of algebraic spaces we will say that

> f is <u>directly factorizable</u> if it is a composition of blowings-up with nonsingular centers.
>
> f is <u>strongly factorizable</u> if it is a composition of the form $g_1 \circ g_2^{-1}$, for g_1 and g_2 directly factorizable morphisms.
>
> f is <u>weakly factorizable</u> if it is an alternating composition $g_1 \circ g_2^{-1} \circ g_3 \circ g_4^{-1} \circ \ldots \circ g_n$ of directly factorizable morphisms and their inverses.

There is a conjecture that every birational morphism is strongly factorizable. It is true for surfaces, and no counterexample has yet been adduced in any higher dimension, but the combinatorial complexities, even in the case of three-folds, have discouraged work in that direction. Even very simple test cases have not yielded a general method.

The prospects of proving a weaker conjecture that every birational morphism is weakly factorizable look much brighter, particularly for threefolds, in light of the success of the "Mori program" for contracting birational morphisms (see Kollar's expository article [5] for an introduction to this theory). Since it is not directly relevant to this work we will give only a brief description.

Starting with a projective algebraic scheme, one can reach a simpler scheme called a "convenient model" by a finite sequence of two operations: divisorial contractions and " curve flips ". The divisorial contractions are allowed to introduce certain mild singularities called terminal singularities. The " flips " are isomorphisms outside of one irreducible curve, which is exchanged for another curve. We will have occasion in this work to investigate a particularly simple kind of " flip " which is called a " flop ". In a flop, a curve of self-intersection 0 is blown up and then contracted along a section to produce another curve of self intersection 0.

To explain the connection between the contraction theorem and the weak factorization conjecture, we will first describe Danilov's proof of the weak factorization theorem for toroidal schemes of dimension 3, beginning from the definition of a toroidal scheme for A^n.

Fix a set of transversal parameters $x_1 \dots x_n$. An affine toroidal scheme of dimension n is a scheme $f: T \to A^n$ with a birational morphism to A^n, such that all components of the exceptional divisor are defined locally by the vanishing of monomials (with positive and negative exponents) in $x_1 \dots x_n$. The simplest examples of toroidal schemes can be obtained by a sequence of blowings up whose centers are intersections of coordinate planes and exceptional divisors. The toroidal scheme can be described combinatorially by a dual graph of the coordinate planes and the exceptional divisors. A brief description of the construction of the dual graph can be found in [9]: Because they can be treated combinatorially via the dual graph, toroidal schemes have become the " white mice " of the geometry laboratory.

Shortly after Mori's first work on contractibility (which did not serve as a general induction step because the scheme being contracted was non-singular), Danilov used Mori's methods to prove that every toroidal morphism of 3-folds could be contracted to A^3 by a finite sequence of divisorial contractions and flips. However, Danilov went further. Choosing a fixed resolution for each of the terminal singularities appearing in this sequence of contractions, he then showed that each flip was weakly factorizable. By Hironaka's resolution of singularities it is possible to find directly factorizable toroidal morphisms $g_1: \tilde{X}_1 \to X_1$ and $g_2: \tilde{X}_2 \to X_2$ such that $\tilde{X}_1$ and $\tilde{X}_2$ are projective. The Danilov theorem proves that the canonical morphisms $h_1: \tilde{X}_1 \to A^3$ and $h_2: \tilde{X}_2 \to A^3$ are weakly factorizable, and thus the total composition $g_2 h_2^{-1} h_1 g_1^{-1}: X_1 \dashrightarrow X_2$ is weakly factorizable.

If the " convenient models " in the Mori program are not contractable to a lower dimensional variety, then they can be transformed into each other by a sequence of flops or their inverses. Thus in this case it is reasonable to hope that they could play the role of A^3 in a general weak factorization theorem for three-folds. In the successful completion of the Mori program for three-folds, it is thus reasonable to see the completion of the global framework for a weak factorization theorem for three-folds. The portion remaining would be to choose fixed resolutions of the terminal singularities and to show that the resulting "small steps", corresponding to a single divisorial contraction or flip, are weakly factorizable.

This article belongs to a different and more modest line of work on the factorization problem, but one which has produced methods well-suited to an attack on the remaining local steps in a weak factorization theorem.

Two decades ago, shortly after Hironaka's work on resolution of singularities had cleared away one great question mark in the theory of birational morphisms, Moishezon embarked on a program, later pursued by his students, to classify birational morphisms with small canonical divisors. He felt that the entire subject would benefit if there would be more information on which phenomena "occur in nature", and which do not. In the following summary of results to date, all spaces are nonsingular algebraic spaces (not necessarily schemes), and all exceptional divisors have normal crossings.

1) (1967) Moishezon proved that every birational morphism whose exceptional divisor contained a single irreducible component was a blowing up with nonsingular center.

2) (1976) Schaps [8] showed that every birational morphism of three-folds collapsing two irreducible components was directly factorizable.

3) (1981) Teicher [10] demonstrated that every birational morphism of four-folds collapsing two components was directly factorizable.

4) (1981) Crauder [1] proved that every birational morphism of three-folds collapsing three components to a point was "locally factorizable" and Schaps [9] obtained the same result for birational morphisms of three-folds collapsing three components to a nonsingular curve.

During this period of the work (2)-(4) described above, there was also work in various directions by Danilov, Kullikov, Pinkham, and Persson, summarized in Pinkham's expository article [7]. Of these efforts the one most directly relevant to this paper was Danilov's result in [2] that a birational morphism with one dimensional fibers is locally factorizable.

At each stage pushing the classification further was difficult and required the introduction of more sophisticated techniques, designed to show that the morphism under consideration was in some sense locally toroidal. These techniques have been carried further and put on a general footing in the current paper. As an illustration of how the general techniques can be applied in practice to analyze a birational morphism, we then settle the next two cases in the classification of birational morphisms of three-folds with small canonical divisor: four components collapsing to a point and three components collapsing to a singular curve.

Although these techniques were developed for nonsingular algebraic spaces, the combinatorial analysis depends only on the generic point of each exceptional divisor and is thus indifferent to possible isolated singular points. Furthermore, the "quasi-blowings up" introduced to get around singularities in the fundamental locus could as well be applied to get around singularities in the space, so the main formulae, like the "additivity formula", should hold just as well for spaces with terminal singularities.

In § 1 and § 2 we develop the new tools and notation required to efficiently use large quasi-factorization sequences. The application of these tools is then illustrated in § 3 and § 4, showing that with one exception a birational morphism of algebraic spaces which collapses four components to a point is locally factorizable. In order to aid the reader in fitting these methods into the context of previous work on the subject, we give a brief introduction to the various methods.

A. Local factorizability (1.1-1.4): This is a transposition into the category of algebraic spaces of a long standard analytic technique of creating nonprojective morphisms by blowing-up a smooth branch of a singular curve in a small neighborhood of the singularity. The operation of taking an etale neighborhood in which the branches of a curve are irreducible will substitute for the analytic operation of taking a small neighborhood.

Historically, the standard example of a locally factorizable morphism is obtained by blowing-up one branch of a double node before the other. Crauder [1] and the author [9] came upon examples of morphisms collapsing three normally crossing components without self-intersections, called the "wagon wheel" in [1] and the "bow-tie" in [9]. For the four component case there were so many different types of examples that it was simpler to define a general class of such examples than to enumerate them.

B. Point Obstructions (1.5-1.8): We make a slight extension of Danilov's theorem in [2] about the factorizability of morphisms with one dimensional fiber, requiring only the "generic" part of the fiber to be one dimensional. The change is made possible by the application of the "transversal test curve" lemmas in [9] to Danilov's basic lemma.

Together with a double induction on the number of curve and point components in the fiber over a point, this permits us to show that all obstructions to local factorizability are of the type we call "point obstructions", in which the "generic" fiber over a bad point y is two dimensional but the morphism does not factor through the blowing-up of y. The set on which the morphism does not factor is called the pinch locus. In [1] and [9] it was necessary to show that the pinch locus was empty. Here the pinch locus is blown-up and a combinatorial analysis of the resulting components restricts the types of components of the canonical divisor in which the pinch locus can lie.

C. Quasi-factorization (1.11-1.13): In [1], [2], [8], [9], and [10], the work of factoring a morphism $f: X \to Y$ proceeds by proposing a quasi-factorization $g: Y_1 \to Y$, generally a blowing-up with non-singular center, and trying to analyse or eliminate the set on which the induced correspondence $f_1: X \dashrightarrow Y_1$ is not welldefined. As f becomes more complicated, it is necessary to use a sequence of blowings-up for the quasi-factorization, and to permit non-singular centers, creating problems which are solved here by the introduction of quasi-blowings-up and accessible components.

D. The weight vector: The multiplicities r_i of the components D_i of the relative canonical divisor K_f of a morphism f have played a crucial role in all attempts to factor f. Crauder, in extending a method pioneered by Kullikov, also introduces the multiplicities s_i of the D_i in $f^*(H)$ for a generic hyperplane H. In this work we must consider the multiplicities in $f^*(H)$ for special H as well, mimicking a "toroidal" analysis of the components. We also convert the "excess" defined in [9] into a measure of the extent to which a component of K_f fails to be toroidal.

E. The additivity formula (2.4-2.5): The central idea of Danilov's [2] is to decompose a morphism into composition of correspondences and to compare the multiplicities obtained on the two sides of the formula

$$K_{f\circ p} = p^*(K_f) + K_p$$

This procedure is formalized in the additivity formula and extended to include the other components of the weight vector. In 2.5 it is generalized to quasifactorization sequences with a number of factors.

F. Well-definedness (2.6-2.7, 2.12): Earlier lemmas on the well-definedness of a map f_1 to a blowing-up are extended to a map f_m to a quasi-factorization sequence of length m.

G. Analysis of point obstructions (2.8-2.11): The major working tools used in this paper for the analysis of point obstructions are the inequalities in lemmas 2.9 and 2.11, which restrict the possible values of r_i and s_i for components D_i of K_f containing the pinch locus. These lemmas represent a considerable strengthening of the lemmas in § 2 of [9].

We will use K_f to denote the canonical divisor of a birational morphism $f: X \to Y$, and S_f for the fundamental locus in Y, the closed subalgebraic space on which f^{-1} is not an isomorphism. Points of K_f will be referred to as singleton points, double points and triple points according to the number of components of K_f containing the point. Since we are working with three-folds and will assume that the canonical divisor has normal crossings, no more than three divisors can come together at one point. The support of a divisor D will be denoted by $|D|$.

Let Y be an algebraic space, obtained by patching together schemes via an etale equivalence relation. Let y be a closed point of Y. Assume that the ground field k is algebraically closed. An <u>etale neighborhood</u> of y is an algebraic scheme together with an etale morphism $e: W \to Y$ such that the inverse image of y is a single point w. (If k were not algebraically closed, we would also have to require that the residue fields at y and w be the same.) Since e is etale, the completion of the structure sheaf at y is isomorphic to the completion of the structure sheaf at w. An <u>etale cover</u> $\{e_i: W_i \to Y\}$ is a set of etale morphisms into Y such that for any morphism $g: Z \to Y$, with Z an affine scheme, the images of $W_i \times_Y Z$ in Z form a Zariski cover of Z.

If $f: X \dashrightarrow Y$ is a birational correspondence which is well-defined at the generic point of an irreducible subspace W of X, then we denote by $f[W]$ the closure of the image of the generic point of W, and call this the <u>strict image</u> of W. If in place of f we have an inverse correspondence $g^{-1}: X \dashrightarrow Y$, then we will call $g^{-1}[W]$ the <u>strict preimage</u>. A <u>test curve</u> $\Gamma \subset Y$ for $f: X \dashrightarrow Y$ is an irreducible curve intersecting the set S_f on which f is not an isomorphism in a single point y and having a unique analytic branch at y. The point x at which the strict preimage $f^{-1}[\Gamma]$ in X intersects the set on which f is not an isomorphism is called the <u>closure point</u> of the test curve.

The following four lemmas from [9] will be used repeatedly in § 3 and § 4, so we quote them here for convenient reference:

<u>Lemma 1.1</u> (of [9]). Let $f: X \to X'$ be a birational morphism of nonsingular algebraic spaces of dimension n, and let W be a nonsingular subspace in the complement of the set on which f is an isomorphism. Let $g: X'_1 \to X'$ be the blowing-up whose center is the ideal I_W of W. Let $x_1 \in X'_1$ be a point on the fiber $g^{-1}(x')$, and let Γ_1 be a closed curve which intersects this fiber only at x_1 and has a single analytic branch there. Let $\Gamma' = g(\Gamma_1)$, and $\Gamma = f^{-1}[\Gamma']$. Let H be a generic hyperplane containing W, and suppose

(a) that Γ contains a point x of $f^{-1}(x')$, which we will call the closure point of Γ_1,

(b) x lies on components $E_1, \ldots, E_r$, each E_i having multiplicity m_i in $f^{-1}(I_W)O_{X,x}$, and

(c) $m_1 +, \cdots, + m_r \geq dg(T' \cdot H) = dg(\Gamma_1\, g^*(H))$.

Then Γ is nonsingular at x, transversal to each of $E_1, \cdots, E_r$, and $f^{-1}(I_W)O_{X,x}$ is invertible at x, being generated by $t_1^{m_1} \cdots t_r^{m_r}$, for t_i a local equation of E_i. Thus $f_1 : X \dashrightarrow X_1'$ is well defined in a neighborhood of x and $f_1(x) = x_1$.

Lemma 1.2 (of [9]). Let $f : X \to X'$ be a proper birational morphism of n-folds. Let W be a nonsingular subspace of D', the set on which f^{-1} is not an isomorphism, and define the blowing up g and $W_1 = g^{-1}(W)$ as in lemma 1.1 with $f_1 = g^{-1}$ of the induced correspondence. Let $D_1, \ldots, D_m$ be the components of the exceptional divisor of f, with $D_1, \ldots, D_r$ contained in $f^{-1}(W)$.

(i) Suppose $f_1^{-1}[W_1]$ is a divisor D_1. Then f_1^{-1} is an isomorphism on the set

$$W_1 - \bigcup_{1<j\leq r} f_1[D_j] - \bigcup_{j>r} f_1(D_j \cap f^{-1}(W))$$

(ii) If $f^{-1}(W)$ is a union of divisors, and in particular if $W = D'$, then $f_1^{-1}[W_1]$ is a divisor, and on $W_1 - \bigcup_{j\neq 1} f_1[D_j]$, f_1^{-1} is an isomorphism.

Notation ([9]). Let $f : X \to X'$ be a birational map of nonsingular n-folds, collapsing a divisor D with normal crossings to a subspace D' of X', of codimension c', greater than 1. Let $z_1', \ldots, z_n'$ be local parameters centered at x' in X'. Let $z_1, \ldots, z_n$ be the liftings to regular functions on X. We define the canonical divisor K_f of f by $K_f = \mathrm{div}(f^*(\omega_{X'}) \otimes \omega_X^{-1})$. Locally at a point of $f^{-1}(x')$ this is the divisor of the form $dz_1 \wedge \ldots \wedge dz_n$.

Let x be a point on the intersection of components $D_1, \ldots, D_s$ of D. Letting t_i be a local equation for D_i, we extend this to a set $t_1, \ldots, t_n$ of local parameters of X at x. Suppose that the order of z_i on D_j is at least a_{ij}, so that we can write

$$z_i = t_1^{a_{i1}} \cdots t_s^{a_{is}} q_i.$$

The canonical divisor at x of the map f is given by

$$t_1^{(\Sigma a_{i1})-1} \cdots t_s^{(\Sigma a_{is})-1} \det(J') \qquad (*)$$

for some matrix J'.

Let r_j be the multiplicity of D_j in the canonical divisor of the map f, and set $e_j = r_j - (\sum a_{ij}) + 1$, which by (*) is nonnegative. We will call it the excess of r_j. Let $e = e_1 + \ldots + e_s$.

Lemma 2.2 [9] Let x be a point lying on a unique component D_1 of D, and suppose that $f(D_1) \subset W$, the subspace defined by the vanishing of $z'_1, \ldots, z'_{c'}$. Let I_W be the reduced ideal of W, and suppose that the multiplicity of D_1 in $f^{-1}(I_W)$ is at least b. Let $f_1 : X \dashrightarrow X'_1$ be the map to the blowing up of W. It is well defined at x if

(i) $r_1 = bc' - 1$, so that $e = 0$, or
(ii) $r_1 = bc'$, and f_1^{-1} doesn't collapse the exceptional divisor.

Lemma 2.3 [9] Let x be a point lying on only two components D_1 and D_2 of D. Suppose that $f(D_i) = W_i$ is defined by the vanishing of local coordinates $z'_1, \ldots, z'_{c_i}$, for $i = 1, 2$, and $c_2 \geq c_1$. Suppose that D_i has multiplicity b_i in the lifting of the ideal of W_i to X. Let f_i be the map to the blowing-up of W_i.

(i) If $e_1 = e_2 = 0$, then f_1 and f_2 are both well defined.
(ii) If $e = e_1 + e_2 = 1$, then either f_1 or f_2 is well defined at x.
(iii) If $c_2 = c_1 + 1$, and $e = 1$, and f_1^{-1} does not collapse the exceptional divisor, then f_1 is well defined at the generic point of $D_1 \cap D_2$.

For ease of reference, before beginning the new definitions and lemmas, we append a list of the terms which will be defined in the body of the paper, and the number of the corresponding definition: root tree, 1.1; partial factorization tree, 1.2; local factorization tree, 1.3; locally factorizable morphism, 1.4; point obstruction, 1.5; strict preimage $f^{-1}[y]$ of a point, 1.7; pinch locus, $P_y(f)$, 1.9; quasi-blowing-up, 1.11; quasi-factorization sequence, 1.13; dominated by f, 1.13; $r_f(F)$, 2.1; $w_f(F)$, 2.1; $s_f(F, H)$, 2.1; $u_f(F; H_1, \ldots, H_r)$, 2.1; excess, $ex_f(F; H_1, \ldots, H_r)$, 2.2; total excess at x, 2.10.

§ 1. Local factorizability.

We wish to call a morphism of algebraic spaces locally factorizable if it "locally" factorizable by blowings-up with nonsingular centers. There are two factors complicating this basically simple idea. The first is that we must work in the etale topology so that the maps from our local neighborhoods are not injective; the second is the process of localization proceeds in fibers over the original base.

EXAMPLE: Local factorization: Suppose Y is a smooth 3-dimensional scheme. First we blow up a point y, giving a space Y'_1 with exceptional divisor M_1. We now want to blow up a curve C in M_1 with a unique nodal singularity at a point y_1. We choose an etale cover of Y'_1, consisting of two Zariski open affine subsets Y_{11}, Y_{12} of Y'_1 not containing y_1, and an etale neighborhood Y_{13} of y_1 in which the preimage of one branch of the node is irreducible. In Y_{11} and Y_{12} we just blow up the curve C getting Y'_{11} and Y'_{12}. In Y_{13} we first blow-up one branch of the node to get Y'_{13}, then cover this with Zariski open affine neighborhoods Y_{13i} and blow up the remaining branches of C to get schemes Y'_{13i}. Then Y'_{11}, Y'_{12} and the Y'_{13i} patch together to form an algebraic space X.

For our purposes we may assume the algebraic space X and the morphism $f: X \to Y$ to be given, so that we avoid arguments about etale patching. Showing that f is locally factorizable means constructing a tree of successively simpler morphisms $f'_\alpha: X_\alpha \to Y'_\alpha$, such that if f'_α is not an isomorphism, then Y'_α has an etale covering $\{e_\beta: Y_\beta \to Y'_\beta\}$ by schemes Y_β with the following property: Let $X_\beta = X_\alpha \times_{Y'_\alpha} Y_\beta$ be the pullback of the pair of morphisms (f'_α, e_β). Let $f_\beta: X_\beta \to Y_\beta$ be the projection onto the second factor. Then f_β can be factored as the composition of a blowing up $g_\beta: Y_{\beta'} \to Y_\beta$ with nonsingular center and a morphism $f'_\beta: X_\beta x_{Y'_\alpha} Y_\beta \to Y'_\beta$. An obstruction to locally factoring f is a morphism $f'_\alpha: X_\alpha \to Y'_\alpha$ for which no such covering exists. If there are no such obstructions, then we will show in lemma 1.6 below that after a finite number of steps all terminal morphisms $f'_\alpha: X_\alpha \to Y'_\alpha$ will be isomorphisms. These X_α will form an etale cover of X.

DEFINITION 1.1.: A connected tree will be called a root tree if it has a distinguished initial vertex $v_\emptyset$. The choice of $v_\emptyset$ implies a unique direction away from $v_\emptyset$ on each edge t, and every other vertex v has a unique entering edge, the last step on the unique path connecting $v_\emptyset$ to v. If the branches leaving each vertex are numbered by natural numbers, then each path of length m out from $v_\emptyset$ is uniquely determined by an m-tuple $\beta = (\beta_1, \dots, \beta_m)$ of numbers listing the branch chosen at each step. We index each vertex v_β and its entering edge t_β by the m-tuple of the unique path connecting it to $v_\emptyset$. We define the predecessor $\beta^- = (\beta_1, \dots, \beta_{m-1})$ of a non-empty multi-index β, and the length $l(\beta) = m$. A vertex with no edges leaving it will be called terminal.

Let us now suppose that we have a morphism $f: X \to Y$ of smooth algebraic spaces, and we wish to discover if it can be factored locally. Blowing-up commutes with etale base extension. If Y has an etale cover $\{e_i: Y_i \to Y\}$ in which the base extensions $f_i: X \times_Y Y_i \to Y_i$ of f all factor locally through blowings-up with smooth centers, then the base extension

$\tilde{f}: X \times_Y \tilde{Y} \to \tilde{Y}$ by the Henselization $\tilde{Y}$ will also factor through a blowing-up with nonsingular center. Conversely, if f factors locally at each point, then since the number of possible centers is limited by the number of smooth points or branches in the fundamental locus S_f, it will be possible to find a finite etale cover $\{Y_i\}$ of Y such that each f_i factors. We could, therefore, give a recursive definition of local factorizability by requiring that the f factor locally at each point of Y, and that the factored morphism be locally factorizable. Instead we take the less canonical approach of choosing etale covers and constructing a factorization tree.

DEFINITION 1.2: A partial factorization is a root tree with

a) Each vertex v_β labelled by a morphism $f'_\beta: X_\beta \to Y'_\beta$

b) Each edge t_β labelled by a pair of morphisms (g_β, e_β), where $g_\beta: Y'_\beta \to Y_\beta$ is a blowing-up with nonsingular center B_β, and $e_\beta: Y_\beta \to Y'_{\beta^-}$ is etale such that, for $\beta \neq \emptyset$, the space X_β is the fiber product $X_{\beta^-} \times_{Y'_{\beta^-}} Y_\beta$ induced by the pair of morphisms (f'_{β^-}, e_β), and the composition $f_\beta = f'_\beta \circ g_\beta$ is the base extension of f'_{β^-} by e_β

$$\begin{array}{ccccc} X_{\beta^-} & \xleftarrow{\pi_\beta} & X_\beta & = & X_\beta \\ {\scriptstyle f'_{\beta^-}}\downarrow & & {\scriptstyle f_\beta}\downarrow & & \downarrow{\scriptstyle f'_\beta} \\ Y'_{\beta^-} & \longleftarrow & Y_\beta & \xleftarrow{g_\beta} & Y'_\beta \end{array}$$

DEFINITION 1.3: A vertex v_α in a partial factorization tree will be called covered if $\{e_\beta: Y_\beta \to Y'_{\beta^-}\}_{\beta^-=\alpha}$ is an etale cover of Y'_α. The tree will be called a local factorization tree if every non-terminal vertex v_α is covered, and if for every terminal vertex v_β, f'_β is an isomorphism.

REMARK: If v_α is a covered vertex, then $\{\pi_\beta: X_\beta \to X_{\beta^-}\}_{\beta^-=\alpha}$ is the pullback to X_α of an etale cover of Y_α, and is therefore an etale cover of X_α. By an induction on path length, a local factorization tree provides an etale cover $\{p_\beta: X_\beta \to X\}$, where the β are the indices of the terminal vertices, and the p_β are compositions of morphisms $\pi_\gamma: X_\gamma \to X_{\gamma^-}$ for the various predecessors $\gamma = \beta^-, \beta^{--}, \dots, \{\beta_1\}$ of β.

DEFINITION 1.4: A birational morphism $f: X \to Y$ of algebraic spaces which can be associated to the initial vertex of a local factorization tree will be called locally factorizable.

REMARK. This is a local property, and thus for all practical purposes we may assume that Y is a scheme.

EXAMPLE. Let Y be affine three-space, A^3, with coordinates x, y, z.

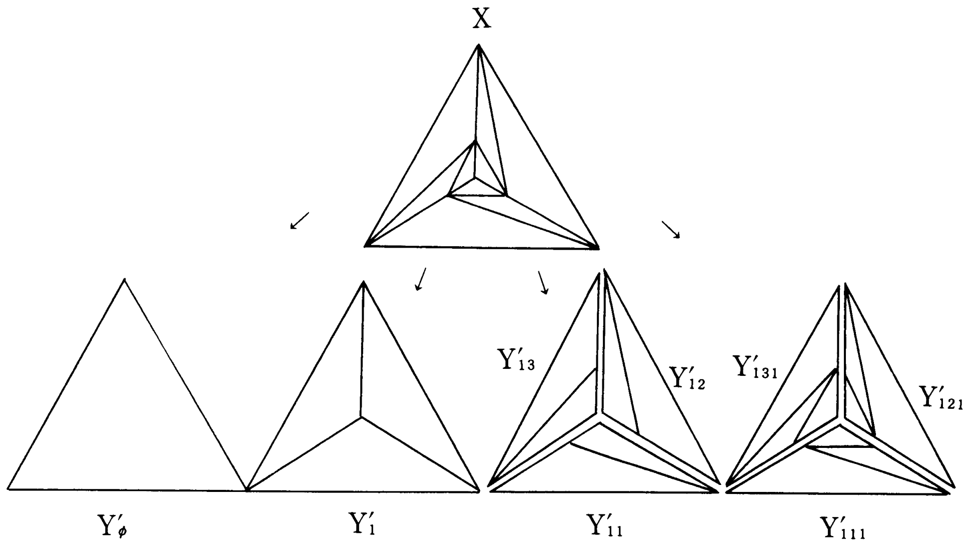

Fig. 1

We construct a smooth non-projective morphism $f:X\to Y$ as follows. First blow up the origin, getting a space Y_1'. It can be covered by three neighborhoods Y_{11}, Y_{12}, Y_{13} obtained by removing the strict preimages of the three coordinate planes, respectively. Each Y_{1i} is again isomorphic to A^3, with two of the coordinate axes contained in the exceptional divisor of the induced morphism from Y_{1i} to Y. Blow up the coordinate axes in cyclic order, getting first $Y'_{1i}=Y_{1i1}$ and then Y'_{1i1}. Set $X_{1i1}=Y'_{1i1}$, and patch together X_{111}, X_{121} and X_{131} to get X. (See figure 1.)

This morphism is strongly factorizable, so it can be obtained without recourse to local neighborhoods, but the local description has the advantage of being symmetrical, and not introducing extraneous components which must later be removed.

We are interested in determining the obstructions to constructing a local factorization tree for a morphism f_1 in the case of three-folds.

DEFINITION 1.5. Let $f:X\to Y$ be a proper birational morphism of algebraic spaces. A point $y\in Y$ will be called a <u>point obstruction</u> if, when $\tilde{Y}$ is the Henselization at y, $\tilde{f}:X\times\tilde{Y}\to\tilde{Y}$ does not factor through the blowing up of any smooth subscheme of $\tilde{Y}$.

LEMMA 1.6. *If $f:X\to Y$ is a proper birational morphism of three-folds which is not locally factorizable, then any partial factorization tree for f can be extended until it encounters a vertex morphism $f':X'\to Y'$ containing a point obstruction at a point $y'\in Y'$.*

PROOF: We first show that any uncovered vertex v_α with morphism $\bar{f}'_\alpha:X_\alpha\to Y'_\alpha$ can be covered unless Y'_α has a point obstruction. The fundamental locus $S'_{\bar f_\alpha}$ in $Y_{\alpha'}$ has dimension ≤ 1. Over the generic point of each curve component of $S'_{\bar f_\alpha}$, we have unique factorization, from the factorization theorem for surfaces. After removing a finite number of points $\{y_1, \dots, y_r\}$, we can find an etale cover $\{e_j : W_j \to Y'_\alpha\}_{j=r+1}^{s}$ of $Y - \{y_1, \dots, y_r\}$ such that for each $j = r+1, \dots, s$, $W_j \times_{Y_{\alpha'}} S_{\bar f'_\alpha}$ is smooth and the base extension of $\bar f'_\alpha$ by W_j is directly factorizable. Either $\bar f'_\alpha$ has a point obstruction at one of the Y_i, or else for each y_i we can find an etale neighborhood e_i: $W_i \to Y'_\alpha$, such that the image of W_i in Y'_α contains none of the other points y_j, $j \neq i$, and such that $\bar f'_\alpha$ factors locally through some blowing up.

It remains only to show that this process of covering uncovered vertices cannot continue indefinitely. Since $f : X \to Y$ is not locally factorizable, we then see that at some point we must encounter a point obstruction.

For each birational morphism $f : X \to Y$ of 3-folds, and each point $y \in S_f$, we define $N_{f,y} = (N_0, N_1)$, where N_0 is the number of irreducible surfaces in $f^{-1}(y)$, and N_1 is the number of irreducible curves. We order these pairs lexicographically, denoting the order relation by "$\leq$", and let

$$N_f = \max_{y \in S_f} N_{f,y}$$

It suffices to show that as one proceeds outward along any branch in a partial factorization tree, N_f decreases. In fact, by applying induction, it suffices to show this for one step.

We proceed from a vertex labelled by $f'_\alpha : X_\alpha \to Y'_\alpha$ to the following vertex $f'_\beta : X_\beta \to Y'_\beta$, with $\beta^- = \alpha$, in two steps: first we take an etale morphism $e_\beta : Y_\beta \to Y'_\alpha$, then we factor through a blowing-up $g_\beta : Y'_\beta \to Y_\beta$. The first step replaces the morphism $f'_\alpha : X_\alpha \to Y'_\alpha$ by a morphism $f_\beta : X_\beta \to Y_\beta$, with $X_\beta = X_\alpha \times_{Y'_\alpha} Y_\beta$. If $\hat y \in Y_\beta$ is such that $e_\beta(\hat y) = y$, then $f_\beta^{-1}(\hat y) \underset{\sim}{\to} f_\alpha'^{-1}(y)$. This follows from the fact that the Henselization $(\tilde Y_\beta)_{\hat y}$ of Y_β at $\hat y$ is isomorphic to the Henselization $(\tilde Y'_\alpha)_y$ of Y'_α at y, and that $f_\alpha'^{-1}(y)$ and $f_\beta^{-1}(\hat y)$ are the closed fibers, respectively, of $X_\alpha \times_{Y'_\alpha} (Y'_\alpha)_y$ and of $X_\beta \times_{Y_\beta} (Y_\beta)_{\hat y}$. Thus $N_{f_\beta, \hat y} = N_{f'_\alpha y}$, whenever $e_\beta(\hat y) = y$, and we conclude that $N_{f_\beta} = N_{f'_\alpha}$.

If remains to show that $N_{f'_\beta} < N_{f_\beta}$. Let $\hat y$ be a point of Y_β for which $N_{f_\beta} = N_{f_\beta, \hat y}$, *and let* y' be a point of the blown-up scheme Y'_β at which $g_\beta(y') = \hat y$. Then $f_\beta'^{-1}(y') \subset f_\alpha'^{-1}(y)$, so $N_{f'_\beta, y'} \leq N_{f'_\alpha, y} = N_{f_\beta, \hat y}$. At least one component of $f_\alpha'^{-1}(y)$ must map onto $g_\beta^{-1}(\hat y)$ under f'_β. Thus $f_\beta'^{-1}(y')$ is a proper subset of $f_\alpha'^{-1}(y)$. $f_\beta'^{-1}(y')$ either has fewer surface components, or, if it has the same number of surface components, then it has fewer curve components.

EXAMPLE: In Fig. 2 we give the dual graph of the minimal toroidal

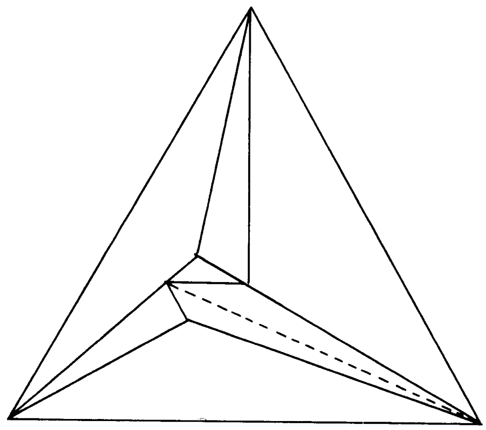

Fig. 2

example of a point obstruction, given by Oda in [6].

In view of lemma 1.6, the only obstructions to local factorizability are point obstructions. We wish to give some restrictions on these obstructions.

DEFINITION 1.7. Let $f: X \to Y$ be a birational morphism, and let y be a point of Y. Let Y_1 be the blowing up of y, with exceptional divisor M_1, and induced correspondence $f_1: X \dashrightarrow Y_1$. We define the strict preimage $f^{-1}[y]$ to be the strict preimage $f_1^{-1}[M_1]$ of M_1. We can similarly define $f^{-1}[W]$ for any subalgebraic space W, by blowing up at the generic point and taking the image.

REMARK. For three-folds, $f^{-1}[y]$ is in fact a component of $f^{-1}(y)$. $f^{-1}[y]$ is irreducible, being the strict preimage of an irreducible divisor, so if $\dim f^{-1}[y]=2$ it is clearly a component. The case $\dim f^{-1}[y] \leq 1$ will be treated below, where we will show that it is a curve contained in a unique exceptional divisor of f whose image is larger than y.

We now turn to the results of Danilov, which will give us additional information about the structure of point obstructions. Given a morphism $f: X \to Y$ of algebraic spaces, we let K_f be the relative canonical divisor of f, $K_f = K_X - f^*(K_Y)$, and we let ξ be the generic point of the strict preimage $f^{-1}[y]$ of a point y. Then Danilov proves, in Prop. 3.4 of [2], the following

PROPOSITION:

Let $f: X \to Y$ be a proper birational morphism of nonsingular schemes of dimension r over an algebraically closed field K. Suppose Y is a local Henselian scheme obtained by Henselization of a smooth K-scheme at the closed point y and $\dim f^{-1}(y) \leq 1$. *Then*

a) The codimension of ξ in X is equal to r-1, i. e. ξ is the generic point of a curve component of $f^{-1}(y)$.

b) K_f is non-singular at ξ.

c) The subscheme $f^{-1}(y)$ is non singular at ξ.

REMARK. We allow X to be an algebraic space, and replace the requirement dim $f^{-1}(y)\leq 1$ by dim $f^{-1}[y]\leq 1$. The entire proof carries over intact. We use this form of the lemma to prove the following variation of Danilov's Theorem 3.1:

Theorem 3.1:

LEMMA 1.8. *Let $f:X\to Y$ be a proper birational morphism of smooth algebraic spaces of dimension 3. Suppose $y\in Y$ is a point for which f^{-1} is not an isomorphism and* dim $f^{-1}[y]\leq 1$. *Then there is an etale neighborhood W of y in which $f_W: X\times W\to W$ factors through the blowing up of a smooth curve $B\subset W$.*

PROOF: We follow the outline of the proof of Danilov's theorem. Letting $\tilde{Y}$ be the Henselization of Y at y, and $\tilde{f}: X\underset{Y}{\times}\tilde{Y}\to\tilde{Y}$ we conclude from the proposition above that the generic point ξ of $f^{-1}[y]$ lies in a single component D_1 of K_f and that $f^{-1}(y)$ is non-singular there. Let c be a general point of $f^{-1}[y]$, and let $\tilde{Z}$ be a curve in $\tilde{X}=X\times\tilde{Y}$ which is transversal to $f^{-1}[y]$ at c. It intersects $f^{-1}(y)$ at a finite number of points. The morphism from $\tilde{Z}$ to $\tilde{Y}$ is quasifinite, $\tilde{Y}$ is Henselian, and hence $\tilde{Z}=\tilde{Z}'\cup\tilde{Z}''$ is a disjoint union with $\tilde{Z}'\cap f^{-1}(y)=c$, a single point, (EGA [4] 18.5.11). We replace $\tilde{Z}$ by $\tilde{Z}'$ and let $\tilde{B}\subset\tilde{Y}$ be the image of $\tilde{Z}$. The induced morphism $\pi:\tilde{Z}\to\tilde{Y}$ is a finite morphism, and $\pi^{-1}(y)=c$ is an isomorphism, so by Nakayama's lemma we conclude that π is a closed embedding and thus $\tilde{B}=\pi(\tilde{Z})\xrightarrow{\sim}\tilde{Z}$ is non-singular. Since $\tilde{Z}\subset D_1$, $\tilde{B}=\tilde{f}(\tilde{Z})\subset\tilde{f}(\tilde{D}_1)$, and since both $\tilde{B}$ and $\tilde{f}(\tilde{D}_1)$ are irreducible curves, $\tilde{B}=\tilde{f}(\tilde{D}_1)$.

For any etale neighborhood $e: W\to Y\circ f\ y$, let $f_W: X_W\to W$ be the induced birational morphism. Let D_1 be the unique component of K_{f_W} containing $f_W{}^{-1}[y]$, $\bar{y}\in e^{-1}(y)$ and $B=f_W(D_1)$, an irreducible curve in W. Since $\tilde{Y}$ is the inverse limit of the W, and $\tilde{B}$ maps to B under the morphism $\tilde{Y}\to W$, there must exist a neighborhood W in which $\tilde{B}$ is the unique preimage of B and thus B has a unique smooth branch at $\bar{y}$. We choose the neighborhood sufficiently small that f_W has a unique factorization over every other point of W. Let $g: W'\to W$ be the blowing-up of B, with exceptional divisor M, and induced morphism $f_1': X_W\to W'$ By lemma 1.1 of [9], the strict preimage $f_1'^{-1}[M]$ is a surface generically isomorphic to M. We wish to show that it is D_1. Let H be a generic hyperplane in W containing B. Over the general point w of B, since $g^{-1}[H]$ intersects M at a point where it is isomorphic to $f_1'^{-1}[M]$, $f_{\bar{W}}^{-1}[H]$ must intersect $f_{\bar{W}}^{-1}[W]$ at a point of $f_1'^{-1}[M]$. On the other hand, it must intersect $f^{-1}(y)$ at a generic point, i.e., at a point of $f^{-1}[y]$.

Let Γ be a curve through y transversal to H. $f^{-1}[y]\subset|f_W^*(\Gamma)|$. By the projection formula deg $f_W^*(\Gamma)\cdot f_{\bar{W}}^{-1}[H]=$deg $\Gamma\cdot H=1$, so $f_{\bar{W}}^{-1}[H]$ is transversal to $f^{-1}[y]$ at c. It intersects D_1 in a curve in a neighborhood of the intersection point, so the remaining points of that curve must belong to fibers of other points y' of B. Thus $D_1=f_{\bar{W}}^{-1}[M]$ as desired.

By Lemma 1.2 of [9], f_1' is an isomorphism at every singleton point of D_1. Since f_1 is well defined over every point of B except y, and on $g^{-1}(y)$ $f_1'^{-1}$ is an isomorphism except at isolated points, we conclude by lemma 1.4 of [9] that f_1' is well-defined, and thus we have the desired factorization.

DEFINITION 1.9. Let $f:X\to Y$ be a morphism of 3-folds, and let y be an element of Y. We let Y_1 be the blowing-up of y, with $f_1:X\dashrightarrow Y_1$, the induced correspondence. The locus in X on which f_1 is <u>not</u> well-defined will be designated by $P_y(f)$, and will be called the <u>pinch locus</u>.

LEMMA 1.10. *In a 3-fold, we have the following alternative characterizations of the pinch locus*:

<u>(*a*)</u>: *If H_1 and H_2 are two generic hyperplanes through y, then* $P_y(f)\cup f^{-1}[H_1\cap H_2]=f^{-1}[H_1]\cap f^{-1}[H_2]$.

<u>(*b*)</u>: *Suppose y is a point obstruction. Let G_f be the graph of f_1, with projection π_1 on X and π_2 the projection on Y_1. Then $P_y(f)=\cup\pi_1(S)$, where S ranges over all the irreducible surfaces in G_f, such that* dim $\pi_1(S)=$dim $\pi_2(S)=1$.

PROOF: Teicher proved in [10] that $P_y(f)=f^{-1}[H_1]\cap f^{-1}[H_2]\cap f^{-1}[H_3]$ for generic H_1, H_2, H_3. In (a) we strengthen that result by eliminating the third hypersurface $f^{-1}[H_3]$.

<u>(a)</u>: If H_1 and H_2 are not tangent at y, then if $g: Y_1\to Y$ is the blowing-up of y, $g^{-1}[H_1]$ and $g^{-1}[H_2]$ do not intersect on the exceptional divisor except at $g^{-1}[H_1\cap H_2]$. Thus if $f_1: X\dashrightarrow Y_1$ is well-defined at $x\in|K_f|$, $f^{-1}[H_1]$ and $f^{-1}[H_2]$ do not intersect there unless $x\in f_1^{-1}(g^{-1}[H_1\cap H_2])$. For generic choice of H_1 and H_2, f_1^{-1} will be well-defined on $g^{-1}[H_1\cap H_2]$ so $f_1^{-1}(g^{-1}[H_1\cap H_2])$ will just be $f^{-1}[H_1\cap H_2]$. We conclude that $f_1^{-1}[H_1]\cap f_1^{-1}[H_2]=P_y(f)\cup f^{-1}[H_1\cap H_2]$.

<u>(b)</u>: Suppose y is a point obstruction, whence, by lemma 1.3, there is a

component D_1 of K_f generically isomorphic to the exceptional divisor M_1 of $g: Y_1 \to Y$, the blowing up of y. The set $P_y(f)$, on which $f_1: X \dashrightarrow Y_1$ is not well-defined, is the fundamental locus S_{π_1} of the first projection π_1 from the graph G_{f_1}. By Zariski's main theorem each component of $P_y(f)$ is the image of a surface S in G_{f_1}. The image of S in Y_1 is also of dimension less than 2, since S is not the unique surface D_1 in G_{f_1} which is generically isomorphic to D_1 in X and to M_1 in Y. Since $S \subset \pi_1(S) \underset{Y}{\times} \pi_2(S)$, both projections must be of dimension 1. Q. E. D.

Our basic approach to analyzing the pinch locus will be to blow-up bad curves on the Y_1 side of the "valley"

$$\begin{tikzcd} X \arrow[dr] & & Y_1 \arrow[dl] \\ & Y & \end{tikzcd}$$

For this purpose regular blowings-up will not always suffice, and we will occasionally need a slightly more general technique.

DEFINITION 1.11: A quasi-blowing up with center B and accessible component M is a locally factorizable morphism $h: V \to Y$ such that $M \subset \operatorname{Supp} K_n$ is an irreducible divisor without self intersections, $B = S_h$ is irreducible, and h is generically the blowing up of B with exceptional divisor M. Furthermore, for every singleton point v of M, i.e. every point contained in no other component of K_n, we presume that after base extension by the Henselization $\tilde{Y}$ of Y at $f(v)$, h factors through the blowing up of a smooth branch $\tilde{B}$ of B, and is isomorphic to this blowing up at v. The singleton points v of M are called accessible points.

LEMMA 1.12: *Let $h: V \to Y$ be a quasi-blowing up, let $f: X \to Y$ be a birational morphism and let $\tilde{B}$ be the local center in the henselization $\tilde{Y}$ of a point y. Let v be an accessible point of $h^{-1}(y)$. Let Γ be a nonsingular curve intersecting $h^{-1}(\tilde{B})$ transversally at v, which we will call a test curve. The closure point of Γ is $x = f^{-1}(B) \cap (f^{-1}(h(\Gamma\text{-}\{v\})))$. The correspondence $f_1: X \dashrightarrow V$ is well-defined at x if and only if after base extension $\tilde{f}: \tilde{X} \to \tilde{Y}$ we have $\tilde{f}^{-1}(I_{\tilde{B}}) O_{\tilde{X},x}$ invertible.*

PROOF: For any accessible point, $V \times \tilde{Y}$ is isomorphic to the blowing up $\bar{V}$ of $\tilde{B}$ at y. $\bar{f}_1: \tilde{X} \to \bar{V}$ is well defined if and only if $\bar{f}^{-1}(I_{\tilde{B}}) O_{\tilde{X},x}$ is invertible. If $\bar{f}_1$ is welldefined at x, then $\bar{f}_1(x)$ is determined by the test curve of which x is the closure point. Thus the image of x must be the point of $\bar{V}$ isomorphic to v, so by composition $\tilde{f}_1: \tilde{X} \to V \times \tilde{Y}$ is well-defined. Similarly if $\tilde{f}_1$ is well-defined, so is $\bar{f}_1$. Finally, since the property of being well-defined is local, f_1 is well-defined at x if and only if $\tilde{f}_1$ is well-defined there.

DEFINITION 1.13: Let $Y_0, \ldots, Y_m$ be a sequence of algebraic spaces such that $b_j : Y_j \to Y_{j-1}$ is a quasi-blowing up with accessible component M_j. The liftings $M_j^{(k)}$ of these components to Y_k for $k \geq j$ will also be called accessible components. A point of Y_k which lies only in accessible components will be called accessible. The sequence will be called a quasifactorization sequence if the generic point of the fundamental locus S_{p_i} is accessible for each i. Letting $h_{kj} = b_{j+1} \circ \cdots \circ b_k$, and $h_k = h_{ko}$, we will say that h_k is dominated by $f : X \to Y$ if each accessible component of Y_k is generically isomorphic to a component of K_f.

To conclude this section, we outline an approach to checking the local factorizability of a morphism $f : X \to Y$ of smooth algebraic spaces of dimension 3. This approach will be applied in § 3 to analyze point obstructions with four components collapsing to a point, and in § 4 to analyze morphisms collapsing 3 components to a curve with a singular point.

By lemma 1.6 if $f : X \to Y$ is not locally factorizable, then every possible local factorization tree for f can be extended until it encounters a point obstruction. By lemma 1.8, this is a point at which the strict preimage of the point is a surface, but the morphism does not factor through the blowing up of the point. We replace the original morphism by the morphism with the point obstruction, and replace the original hypotheses about the morphism by hypothesis stable under progress out the branches of a local factorization tree.

We then proceed to deduce the possible structures for the exceptional divisor K_f of our new morphism $f : X \to Y$. We blow up the bad point y, obtaining a space y_1 and a correspondence $f_1 : X \to Y_1$ which is not well defined at the pinch locus $P_y(f)$. $P_y(f)$ is a union of curves, each the image of a surface S in the graph of f_1 which collapses to a "bad" curve in Y_1. By successively blowing-up such bad curves, first on the Y side and then on the X side, we produce a diagram as in figure 3, in which the generically isomorphic components N_l and M_k "bridge" the gap between the two towers.

In the diagram in Fig. 3, both $X_0, \ldots, X_l$ and $Y_0, \ldots, Y_k$ will be quasifactorization sequences. The centers of the quasi-blowings up will be the images N_l and M_k respectively. In § 2 we will assign "weights" to different components of the exceptional locus of the morphisms. By following the changes in these numbers as we go up the right tower to Y_k, across the bridge to X_l and down the left tower to X, we will obtain

$$\begin{tikzcd}
X_l \arrow[r, phantom, "\supset"] \arrow[d, "a_l"] & N_l \arrow[r, phantom, "\sim"] & {} \arrow[r, phantom] & M_k \arrow[r, phantom, "\subset"] & Y_k \arrow[d, "b"] \\
X_{l'} \arrow[d, dashed] \arrow[d, bend right, "g_{l'l}"'] & & & & Y_{j'} \arrow[d, dashed] \arrow[d, bend left, "h_{j'j}"] \\
X_l \arrow[rrrr, dashed, "f_{lj}"] \arrow[d, dashed] \arrow[ddrr, "f_{l0}"] & & & & Y_j \arrow[d, dashed] \arrow[ddll, "h_j"'] \\
X_0 \arrow[drr, "f"'] & & & & Y_1 \arrow[dll, "b_1"] \\
& & Y_0 & &
\end{tikzcd}$$

Fig. 3

information about those components of K_f containing components of the pinch locus of f.

In order to carry out this program, we must be able to construct a factorization sequence which blows up the successive images of a divisor F under the correspondences induced by a morphism $q: W \to Y$. When the image is a point or a non-singular curve there is no problem. Thus the only problem comes when the image B is a singular curve. To this end we prove the following lemma:

LEMMA 1.14: *Let Y be an algebraic space which has a finite etale cover, and let $B \subset Y$ be an irreducible curve. Then there exists a quasi blowing-up with center B, and we may specify that the quasi-blowing up factors locally through designated smooth branches.*

PROOF: Let $\{y_1, \dots, y_m\}$ be the set of singular points of B. Since each member of the finite cover of Y is quasi-compact, we can find a finite etale cover $\{e_j: W_j \to Y - \{y_1, \dots, y_m\}\}_{j=m+1}^{m'}$ or $Y - \{y_1, \dots, y_m\}$. For each $j = 1, \dots, m$, choose an etale neighborhood $e_j: W_j \to Y$ such that there is a unique point w_j in $e_j^{-1}(y_i)$, and the image of W_j in Y does not contain any of the other singular points of B. At those singular points y_i at which we have designated a particular smooth branch of B, we choose W_j sufficiently fine that W_j contains a subscheme which is smooth at W_j and whose preimage in the Henselization $\widetilde{Y}$ of Y at Y_i is the desired branch. It is possible to find such a W_j since $(\widetilde{Y}, y_i)$ is the direct limit of the etale neighborhoods of y_i.

For each $j=1,\dots,m'$ we construct a blowing-up $g_j: W'_j \to W_j$. For $j>m$, $e_j^{-1}(B)$ is nonsingular, and we let g_j be the canonical blowing up of $e_j^{-1}(B)$. For $j \leq m$, $e_j^{-1}(B)$ has a unique singular point at w_1. If there is no designated branch at W_j, we blow up points until the strict preimage of $e_j^{-1}(B)$ is nonsingular, then blow up this nonsingular curve. If there is a designated branch, we first blow it up, then blow up points until the remaining branches of $e_j^{-1}(B)$ are nonsingular and separated from the exceptional divisor over the designated branch. We then blow up the remaining branches of the curve.

We now wish to construct the quasi-blowing-up $\overline{Y}$ as a quotient of the disjoint union $\coprod_{j=1}^{m'} W'_j$. We want to construct an appropriate etale equivalence relation R which will "patch" the pieces together. Letting $Y_j = e_j(W_j)$, and $Y_{ij} = Y_i \times_Y Y_j \rightrightarrows Y_i \cap Y_j$, we claim that $W'_i \times_{Y_i} Y_{ij} \rightrightarrows W_i \times_{Y_i} Y'_{ij}$. The existence of morphisms in each direction are insured by the following two commutative diagrams:

$$\begin{array}{ccc} W'_i \times_{Y_i} Y_{ij} & \dashrightarrow & Y'_{ij} \\ \downarrow & & \downarrow \\ W_i \times_{Y_i} Y_{ij} & \longrightarrow & Y_{ij} \end{array} \qquad\qquad \begin{array}{ccc} W'_i & \dashleftarrow & W_i \times_{Y_i} Y'_{ij} \\ \downarrow & & \downarrow \\ W_i & \longleftarrow & W_i \times_{Y_i} Y_{ij} \end{array}$$

where the dotted arrows are induced by the universal mapping property of the blowing up. R must be a closed immersion with etale projections, satisfying reflexivity, symmetry and transitivity.

Now for $i \neq j$, we define

$$\begin{aligned} R_{ij} = (W_i \times_{Y_i} Y'_{ij}) \times_{Y'_{ij}} (Y'_{ij} \times_{Y_j} W_j) &\hookrightarrow (W_i \times_{Y_i} Y'_{ij}) \times_{Y_{ij}} (Y'_{ij} \times_{Y_j} W_j) \\ &\rightrightarrows (W'_i \times_{Y_i} Y_{ij}) \times_{Y_{ij}} (Y_{ij} \times_{Y_j} W'_j) \\ &\rightrightarrows W'_i \times_{Y_i} Y_{ij} \times_{Y_j} W_{j'} \\ &\rightrightarrows W'_i \times_Y W'_j \end{aligned}$$

The local properties of being a closed immersion, having etale projections and symmetry are induced from the fact that it is the equivalence relation on two etale neighborhoods, $W_i \times_{Y_i} Y'_{ij}$ and $Y'_{ij} \times_{Y_j} W_j$ of Y'_{ij}.

Before defining R_{ii}, we first note that in $W_i \times_{Y_i} W_i$ we have a closed subscheme $W_i \times_{W_i} W_i$, since W_i is a scheme and thus separated. The closed immersion $\triangle: W_i \times_{W_i} W_i \hookrightarrow W_i \times_{Y_i} W_i$ gives a section of the etale projection $W_i \times_{Y_i} W_i \xrightarrow{\pi_1} W_i$. Because π_1 is a local isomorphism in the etale topology, the section $\triangle$ is also an open immersion. Since the diagonal is both open and closed, and W_i is connected, we conclude that the diagonal is a connected component, whence $W_i \times_{Y_i} W_i \xrightarrow{\sim} W_i \times_{W_i} W_i \coprod D$. Let $\overline{W}_i = \mathrm{W}_i - \{\mathrm{w}_i\}$, and $\overline{Y}_i = Y_i - \{y_i\}$. Then $D \subset \overline{W}_i \times_{Y_i} \overline{W}_i$, since w_i is the only point of W_i lying over y_i, and $(w_i, w_i) \in W_i \times_{W_i} W_i$.

We now let $\overline{W}'_i = W'_i - g_i^{-1}(w_i)$, and set

$$R_{ii} = W'_i \times_{Y_i} W_i \xrightarrow{\sim} (W'_i \times_{W_i} W_i) \cup (\overline{W}'_i \times_{Y_i} W_i)$$

$W'_i \times_{W_i} W_i$ is actually a component, and there is a complement $D' \subset \overline{W}'_i \times_{Y_i} W_i$ whose image in $W_i \times_{Y_i} W_i$ is D. The immersion $W'_i \times_{W_i} W_i \xrightarrow{\sim} W'_i \times_{W'_i} W'_i \hookrightarrow W'_i \times_{Y_i} W'_i$, and the immersion $\overline{W}'_i \times_{Y_i} W_i \xrightarrow{\sim} W'_i \times_{\overline{Y}_i} \overline{W}_i$

$$\xrightarrow{\sim} \overline{W}'_i \times_{\overline{Y}'_i} \overline{Y}'_i \times_{\overline{Y}_i} W_i$$

$$\xrightarrow{\sim} \overline{W}'_i \times_{\overline{Y}'_i} \overline{W}'_i \hookrightarrow \overline{W}'_i \times_{\overline{Y}_i} \overline{W}'_i$$

induce an immersion $R_{ii} \hookrightarrow W'_i \times_{Y_i} W'_i$, since the images of $W'_i \times_{Y_i} W_i$ and D' are disjoint. Since the composition with the proper morphism $W'_i \times_{Y_i} W'_i \to W'_i \times_{Y_i} W_i$ is an isomorphism, we conclude that $R_{ii} \hookrightarrow W'_i \times_{Y_i} W'_i$ is proper and thus must be a closed immersion.

The image of R_{ii} in $W'_i \times_{Y_i} W'_i$ is symmetric. The first projection $R_{ii} \to W'_i$ is the base extension of an etale morphism and is thus etale. By symmetry the other projection is also etale. We have already shown that the diagonal map factors through R_{ii}, giving reflexivity.

It remains to check the global property of transitivity. We need to show that $R_{ij} \times_{W_{j'}} R_{jk}$ factors through R_{ik}. We begin with the case $i \neq j \neq k \neq i$, and let $Y_{ijk} = Y_{ij} \times_{Y_i} Y_{jk}$.

We will make frequent use of various versions of the isomorphism $W'_i \times_Y Y_j \xrightarrow{\sim} W_i \times_Y Y'_j$ and of standard fiber product isomorphisms like $W \times_Y Y \xrightarrow{\sim} W$.

$$\begin{aligned}
R_{ij} \times_{W_{j'}} R_{ik} &\xrightarrow{\sim} [(W_i \times_{Y_i} Y'_{ij}) \times_{Y'_{ij}} (Y_{ij} \times_{Y_j} W'_j)] \\
&\quad \times_{W_{j'}} [(W'_j \times_{Y_j} Y_{jk}) \times_{Y'_{jk}} (Y'_{jk} \times_{Y_k} W_k)] \\
&\xrightarrow{\sim} W_i \times_{Y_i} [Y_{ij} \times_{Y_j} W'_j \times_{Y_j} Y_{jk}] \times_{Y_k} W_k \\
&\xrightarrow{\sim} W_i \times_{Y_i} [W_j \times_{Y_j} Y'_{ijk}] \times_{Y_k} W_k
\end{aligned}$$

The etale morphism $e_j \colon W_j \to Y_j$ and the open immersion $Y'_{ijk} \to Y'_{ik}$ then induce an etale morphism

$$\begin{aligned}
R_{ij} \times_{W'_j} R_{jk} &\to W_i \times_{Y_i} Y'_{ik} \times_{Y_k} W_k \\
&\xrightarrow{\sim} (W_i \times_{Y_i} Y'_{ik}) \times_{Y'_{ik}} (Y'_{ik} \times_{Y_k} W_k) \xrightarrow{\sim} R_{ik}
\end{aligned}$$

This gives the desired factorization.

The various degenerate cases follow the same general procedure, but require more care because of the more complicated definition of R_{ii}. As an example, in the case $i=k$, we have an isomorphism as before

$$R_{ij} \times_{W_{j'}} R_{ij} \xrightarrow{\sim} W_i \times_{Y_i} (W_j \times_{Y_j} Y'_{ijk}) \times_{Y_i} W_i$$

Applying the etale morphism $e_j \colon W_j \to Y_j$ and the isomorphism $Y'_{iji} \xrightarrow{\sim} Y'_{ij}$ we get a morphism

$$R_{ij} \times_{W_j} R_{ji} \to W_i \times_{Y_i} (Y'_{ij}) \times_{Y_i} W_i \xrightarrow{\sim} (W'_i \times_{Y_i} W_i) \times_{Y} Y_j$$

The latter space is an open subset of R_{ii}.

We have a diagram

$$\begin{tikzcd}
R_{ij} \times_{W'_j} R_{jk} \arrow[r] \arrow[d] & (W'_i \times_Y W'_j) \times_{W'_j} (W'_j \times_Y W'_k) \arrow[d, "(\pi_1,\, \pi_4)"'] \\
R_{ik} \arrow[r] & (W'_i \times_Y W'_k)
\end{tikzcd}$$

At every stage in the transformation of $R_{ij} \underset{W'_j}{\times} R_{jk}$, the first and fourth projections can be defined via canonical isomorphisms of the type $W'_i \underset{Y_i}{\times} Y_{ij} \xrightarrow{\sim} W_i \underset{Y_i}{\times} Y'_{ij}$. The diagram thus commutes, and the equivalence relation R is transitive. We define X to be the quotient of $S=\coprod W'_j$ by R, and the induced morphism $f: X \to Y$ to the base Y gives the desired quasi-blowing-up.

§ 2 Combinatorial analysis of K_f.

We begin the quantitative analysis of the components of the exceptional divisor with definitions of a few of the basic functions we will be using.

DEFINITION 2.1: Let $f: X \to Y$ be a birational morphism, let F be an irreducible component of K_f, and let $H_1, \dots, H_r$ be divisors in Y, i.e., integral combinations of irreducible divisors. We denote by

$r_f(F)$, the multiplicity of F in K_f
$w_f(F)$, the number $r_f(F)+1$, called the weight of F.
$s_f(F, H_i)$, the multiplicity of F in $f^*(H_i)$.
$u_f(F; H_1, \dots H_r) = (w_f(F); s_f(F, H_1), \dots, s_f(F, H_r))$,

called a weight vector.
For B a smooth irreducible subscheme of Y, we can define the canonical B-pair $u_f(F, B) = (w_f(F), s_f(F, H))$ for H a generic hyperplane containing B. When S_f, the fundamental locus of f, consists of a single point y, we will abbreviate $u_f(F, y)$ by $u_f(F)$, and will simply call it the canonical pair.

REMARK: Note that $s_f(F, H)$ is an additive function of H.

REMARK: If $\hat{f}: \hat{X} \to Y$ is a birational correspondence, and F is a component of $K_{\hat{X}}$, let $f: X \to Y$ be a morphism obtained by resolving the fundamental points of $\hat{f}$. Since $\hat{f}$ is well-defined at the generic point of F, we have X generically isomorphic to $\hat{X}$ at the generic point of F. Thus the multiplicities given in $u_f(F_1; H_1, \dots, H_r)$ will be independent of the choice of f, We can define

$$u_{\hat{f}}(F; H_1, \dots, H_r) = u_f(F; H_1, \dots, H_r)$$

and this will be independent of the choice of f.

EXAMPLE: If $f: X \to Y$ is a toroidal morphism, with $Y \cong A^n$, then each component F of K_f is uniquely determined by an integral vector $(a_1, \dots, a_n)$, where a_i is the order of $f^*(x_i)$ on F. Choose a general point x of F, and a set of local toroidal coordinates $t_1, \dots, t_n$ in a neighborhood $U \subset X$, such that t_1 is a local coordinate for F, and

$$x_i = t_1^{a_{i1}} \cdots t_n^{a_{in}}, \text{ with } \det [a_{ij}] = \pm 1$$

By 1.1 of [9], if $r_j = (\sum_i a_{ij}) - 1$

$$\begin{aligned} f^*(dx_1 \wedge \dots \wedge dx_n) &= t_1^{r_1} \dots t_n^{r_n} \det[a_{ij}]\, dt_1 \wedge \dots \wedge dt_n \\ &= \pm t_1^{r_1} \dots t_n^{r_n} dt_1 \wedge \dots \wedge dt_n \end{aligned}$$

Thus $r_f(F) = r_1 = (\sum a_{ij}) - 1$
$w_f(F) = \sum a_{ij}$

Letting H_i be the hyperplane determined by $x_i = 0$, we have

$$u_f(F; H_1, \dots, H_n) = (\sum_{i=1}^n a_{i1}; a_{11}, \dots, a_{n1})$$

Let $p: W \to X$ be the blowing up of an intersection $E_1 \cap \dots \cap E_r$ of components of K_f. The integral vector of the resulting exceptional divisor F is just the vector sum of the integral vectors of the components E_j. We thus have

$$u_{f \circ p}(F; H_1, \dots, H_n) = \sum_{j=1}^n u_f(E_j; H_1, \dots, H_n)$$

Let us now consider the behavior of the weight vector under composition. We let $p: W \to X$ be a toroidal morphism, and let F be a component of the exceptional divisor. Let $E_1, \dots, E_r$ be the components of K_f containing $p(F)$ with local toroidal coordinates $t_1, \dots, t_r$ and let q_i be a local toroidal coordinate for $f^{-1}[H_i]$. Let $t_1, \dots, t_n$ be the complete set of toroidal coordinates in a neighborhood of the general point of $p(F)$. If $f^{-1}[H_i]$ intersects this neighborhood, then its local parameter is a toroidal coordinate. Thus each q_i either equals some t_j for $j > r$ or else is 1.

For each $j = 1, \dots, n$ let s_j be the order of $p^*(t_j)$ on F, so that if t is a local toroidal parameter for F,

$$\begin{aligned} p^*(t_j) &= t^{s_j} p_j, \quad j = 1, \dots, n \\ f^*(x_i) &= t_1^{a_{i1}} \dots t_r^{a_{ir}} q_i \end{aligned}$$

Therefore, $(f \circ p)^*(x_i) = p^*(f^*(x_i))$

$$= \prod_{j=1}^r (t^{s_j} p_j)^{a_{ij}} p^*(q_i)$$

By definition, $s_{f\circ p}(F, H_i)$ is the multiplicity $\nu(F, (f\circ p)^*(H_i))$ of F in the divisor $(f\circ p)^*(H_i)$, defined locally by $(f\circ p)^*(x_i)=0$.
Thus

$$\begin{aligned} s_{f\circ p}(F, H_i) &= \sum_{j=1}^{r} s_j \cdot a_{ij} + s_p(F, f^{-1}[H_i]) \\ &= \sum_{j=1}^{r} s_p(F, E_j) s_f(E_j, H_i) + s_p(F, f^{-1}[H_i]) \end{aligned}$$

Taking the sum over all $i=1, \dots, n$ we get

$$w_{f\circ p}(F) = \sum_{j=1}^{r} s_p(F, E_j) w_f(E_j) + \sum_{i=1}^{n} s_p(F, f^{-1}[H_i])$$

Combining these equations, we have

$$\begin{aligned} u_{f\circ p}(F; H_1, \dots, H_n) = \sum_{j=1}^{k} s_p(F, E_j) u_f(E_j; H_1, \dots, H_n) \\ + \Big(\sum_{i=1}^{n} s_p(F, f^{-1}[H_i]); s_p(F, f^{-1}[H_1]), \dots, \\ s_p(F; f^{-1}[H_n])\Big) \end{aligned}$$

We wish to use the best approximation possible to this formula in the nontoroidal case. To this end we need a function which will measure the extent to which a component fails to mimic the toroidal case, that is, to be determined by the blowings-up of normally crossing hyperplanes.

DEFINITION 2.2: Let $f: X \to Y$ be a birational morphism, and let $H_1, \dots, H_c$ be divisors in Y. Let $H = H_1 + \dots + H_c$. Then the excess of a component F with respect to $H_1, \dots, H_c$ will be

$$ex_f(F; H_1, \dots, H_c) = w_f(F) - s_f(F, H) = w_f(F) - \sum_{i=1}^{c} s_f(F, H_i)$$

REMARK: In a toroidal scheme, if $c=$n and $H_1, \dots, H_n$ correspond to the coordinates of the torus, $ex(F; H_1, \dots, H_n)=0$. For any morphism, if $H_1, \dots, H_n$ are irreducible and normally crossing, we have, by 2.1 of [9], that, for $c=n$

$$ex_f(F; H_1, \dots, H_c) \geq 0,$$

and the inequality will surely still hold if we take $c \leq n$ under the same conditions.

LEMMA 2.3 (*the additivity formula*): *Let $p: W \to X$ and $f: X \to Y$ be birational morphisms, with Y a scheme and let H be an irreducible hypersurface. Then if F is an irreducible component of $K_{f\circ p}$, and $E_1, \dots,$*

E_r are the components of K_f, all crossing normally, we have

$$u_{f\circ p}(F\,;H)=\sum_{j=1}^{r}s_p(F,E_j)u_f(E_j\,;H)+(ex_p(F\,;E_1,\ldots,E_r)\,;\\ s_p(F,f^{-1}[H])).$$

PROOF: We calculate the components of $u_f(F,H)$, for H an irreducible hypersurface. We calculate $w_f(F)$ and $s_f(F,H)$. Let $\nu(F,D)$ denote the multiplicity of a component F in a divisor D, and let $r_i=w_f(E_i)-1=r_f(E_i)=\nu(E_i,K_f)$

$$\begin{aligned}
w_f(F)&=r_f(F)+1\\
&=\nu(F,K_{f\circ p})+1\\
&=\nu(F,p^*(K_f)+K_p)+1\\
&=\nu(F,p^*(\sum_{i=1}^{r}r_iE_i))+\nu(F,K_p)+1\\
&=\sum_{i=1}^{r}r_i\nu(F,p^*(E_i))+r_p(F)+1\\
&=\sum_{i=1}^{r}r_is_p(F,E_i)+w_p(F)\\
&=\sum_{i=1}^{r}w_f(E_i)s_p(F,E_i)+(w_p(F)-\sum_{i=1}^{r}s_p(F,E_i))\\
&=\sum_{i=1}^{r}s_p(F,E_i)w_f(E_i)+ex_p(F\,;E_1,\ldots,E_r)\\
s_{f\circ p}(F\,;H)&=\nu(F,(f\circ p)^*(H))\\
&=\nu(F,p^*(\sum_{i=1}^{r}s_f(E_i,H)E_i+f^{-1}[H]))\\
&=(\sum_{i=1}^{r}s_p(F,E_i)s_f(E_i,H))+s_p(F,f^{-1}[H])
\end{aligned}$$

REMARK: We may note from the formula given in the toroidal example, that in the toroidal case $ex_p(F\,;E_1,\ldots,E_r)$ measures the contribution to F of the liftings $f^{-1}[H_i]$ of coordinate hyperplanes. We will analyze this excess more carefully in lemmas 2.4 and 2.8.

LEMMA 2.4: *If $p:W\to X$, $f:X\to Y$ are birational morphisms, such that K_f has normal crossings, k' is the codimension of $p(F)$, k is the number of components of K_f containing $p(F)$, and H is an irreducible hypersurface, then*

$$ex_p(F\,;E_1,\ldots,E_r)\geq k'-k$$

If p is a blowing up with center $p(F)$, equality holds and the additivity formula becomes

$$u_{f\circ p}(F\,;H)=\sum_{p(F)\subset E_i}u_f(E_i,H)+(k'-k,s_p(F,f^{-1}[H]))$$

PROOF: Let the E_i be so numbered that $E_1, \dots, E_k$ are the components of K_f containing $p(F)$. Localizing we can assume that $p(F)$ is smooth, without affecting the quantities we are calculating. We can thus add hypersurfaces $E'_1, \dots, E'_{k'-k}$ crossing normally with $E_1, \dots, E_k$ such that the intersection of all k' hypersurfaces is $p(F)$.

$$\begin{aligned} 0 &\le ex_p(F\,; E_1, \dots, E_k, E'_1, \dots, E'_{k'-k}) \\ &= w_p(F) - \sum_{i=1}^{k} s_p(F, E_i) - \sum_{i=1}^{k'-k} s_p(F, E'_i) \\ &= ex_p(F\,; E_1, \dots, E_k) - \sum_{i=1}^{k'-k} s_p(F, E'_i) \end{aligned}$$

We have $s_p(F, E'_i) \ge 1$ for each i, and furthermore, we have equalities when p is a blowing up. Since $s_p(F, E_i) = 0$ for $i > k$, $ex_p(F\,; E_1, \dots, E_r) = ex_p(F\,; E_1, \dots, E_k) \ge k' - k$.

We now generalize a case of lemma 1.1 of [9] for quasiblowings-up, in preparation for an investigation of the properties of quasi-factorization sequences.

LEMMA 2.5: *Let $f: X \to Y$ be a proper birational morphism, and let $h: Y_1 \to Y$ be a quasi-blowing-up dominated by f, i.e. such that every accessible component is generically isomorphic to a component of K_f. Let y_1 be an accessible point of M_1, and let Γ_1 be a test curve transversal to M_1 at y_1. Let x be the closure point in X. It there is a hypersurface H containing the center B of h, such that*

$$\sum_{x \in D_i} s_f(D_i, H) \ge \deg(h(\Gamma_1) \cdot H) = 1,$$

then x belongs to a unique component D_i of $f^{-1}(H)$, $s_f(D_i, H) = 1$, and $f_1: X \dashrightarrow Y_1$ is well-defined at x if and only if after base extension by the Henselization of Y at $f(x)$, $f(\tilde{D}_i)$ is contained in the local center $\bar{B}$ of h.

PROOF: Since $H \supset B$, $I_H \subset I_B$, so $f^{-1}(I_H) O_{X,x}$. Let $\Gamma = f^{-1}[h(\Gamma_1)]$. Letting $s_i = s_f(D_i, H)$, and letting t_i be a local equation for D_i at x, we have

$$f^{-1}(I_H) O_{X,x} = (\Pi^{s_i}_{t_i}) J_{X,x},$$

for some ideal $J_{X,x}$. Since $\deg(\Gamma \cdot D_i) \ge 1$ for each i, we have

$$\Gamma \cdot f^*(H) = \sum_{x \in D_i} s_i(\Gamma \cdot D_i) \ge (\sum_{x \in D_i} s_i) x$$

By the projection formula, since f is proper, deg $\Gamma \cdot f^*(H) = \deg f_*(\Gamma \cdot H = \deg h_*(\Gamma_1) \cdot H = 1$. Thus $1 \le \sum s_i \le 1$, whence all the s_i are 0 except for one $s_i=1$. Thus

$$f^{-1}(I_H)O_{X,x}=t_i\,J_{X,x}.$$

Since $1=\deg\Gamma\cdot f^*(H)$ is the order of the ideal induced by $f^{-1}(I_H)O_{X,x}$ in O_Γ, we conclude that Γ intersects B_i transversally at x and that $J_{X,x}$ is trivial.

We will indicate by "$\sim$" base extension by the Henselization of Y at $f(x)$. It $\tilde f(\tilde D_i)\subset\bar B$, where $\bar B$ is the local center of the quasiblowing-up h, then we have

$$(\tilde t_i)O_{\tilde X,\tilde x}=\tilde f^{-1}(I_{\tilde H})O_{\tilde X,\tilde x}\subset f^{-1}(I_{\bar B})O_{\tilde X,\tilde x}\subset(\tilde t_i)O_{\tilde X,\tilde x}$$

All the inclusions are then equalities, and since $\tilde x$ is a closure point for $\tilde y_1$, we conclude from Lemma 1.12 that $\tilde f_1:\tilde X\dashrightarrow\tilde Y_1$ is well-defined at $\tilde x$.

Suppose, on the other hand, that $\tilde f(\tilde D_1)\not\subset\bar B$. Since $\tilde D_i$ is the only component of K_f contained in $\tilde f^{-1}(H)$ which passes through $\tilde x$, we conclude that $\tilde f^{-1}(B)$ is of codimension greater than one at $\tilde X$. Since $x\in\tilde f^{-1}(\bar B)$, it is non-empty. Thus $\tilde f^{-1}(I_{\bar B})O_{\tilde X,\tilde x}$ cannot be invertible, and we conclude that $\tilde f_1:\tilde X\to\tilde Y_1$ is not well-defined at x, by applying lemma 1.12 again.

LEMMA 2.6: *Let $Y_0,\dots,Y_k$ be a quasi-factorization sequence of three-folds suppose $y_k\in Y_k$ is an accessible point. Suppose there is a hypersurface H in Y_0 such that $\sum_{y_k\in M_j^{(k)}}s_{h_k}(M_j^{(k)},H)=1$. Then for any transversal test curve Γ_k, with closure point x on a curve Γ in X, either $f_k:X\to Y_k$ is well defined at x, or, if $j<k$ is the largest index for which f_1 is well defined at x, we have (*) After base extension by the Henselization $\tilde Y_j$ of Y_j at y_j, $\tilde x$ is contained in a component $\tilde D_i$ such that $\tilde f_j(\tilde D_i)$ is not contained in the local center of $\tilde b_j$ at $\tilde y_j$, but $\tilde y_j$ is contained in the local center.*

PROOF: We proceed by induction, showing that if (*) does not hold, then f_j is welldefined at x implies that f_{j+1} is well defined at x. We let $\Gamma_j=h_{kj}(\Gamma_k)$. Each h_{kj} is proper, and thus by the projection formula

$$\begin{aligned}\deg\Gamma_0\cdot H=\deg\Gamma_j\cdot h_j^*(H_0)&=\deg\Gamma_k\cdot h_k^*(H_0)\\&=\deg\textstyle\sum s_{h_k}(M_j^{(k)},H_0)\cdot(\Gamma\cdot M_j^{(k)})\\&=\sum_{y_k\in M_j^{(k)}}s_{h_k}(M_j^{(k)},H_0)\\&=1\end{aligned}$$

We conclude that each y_j is contained in a unique $M_i^{(j)}$ for which $s_{h_j}(M_i^{(j)},C)=1$. If (*) does not hold, either y_j is not contained in the center of b_j, in which case $f_{j+1}=b_j^{-1}\circ f_j$ at x, or else y_{j+1} is contained in M_{j+1}, and we can apply lemma 2.5 with $H_j=M_i^{(j)}$ as the hypersurface satisfying deg $\Gamma_j\cdot H_j=1$. Since deg $\Gamma\cdot f_j^*(H_j)=1$, we see that x is contained in some component D_i with $s_{f_j}(D_i, H_j)\geq 1$. We conclude that f_{j+1} is well-defined at x, by lemma 2.5.

We wish to use this lemma in the specific case in which we are analyzing the pinch locus of a morphism $f: X\rightarrow Y$.

LEMMA 2.7: *Let $Y_0, Y_1, \ldots, Y_k$ be a quasifactorization sequence dominated by a proper birational morphism $f: X\rightarrow Y$, such that Y_1 is the blowing-up of a point $y_0\in Y_0$, and each center B_j of b_{j+1} satisfies dim $h_{j1}(B_j)=1$. For each $j<k$, let $\hat{C}_k$ be the finite set of singular points of the locus on which f_j^{-1} is not well-defined. Let $\hat{C}=\hat{C}_1\cup h_{21}(\hat{C}_2)\ldots\cup h_{k-11}(\hat{C}_{k-1})$. Let y_k be any singleton accessible point of Y_k such that its unique accessible component $M_{j'}^{(k)}$ has order 1 in $h_k^*(H)$, for a generic H through y_0. Then, for $f_k: X$---> Y_k, one of the following holds:*

(1) *f_k^{-1} is an isomorphism at y_k or*

(2) *there is a component D_i of K_f such that $D_i\supset f_k^{-1}[y_k]$ and D_i is generically isomorphic to the blowing up of $f_k[D_i]$, or*

(3) *for some $j<k$, $f_j^{-1}[y_j]$ lies in a D_i which does not map locally to the local center of b_{j+1}. ($h_{k1}(y_k)\in\hat{C}$, in this case.)*

PROOF. Let Γ_k be a generic transversal test curve through y_k. Then $\Gamma_k\cdot h_k^*(H)=s_{h_k}(M_{j'}^{(k)}, H)\Gamma\cdot M_{j'}^{(k)}=1$. Let $x\in f^{-1}[y_k]$ be the closure point of Γ_k in X, with corresponding curve $\Gamma=f_k^{-1}[\Gamma_k]$. We first suppose that f_k is not well defined at x, and prove (3). By lemma 2.6, for some j, f_j is well defined at x, and $y_j\in B_j$, but after base extension $\tilde{x}$ is contained in a unique component $\tilde{D}_i$, and $\tilde{f}_i(\tilde{D}_i)$ is not contained in the local center. Since we thus have two different branches of the fundamental locus of f_j passing through y_j, we see that y_j is a singular point of the fundamental locus of f_j, and thus $y_j\in C_j$, proving that (3) holds.

Let us now assume that (3) does not hold, and show that either (1) or (2) then holds. From the previous paragraph, we can conclude that f_k is well-defined at x. Since Γ_k was generic, x must lie on $f^{-1}[y]$. Consider the possible dimensions of $f^{-1}[y]$. If it is zero dimensional, f_k^{-1} is an isomorphism at x, so (1) holds. If $f^{-1}[y]$ is a surface, then that surface is the desired D_i in (2), being generically isomorphic to the blowing up of y_k. If $f^{-1}[y]$ is a curve, then by the modified Danilov result, lemma 1.8, D_i is generically isomorphic to the quasi-blowing up of its image in Y_k, which contains y_k.

We now consider the case of p a quasifactorization, and try to analyze the terms $ex_p(F; E_1, \dots, E_r)$ and $s_f(F, f^{-1}[H])$ appearing in the additivity formula. We assume that $p = a_l \circ \dots \circ a_1$, with $a_i : X_i \to X_{i-1}$ a quasi-blowing-up, $S_{b_i} = A_i$. Over the generic point of A_i we assume that a_i is a blowing up with exceptional divisor N_i and we assume that the generic point of A_i is contained only in the liftings $N_{i'}^{(i)}$ of earlier $N_{i'}$.

We denote by $g_{ii'}$ the composition $a_{i'+1} \circ \dots \circ a_i$ as in Fig. 3 of § 1.

LEMMA 2.8: *Let $p : W \to X$ be a quasifactorizable morphism with factors a_i, $i=1, \dots, l$, let F be an accessible component of K_p, and let $f : X \to Y$ be a birational morphism. Let k'_i be the codimension of A_{i-1}, let k_i be the number of exceptional components of $f_{(i-1)0} = f \circ a_1 \dots \circ a_{i-1}$ containing the generic point of A_{i-1} and let d_i be the multiplicity of $f_{(i-1)0}^{-1}[H]$ along A_{i-1}, which equals $s_{a_i}(N_i, f_{(i-1)0}^{-1}[H])$. Then*

$$u_{f \circ p}(F, H) = \sum_{i=1}^{k_0} s_p(F, E_i) u_f(E_i, H) + \sum_{i=1}^{l} s_{g_{li}}(F, N_i)(k'_i - k_i, d_i)$$

PROOF: Since we may replace f by f_{i0}, we can prove the theorem by induction on l, assuming it is true for $l-1$. We therefore assume that the theorem is known to be true for $g_{l1} : W \to X_1$, and $f_{10} : X_1 \to Y$. We want to show it for $p : W \to X$, $f : X \to Y$. By lemma 2.4 the additivity formula for blowing up, we know that if the components are numbered so that $E_1, \dots, E_{k_1}$ are the components of K_f containing A_0

$$u_{f_{10}}(N_1, H) = \sum_{i=1}^{k_1} u_f(E_i, H) + (k'_1 - k_1, d_1)$$

Letting $E_i^{(1)}$ *be the lifting of* E_i to X_1,

$$s_{g_{l0}}(F, E_i) = s_{g_{l1}}(F, E_i^{(1)}) + s_{g_{l1}}(F, N_1),$$

since $E_i^{(1)}$ and N_1 are the only components of K_{b_1}, whose image is in E_i. Finally $u_{f_{10}}(E_i^{(1)}, H) = u_f(E_i, H)$, since b_1 is an isomorphism at the generic point of E_i for each i.

$$\begin{aligned} u_{p \circ f}(F, H) &= g_{g_{l1} \circ f_{10}}(F, H) \\ &= \{\sum_{j=1}^{k_1} s_{g_{l1}}(F, E_j^{(1)}) u_{f_{10}}(E_j^{(1)}, H) \\ &+ s_{g_1}(F, N_1) u_{f_{10}}(N_1, H)\} + \sum_{i=2}^{l} s_{g_{li}}(F, N_i)(k'_i - k_i, d_i) \\ &= \sum_{j=1}^{k_1} s_{g_{l1}}(F, E_j^{(1)}) u_{f_{10}}(E_j^{(1)}, H) + \{s_{g_1}(F, N_1)(\sum_{j=1}^{k_1} u_f(E_j, H) \\ &+ s_{g_{l1}}(F, N_1)(k'_1 - k_1, d_1)\} + \sum_{i=2}^{l} s_{g_{li}}(F, N_i)(k'_i - k_i, d_i) \\ &= \sum_{i=1}^{l} s_{g_{l0}}(F, E_i) u_f(E_i, H) + \sum_{i=1}^{l} s_{g_{li}}(F, N_i)(k'_i - k_i, d_i) \end{aligned}$$

This combined additivity formula and the resulting " linear programming problem " which will be defined in the following lemma from the technical heart of the combinatorial analysis of the exceptional divisor. We therefore pause to give an illustrative example which should provide some orientation to both Lemmas 2.8 and 2.9.

EXAMPLE: Let $Y=A^3$, and let H_1, H_2 and H_3 be three transversally intersecting coordinate planes. Let $f: X\rightarrow Y$ be the composite of the five blowings-up $p_1, \dots, p_5$ with the following centers and compositions $p_{ij}=p_j\circ \dots \circ p_i$

(1) The line $H_2\cap H_3$, giving E_1 in X; $u_f(E_1; H_1H_2H_3)=(2; 0, 1, 1)$
(2) The line $p_{52}(E_1)\cap p_1^{-1}[H_3]$, giving E_2 in X; $u_f(E_2; H_1H_2H_3)=(3; 0, 1, 2)$
(3) The line $p_{53}(E_1)\cap p_{53}(E_2)$, giving E_3 in X; $u_f(E_3; H_1H_2H_3)=(5; 0, 2, 3)$
(4) The line $p_{54}(E_1)\cap p_{31}^{-1}[H_1]$, giving E_4 in X; $u_f(E_4; H_1H_2H_3)=(3; 1, 1, 1)$
(5) The line $p_5(E_2)\cap p_{41}^{-1}[H_1]$, giving E_5 in X; $u_f(E_5; H_1H_2H_3)=(4; 1, 1, 2)$

Let $y\in Y$ be the origin and let $h_1: Y_1\rightarrow Y$ be the blowing-up of y. (See Fig. 4) where antipodal points of each tube are identified.)

Now suppose that we were given $f: X\rightarrow Y$ without being given its factorization. Let $h_1: Y_1\rightarrow X$ be the blowing-up of the origin, with exceptional divisor M_1. Let $h_1: X$---> Y_1 be the induced correspondence. This map is well-defined at every point of X except the irreducible curve $A_0=E_3\cap H_1$, which is thus, by definition, the pinch locus $P_y(f)$. The question then is, how much information can we obtain about the irreducible component of K_f containing A_0 by constructing quasi-factorization sequences on X and Y which form a bridge between A_0 and its image?

The image of each point of A_0 is the irreducible curve $B_1=M_1\cap h_1^{-1}[H_3]$. We construct a quasi-factorization sequence on the Y_1-side by blowing up B_1 to get Y_2 with exceptional divisor which is the sum of $M_1^{(2)}$ and M_2. Under the correspondence $f_2: X$---> Y_2 the general points of A_0 all correspond to the same curve $B_2=M_1^{(2)}\cap M_2$. The final blowing-up b_3 with center B_2 will produce an exceptional component M_3 which is the image of A_0 under $f_3: X$---> Y_3.

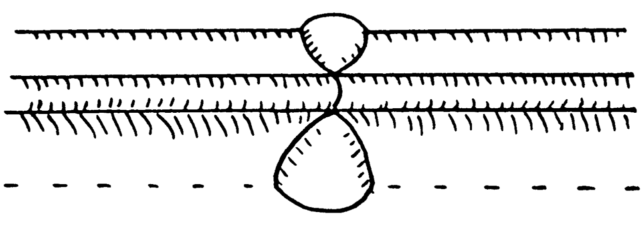

Fig. 4

Now we build the quasi-factorization sequence over X by blowing up A_0 to get a space X_1 with exceptional divisor N_1. $u_{f_{10}}(N_1; H_1, H_2, H_3)=(6; 1, 2, 3)$

The image of M_3 under the induced correspondence $f_{13}^{-1}: Y_3 \dashrightarrow X_1$ is the curve $A_1=N_1 \cap f_{10}^{-1}[H_1]$, where $f_{10}: X_1 \to Y$. Finally, by blowing up A_1, we get a space X_2 with exceptional divisor N_2 generically isomorphic to M_3.

Having constructed the two quasi-factorization "towers" and the bridge between them, we now consider a generic hyperplane through y, and calculate the canonical pair $(w, s)=u_{f_{20}}(N_2; H)$ from the two different quasi-factorization sequences, using the additivity formula in Lemma 2.8.

From the quasi-factorization sequence $Y=Y_0$, Y_1, Y_2, Y_3 we calculate the canonical pairs for the exceptional divisors M_1, M_2, M_3 using the additivity formula. For the first blowing up b_1, the codimension k_1' of the center $B_0=\{y\}$ is 3; there are no exceptional components, so $k_1=0$, and H is smooth at y, so $d_1=1$. Therefore

$$u_{h_1}(M_1, H)=(k_1'-k_1, d_1)=(3, 1)$$

For the second blowing up b_2, the codimension k_2' of $B_1=M_1 \cap b_1^{-1}[H]$ is 2, B_1 is contained in M_1, so $k_1=1$, and $h_1^{-1}[H]$ does not contain B_1, so $d_1=0$. Therefore

$$\begin{aligned} u_{h_2}(M_2, H) &= u_{h_1}(M_1, H)+(k_2'-k_2, d_2) \\ &= (3, 1)+(1, 0) \\ &= (4, 1) \end{aligned}$$

For the third blowing-up, the center B_2 is again a curve so $k_3'=2$, the center is contained in two components, so $k_3=2$, and again $h_2^{-1}[H]$ does not contain B_2, so $d_3=0$. We conclude that

$$\begin{aligned} u_{h_3}(M_3, H) &= u_{h_2}(M_2, H)+u_{h_2}(M_1^{(2)}, H)+(0, 0) \\ &= (4, 1)+(3, 1) \\ &= (7, 2) \end{aligned}$$

We now give a preview for this numerical example of the results of the next lemma, 2.9. Suppose that we know only that some quasi-factorization sequence $X_0 \ldots X_l$ leads to a component N_l generically isomorphic to M_3, and therefore having canonical pair $(w, s)=(7, 2)$. We are interested in determining as much information as possible about the canonical pairs (w_i, s_i) of the components $E_1, \ldots, E_r$ of k_f containing centers of the quasi-factorization sequence. Let us denote $S_p(M_l, E_j)$ by e_j and

$S_p(M_l, N_i)$ by c_i. Then the additivity formula in Lemma 2.8 becomes

$$(w, s)=\sum_{i=1}^{k_0} e_i(w_i, s_i)+\sum_{i=1}^{l} c_i(k'_i-k_i, d_i)$$

Now we note that in dimension 3, every center A_i had codimension k'_i no greater than 2, and is contained in at least one exceptional divisor, so that $k'_i-k_i\leq 1$. Furthermore, the pinch locus is chosen to lie in the strict preimage of H so $d_i\geq 1$.

Therefore, as will be deduced in the proof of lemma 2.9 below, we conclude that

$$w-s\leq\sum_{i=1}^{k_0} e_i(w_i-s_i)$$

On the other hand, since $d_i\geq 1$ for all i, we have

$$s\geq\sum e_i s_i+l$$

We now show how these inequalities can be used to analyse the image of M_3 under f_3^{-1}. We have $(w, s)=(7, 2)$, and since the image of M_3 is in the pinch locus, at least one blowing-up is required to resolve it, so $l\geq 1$. We substitute in the second inequality to get

$$2\geq\sum e_i s_i+1,$$

and conclude that $1\geq\sum e_i s_i$. If $\sum e_i s_i=0$, then by the first inequality some $e_i\neq 0$, so some $s_i=0$. If $\sum e_i s_i=1$, and no $s_i=0$, then $k_0=1$, and we must have $5=w-s\leq w_i-s_i$, whence we conclude that $w_i\geq 6$. Since in fact $\sum e_i s_i=1$ would require $l=1$, and $d_1=1$, we can in fact conclude that the image of M_3 either lies in a component E_i with $s_i=0$, or else lies in a single component with canonical pair $(6, 1)$. For the particular map we gave at the beginning of this example the pinch locus lay in a component with canonical pair $(5, 0)$; as will be illustrated in the final section of this paper, assembling a little more information about the map will allow one to choose between alternative solutions to the equations.

We will now prove lemma 2.9.

LEMMA 2.9: *Let $f: X\to Y_0$ be a proper birational morphism of n-dimensional spaces with normally crossing exceptional divisor and let $X_0, \dots, X_l$ with accessible components $N_i\subset X_i$, $i=1, \dots, l$, be a quasifactorization sequence with $p=g_{l0}: X_l\to X$. Let $(w, s)=u_{f\circ p}(N_l, H)$ for generic H through a point $y\in Y_0$, let $E_1, \dots, E_r$ be the components of K_f, and let (w_i, s_i) be the canonical y-pair $u_f(E_i, H)$. Let A_i be the center of the quasi-blowing-up a_{i+1}, and define*

$k_i' = codim\ A_{i-1}$
$k_i = number\ of\ components\ of\ K_{f\circ g_{(i-1)0}}\ containing\ A_{i-1}$
$d_i = multiplicity\ of\ (f\circ g_{(i-1)0})^{-1}[H]\ along\ A_{i-1}$

Then there exist non-negative integers $e_1, \dots, e_r$ *and* $c_1, \dots, c_l$ *such that*

(i) $w = \sum_{j=1}^{r} e_j w_j + \sum_{i=1}^{l} c_i (k_i' - k_i)$

(ii) $s = \sum_{j=1}^{r} e_j s_j + \sum_{i=1}^{l} c_i d_i$

Suppose that each A_i, $i=0, \dots, l-1$, *is in the pinch locus* $P_y(f_{i0})$ *for* $f_{i0} = f\circ g_{i0} : X_i \to Y_0$. *Then for any number* c *with* $n \geq c \geq \max_i (k_i' - k_i)$, *and in particular for* $c = n-1$,

(iii) $w - cs \leq \sum e_j (w_j - cs_j)$

Letting $H_1, \dots, H_c$ *be a normally crossing generic set of hyperplanes containing y, this may be restated as*

(iv) $ex_{f\circ p}(N_l; H_1, \dots, H_c) \leq \sum_{j=1}^{r} e_j ex_f(E_i; H_1, \dots, H_c)$

If $d = \min_i d_i$, *and* $A_i = g_{li}(N_i)$ *for* $i = 0, \dots, l-1$,

(v) $s \geq (\sum_{n=1}^{r} e_j s_j) + dl$

Proof: (i) and (ii) are simply restatements of lemma 2.8, with $e_j = s_p(N_l, E_j)$ and $c_i = s_p(N_l, N_i)$. For all i, $c \geq k_i' - k_i$ and $d_i \geq 1$, we have

$$w \leq \sum e_j w_j + c \sum c_i$$
$$s \geq \sum e_j s_j + \sum c_i .$$

Thus multiplying s by c and subtracting gives the desired inequality (iii). Since each of the H_i in (iv) is generic, $s_f(E_j, H_i) = s_j$, and thus $ex_f(E_j; H_1, \dots, H_c) = w_f(E_j) - \sum s_f(E_j; H_i) = w_j - cs_j$. Similarly $ex_{f\circ p}(N_l; H_1, \dots, H_c) = w - cs$. Finally, for (v) if each $A_{i-1} = g_{li-1}(N_l)$, then $g_{li}[N_l] \subset N_i$, so $c_i = s_{g_{li}}(N_l, N_i) \geq 1$ for each i. Thus

$$\sum_{i=1}^{l} c_i d_i \geq \sum_{i=1}^{l} d_i \geq \sum_{i=1}^{l} d = l \cdot d.$$

Remark: These last two equations provide a linear programming problem for the values of the e_j, and thus provide restrictions on the components which can contain $p(F)$ if (w, s) is known.

Definition 2.10: Let the total excess of a point x of X be the sum of excesses of each of the components of K_f with respect to a generic coordinate system at $f(x)$.

$$ex_f(x) = \sum_{x \in D_i} ex_f(D_i : H_1, \dots, H_n)$$

LEMMA 2.11: *If x is a generic point of a curve component of the pinch locus of a morphism of 3-folds over a point obstruction y, then*
(a) *if x is a singleton point, $ex_f(x) \geq 3$*
(b) *if x is a double point $ex_f(x) \geq 4$.*

PROOF: Let A be the component of $P_y(f)$ of which x is a general point, and let $p : X_1 \to X$ be a quasi-blowing-up of A, with exceptional divisor F_1. Then, if k components of K_f contain A,

$$u_{f \circ p}(F_1) = \sum_{i=1}^{k} u_f(D_i) + (2\text{-}k, d), \quad d \geq 1$$

$$(w, s) = \sum_{i=1}^{k} (w_i, s_i) + (2-k, d)$$

$$1 \leq w - 3s = \sum_{i=1}^{k} w_i - 3s_i + (2 - 3d - k)$$

$$\sum_{i=1}^{k} w_1 - 3s_i \geq 3d + k - 1$$

$$\geq 2 + k$$

Substituting $k=1$, 2 gives the desired result.

REMARK: This lemma is an improvement on lemmas 2.2 and 2.3 of [9].

We conclude with a generalization of 1.3 of [9] to quasi-factorization sequences

LEMMA 2.12: *Let $Y_0, \dots, Y_k$ be a quasi-factorization sequence, let $f : X \to Y_0$ be a birational morphism and let y_k be an accessible point of Y_k. Let x be the closure point of a transversal test curve Γ_k through y_k. If for every accessible component M_j^k of Y_k containing y_k there is a generically isomorphic component D_j of K_f containing x, then $f_k : X \to Y_k$ is well defined at x. If these are the only components of K_f containing x, then f_k is an isomorphism at x.*

PROOF: As in lemma 1.3 of [9], if f_k can be shown to be well-defined, and we can show that these are the only components of K_f containing x, then because there are no components available which can collapse, we can conclude from Zariski's Main Theorem, that f_k is an isomorphism. We proceed inductively on $f_0, f_1, f_2, \dots, f_k$ assuming f_j has been shown to be well defined at x. We localize at y_j so that the local center $\bar{B}_j$ of the blowing up b_{j+1} is smooth. We let H_j, for $j = 0, \dots, k-1$, be a generic hypersurface containing $\bar{B}_j$, and we let y_j and Γ_j be the images of y_k, Γ_k in Y_j. By the generic isomorphism $M_i^{(k)} \sim D_i$, we have $s_{f_j}(D_i, H_j) = s_{h_{kj}}(M_i^{(k)}, H_j)$. We will denote this number by s_{ij}. Let $\mathrm{I} \subset \{1, \dots, k\}$ be the subset of indices of accessible components containing y_k. Let Γ be $f_k^{-1}[\Gamma_k]$. We assume f_j well-defined for $j < k$, and prove that f_{j+1} is well-defined. f_j is proper, and thus by the projection formula

$$\begin{aligned}\deg \Gamma \cdot f_j^*(H_j) &= \deg \Gamma_j \cdot H_j = \deg \Gamma_k \cdot h_{kj}^*(H_j) \\ &= \sum_{i \in \mathrm{I}} \Gamma_k \cdot s_{h_{kj}}(M_i^{(k)}, H_j) M_i^{(k)} + \Gamma_k \cdot h_{kj}^{-1}[H_j]\end{aligned}$$

Since our H_j was generic, and h_{jk} is a composition of blowings up, we may assume that $h_{kj}^{-1}[H_j]$ does not contain y_k. Furthermore $\deg \Gamma_k \cdot M_i^{(k)} = 1$. Thus $\deg \Gamma \cdot f^*(H_j) = \sum_{i \in \mathrm{I}} s_{ij}$. Since $\Gamma \cdot f^*(H_j) = \Gamma \cdot \sum_{x \in D_i} s_{f_j}(D_i, H_j) D_i + f_j^{-1}[H_j]$, and $\deg \Gamma \cdot s_{ij} D_i \geq s_{ij}$, for $i \in \mathrm{I}$, we conclude that $\Delta \cdot f_j^{-1}[H_j] = 0$, $\Gamma \cdot D_i = 1$ for $i \in \mathrm{I}$, and $s_{f_j}(D_i, H_j) = 0$ if $x \in D_i$ but $i \notin \mathrm{I}$. Since $x \notin f_j^{-1}[H_j]$ for a generic hypersurface H_j containing the local center, we see that f_{j+1} is well defined at x.

§ 3 Four components collapsing to a point.

Let $f: X \to Y$ be a proper birational morphism collapsing four normally crossing surfaces to a point. We will show, in this section and the next, that with one exception such a morphism is locally factorizable. In order for f to be locally factorizable, it would have to factor through the blowing up of the point. The problem thus splits immediately into two parts. In this section we will show that if it does not factor through the blowing up of the point, then it is Oda's [6] example of a point obstruction, given in § 1 after lemma 1.6. In § 4, we will show that if it does factor through the blowing up, then the resulting morphism, collapsing three surfaces, is locally factorizable.

PROPOSITION 1: *If $f: X \to Y$ is a proper birational morphism of smooth algebraic spaces of dimension* 3 *collapsing four surfaces to a point y_0, and f does not factor through the blowing up of y_0, then the surfaces have canonical pairs* $(3, 1)$, $(4, 1)$, $(5, 1)$ *and* $(6, 1)$, *and after blowing up one smooth curve A_0 in the* $(6, 1)$ *component, the resulting morphism is directly factorizable.*

PROOF: In order to analyze K_f, we first build a bridge between X and Y_1, the blowing up of the point y_0. We may assume that Y is a scheme.

LEMMA 3.1: *Let $f: X \to Y$ be a proper birational morphism of 3-folds. Suppose Y_1 is obtained by blowing-up a point $y \in S_f$ for which*

$f^{-1}[y]$ is a surface. If f_1 is not well-defined, then there is a curve $B_1 \subset M_1$, and factorization sequences $Y, Y_1, \dots, Y_k$, and $X, X_1, \dots, X_l$, with accessible components M_j and N_i as in Fig. 3, such that for generic H through y,

(i) *M_k is generically isomorphic to N_l, $B_j \subset M_j$, and $h_{j1}[B_j]=B_1$.*
(ii) *$f_j^{-1}[M_j]$ is a surface for $j<k$.*
(iii) *$g_{ii'}(N_i) \subset f_{i0}^{-1}[H]$ for $i'<l$.*

PROOF: (i) Let G_{01} be a desingularization of the graph of the correspondence $f_1 : X \to Y_1$. Let $F \subset G_{01}$ be a surface of minimal weight collapsing to a curve B_1 in M_1, such that its image in X is contained in $f^{-1}[H]$. Such a surface exists by lemma 1.10 above.

Let $q_1 : G_{01} \to Y_1$ be the projection from the graph, with $B_1 = q_1(F)$. By Lemma 1.14, we can construct a quasi-blowing up $b_2 : Y_2 \to Y_1$ with center B_1 and accessible component M_2 generically isomorphic to the blowing-up of B_1. Let $q_2 : G_{01} \dashrightarrow Y_2$ be the induced birational correspondence, and let $B_2 = q_2[F]$. Since q_2, being birational, is well-defined on points of codimension 1, we have $b_2 \circ q_2[F] = q_1(F) = B_1$, whence $b_2(B_2) = B_1$. B_2, being the strict image of an irreducible divisor, is irreducible. Since, over the generic point of B_1, $b_2^{-1}(B_1)$ is contained in M_2, we conclude that $B_2 \subset M_2$.

Let us now suppose that we have constructed steps $Y_0, Y_1, \dots, Y_j$ in a factorization sequence, such that $q_{j'} : G_{01} \dashrightarrow Y_{j'}$ is the induced birational correspondence, and when $j'<j$, $q_{j'}[F]=B_{j'}$ is the center of the following quasi-blowing-up $b_{j'+1}$. As in the diagram in Fig. 3, we let $h_{ij} : Y_i \to Y_j$ denote the composition of quasi-blowing-up and let $h_j = h_{j0}$. If $q_j[F]$ is not a surface, we define $B_j = q_j[F]$, and apply lemma 1.14 to construct a quasi-blowing-up $b_{j+1} : Y_{j+1} \to Y_j$ with center B_j. As in the case $j=2$ above, we find that $b_j \circ q_j[F] = q_{j-1}[F] = B_{j-1}$ implies that $B_j \subset M_j$, since M_j is generically isomorphic to the blowing-up of B_{j-1}. Similarly, since $h_{j1} \circ q_j[F] = q_1[F] = B_1$, we find that $h_{j1}(B_j) = B_1$. If d is the degree of B_1, then for any generic hyperplane H, $h_{j+1}^{-1}[H]$ intersects M_{j+1} transversally along md contractible curves, where m is the degree of B_j over B_1. We need to show that after a finite number of such steps $q_k[F]$ is the surface M_k, and thus M_k is generically isomorphic to F. We surely have the weight $w_{h_j}(M_j)$ bounded above by the weight $w_{q_0}(F)$, by lemma 2.3, since q_0 is equivalent to $h_j \circ q_j$, and $q_j(F) \subset M_k$. However, by lemma 2.4, since $B_{j'} \subset M_{j'}$ for each $i'<k$, we find that the sequence $w_{h_j}(M_j)$ is strictly increasing. We conclude that for some k, $w_{h_k}(M_k) = w_{q_0}(F)$. From lemma 2.3 we see that M_k is the only component of K_{h_j} containing $q_k(F)$,

with excess 0, and from lemma 2.4 we then conclude that the codimension of $q_k(F)=1$, i.e. that F is generically isomorphic to M_k.

To complete the proof of (i) we construct a quasi-factorization sequence $X=X_0, X_1, X_2, \dots, X_l$, with accessible components N_i, and centers $A_i \subset N_i$ which are the projections of F to X_i. The details and the proof of finiteness proceed as in the construction of the Y_j sequence, the only difference being that $A_0, A_1, \dots$ can be points. This can occur only when the image of F in the non-desingularized graph of f_1 is a singular curve, projecting to A_0 in X and to B_1 in Y_1. However, once one of the A_i is a curve, all subsequent $A_{i'}$ will also be curves.

We continue the sequence $X_0, \dots, X_l$ until N_l is generically isomorphic to F, and thus to M_k. This completes the proof of (i).

(ii): Let j be the lowest number for which $f_{j+1}^{-1}[M_{j+1}]$ is not a surface. We want to use the minimality of the weight of F to show that $j=k-1$. Since $q_i[F] \subsetneqq M_i$, for $i<k$ and thus the weight $w_{q_0}(F) > w_{h_i}(M_i)$, it will suffice to find a surface F' in the desingularized graph G_{01} such that F' is generically isomorphic to M_{j+1} and the image of F' in X is contained in $f^{-1}[H]$.

Let $\overline{Y}_1$ be the localization of the scheme Y_1 along B_1 in the Zariski topology on Y_1. Using base extension by $\overline{Y}_1$, we get a morphism of surfaces $\bar{q}_1 : G_{01} \underset{Y_1}{\times} \overline{Y}_1 \to \overline{Y}_1$. Let $\overline{W} = G_{01} \underset{Y_1}{\times} \overline{Y}_1$ and let $\overline{F}$ be the curve induced by F in $\overline{W}$.

By the Zariski factorization theorem for surfaces, $\bar{q}_1$ must factor into a sequence of blowings up of points, and at each step we may choose an arbitrary point of the fundamental locus as the center of the blowing up. If we consistently choose the image of $\overline{F}$ as our center, then we construct a sequence of spaces $\overline{Y}_j$ with $\overline{Y}_j \simeq Y_j \underset{Y_1}{\times} \overline{Y}_1$. These will actually all be schemes, since the special points at which etale neighborhoods were needed will drop out in the process of localizing along B_1. The centers of the blowings-up will be $\overline{B}_j \simeq B_j \underset{Y_1}{\times} \overline{Y}_1$, and each $\overline{M}_j \simeq M_j \underset{Y_1}{\times} \overline{Y}_1$ will be generically isomorphic to a curve F_j in the exceptional divisor of $\bar{q}_1$. Thus if F' is the surface in G_{01} which induces $\overline{F}_{j+1}$, F' is generically isomorphic to M_{j+1}. Let $p : G_{01} \to X$ be the projection of the graph onto X. If we can show that $p(F') \subset f^{-1}[H]$, then we can conclude that $F=F'$ and $j+1=k$.

Let $C=p(F')$, and suppose $C \not\subset f^{-1}[H]$ for a generic hyperplane H. If so, $f_1 : X \dashrightarrow Y_1$ is well defined at the generic point of C, and thus X is isomorphic to G_{01} almost everywhere along C. This would imply that C is a surface generically isomorphic to F' and thus to M_{j+1}, contradicting the assumption that $f_{j+1}^{-1}[M_{j+1}]$ is not a surface. We conclude, as desired, that $j+1=k$, and thus that for each $j<k$, M_j is generically isomorphic to a surface D_j in K_f, completing the proof of (ii).

(iii): We want to show that if $g_{ii'}: X_i \rightarrow X'_i$, with $i>i'$ is a composition of blowings-up from the factorization sequence, then $g_{ii'}(N_i)$ lies in the pinch locus $f_{i'0}^{-1}[H]$ of f_{i0}. We first reduce to the case $i=l$ by noting that $g_{i(i-1)}(N_i)=A_{i-1}=g_{l(i-1)}(N_l)$, since N_l is generically isomorphic to F, and A_{i-1} is the image of F.

We now reduce further to the case $i'=l-1$, by noting that $f_{i'0}^{-1}[H]=g_{(l-1)i'}(f_{(l-1)0}^{-1}[H])$. It thus suffices to prove that $g_{l(l-1)}(N)\subset f_{(l-1)0}^{-1}[H]$. Assuming this is not the case, we will show that N_{l-1} is generically isomorphic to M_{k-1} and derive a contradiction.

By our assumption, the surfaces $f_{(l-1)0}^{-1}[H]$ do not entirely contain $A_{l-1}=g_{ll-1}[N_l]$, However, they must intersect A_{l-1} at some point, since $f_{l0}^{-1}[H]$ intersects N_l at a general point. We conclude that A_{l-1} is a curve, and intersects $f_{(l-1)0}^{-1}[H]$ in isolated points. Furthermore, f $_{l0}^{-1}[H]$ is obtained from $f_{(l-1)0}^{-1}[H]$ by blowing up these points, with exceptional curves C_i. f_{lk} gives an isomorphism at the generic point of each C_i, mapping it to some component C'_i of $M_k \cap h_k^{-1}[H]$. As described in the definition of the factorization sequence in (i) above, $h_k^{-1}[H]$ is smooth and transversal to M_k along C'_i. We conclude that $f_{l0}^{-1}[H]$ is smooth and transversal to N_l along C_i, whence $f_{(l-1)0}^{-1}[H]$ is smooth and transversal to A_{l-1} at each intersection point p_i. C_i is thus a contractable curve with self intersection -1, isomorphic to C'_i, another contractable curve with self intersection -1. The complete image of p_i under the induced correspondence $f_{(l-1)(k-1)}$ is thus $p'_i=h_{k(k-1)}(C'_i)$, a single point, so $f_{(l-1)(k-1)}$ is well-defined at p_i. This induces, locally, a morphism from $f_{(l-1)0}^{-1}[H]$ to $h_{k-1}^{-1}[H]$ which is a birational morphism of surfaces.

p_i lies in $f_{(l-1)0}^{-1}[H] \cap N_{l-1}$ and at most one other component of $f_{(l-1)0}^{-1}[H] \cap \operatorname{supp}(K_f)$. If either of these components collapses to a point under the morphism of surfaces then the total multiplicity (in the surface canonical class) of components containing p_i will be higher than the total multiplicity of components containing p'_i, whence C_i would not generically be isomorphic to C'_i. Thus we must have a local isomorphism of surfaces. We conclude that N_{l-1} cannot collapse under $f_{(l-1)(k-1)}$. Since it is the highest weight component of $K_{f_{(l-1)0}}$ containing p_i, it must be isomorphic to the highest weight component of K_{h_k} containing p'_i, which is M_{k-1}. However, M_{k-1} is generically isomorphic to a component of D_{k-1} of K_f, while $g_{(l-1)0}(N_{l-1})\subset A_0$, of codimension at least 2. Contradiction.

REMARK: M_1 is isomorphic to a projective plane, so B_1 is a projective curve and therefore has a degree. If B_1 is a curve of degree d', then $h_1^{-1}[H]$ intersects B_1 in d' points, and thus $h_k^{-1}[H]$ will intersect M_k in d' fibers, each of which maps onto A_0 in X. We conclude that $f^{-1}[H]$ has at least d' branches along A_0. The degree of B_1 is thus bounded by the multiplicity of $f^{-1}[H]$ along A_0. In fact, it is bounded by all the multiplicities d_i of $f_{i0}^{-1}[H]$ along A_i.

To continue the proof of Prop. 1, we now divide into cases.

Let $(w, s)=u_{h_k}(M_k, H)$ and let $(w_i, s_i)=u_f(D_i, H)$ for all components $D_1, \ldots, D_4$ of K_f. Note that $k\leq 5$, since K_f has at most four components. If $B_j\subset M_j\cap M_{j'}^{(j)}$, then by lemma 2.4,

$$u_{h_{j+1}}(M_{j+1}, H)=(w_j, s_j)+(w_{j'}, s_{j'})$$

If B_j is not contained in an intersection, then by the same lemma

$$u_{h_{j+1}}(M_{j+1}, H)=(w_j, s_j)+(1, 0).$$

We note in particular that these conditions limit the possible increments in the sequence $\bar{s}=(s_1, \ldots s_{k-1}, s)$. For any j, if $s_{j+1}\neq s_j$, then $s_{j+1}=s_j+s_{j'}$ where $s_j=s_{j-1}+s_{j'}$ or $j'=j-1$. We now divide into cases according to the various sequences s which can be built up this way, starting with $s_1=s_2=1$. Since s_f is a point, each $s_i\geq 1$, and thus since $s_{g_l}(N_l, f^{-1}[H])\geq 1$, we have, from Lemma 2.3 applied to $f\circ g_l$, that $s\geq 2$.

We consider four vectors:

$\bar{s}=(s_1, \ldots, s_{k-1}, s)$
$\bar{w}=(w_1, \ldots, w_{k-1}, s)$
$\bar{e}_2=\bar{w}-2\bar{s}$
$\bar{e}_3=\bar{w}-3\bar{s}$

Lemmas 2.9 and 2.11 give a number of restrictions on these vectors. We now divide the problem into cases. We will discover that the smaller k is, the fewer cases there are and the harder they are to deal with.

<u>$k=5$</u>: We want to eliminate all cases with $k=5$. Our main tool will be lemma 2.11, saying that a component of the pinch locus must be in a single surface of excess≥ 3, or an intersection of excess≥ 4. We generate the vector $\bar{e}_3$ of excesses.

1 $\bar{s}=(1, 1, 1, 1, 2)$, $\bar{w}=(3, 4, 5, 6, 11)$, $\bar{e}_3=(0, 1, 2, 3, 5)$
2 $\bar{s}=(1, 1, 1, 2, 2)$, $\bar{w}=(3, 4, 5, 9, 10)$, $\bar{e}_3=(0, 1, 2, 3, 4)$
3 $\bar{s}=(1, 1, 1, 2, 3)$, $\bar{w}=(3, 4, 5, 9, 13)$, $\bar{e}_3=(0, 1, 2, 3, 4)$
4 $\bar{w}=(3, 4, 5, 9, 14)$, $\bar{e}_3=(0, 1, 2, 3, 5)$
5 $\bar{s}=(1, 1, 2, 2, 2)$, $\bar{w}=(3, 4, 7, 8, 9)$, $\bar{e}_3=(0, 1, 1, 2, 3)$
6 $\bar{s}=(1, 1, 2, 2, 4)$, $\bar{w}=(3, 4, 7, 8, 15)$, $\bar{e}_3=(0, 1, 1, 2, 3)$
7 $\bar{s}=(1, 1, 2, 3, 3)$, $\bar{w}=(3, 4, 7, 10, 11)$, $\bar{e}_3=(0, 1, 1, 1, 2)$

8 $\bar{s}=(1,1,2,3,3)$, $\bar{w}=(3,4,7,11,12)$, $\bar{e}_3=(0,1,1,2,3)$
9 $\bar{s}=(1,1,2,3,4)$, $\bar{w}=(3,4,7,10,13)$, $\bar{e}_3=(0,1,1,1,1)$
10 $\bar{s}=(1,1,2,3,4)$, $\bar{w}=(3,4,7,11,15)$, $\bar{e}_3=(0,1,1,2,3)$
11 $\bar{s}=(1,1,2,3,5)$, $\bar{w}=(3,4,7,10,17)$, $\bar{e}_3=(0,1,1,1,2)$
12 $\bar{s}=(1,1,2,3,5)$, $\bar{w}=(3,4,7,11,18)$, $\bar{e}_3=(0,1,1,2,3)$

Except in (1)-(5) we do not have, in $D_1, \dots, D_4$, a component or pair of components is K_f satisfying lemma 2.11. We now apply lemma 2.9(v) to (1)-(5), getting $s<\sum_{A_0\subset D_j} e_i\ s_i$. In cases 1,2, and 5 where $s=2$, we would have to have a single component D_i of excess≥ 3, which doesn't exist. In cases 3 and 4, we don't have an intersection $D_i\cap D_j$ with $s_i=s_j=1$ and total excess at least 4. Thus we would have $A_0\subset D_4$ with excess 3, and $e_4=1$. However, by 2.9(iii), the excess of M_k, 4 or 5, would have to be smaller than the excess of D_4, which is 3. Contradiction.
$\underline{k<5}$: Here we have other components in K_f whose canonical pairs are not known from the factorization sequence. We will divide into four cases according to $\bar{s}$.

A. $\bar{s}=(1,1,1,2)$, $\bar{w}=(3,4,5,9)$, $\bar{e}_3=(0,1,2,3)$
B. $\bar{s}=(1,1,2,2)$, $\bar{w}=(3,4,7,8)$, $\bar{e}_3=(0,1,1,2)$
C. $\bar{s}=(1,1,2,3)$, $\bar{w}=(3,4,7,10)$, $\bar{e}_3=(0,1,1,1)$
$\bar{w}=(3,4,7,11)$, $\bar{e}_3=(0,1,1,2)$
D. $\bar{s}=(1,1,2)$, $\bar{w}=(3,4,7)$, $\bar{e}_3=(0,1,1)$

Case A: $\bar{s}=(1,1,1,2)$. M_k has canonical pair (9, 2) with excess 3, and $k=4$. Applying lemma 2.9(iii) with $c=2$, we see that A_0 is contained in a single component D_4 with $s_4=1$, and $5=9\text{-}2\cdot 2\leq w_4-2\cdot s_4$. Thus $w_4\geq 7$. Furthermore, $2=s=e_4s_4+d_1$, so $e_4=s_4=d_1=1$. Thus, by the Remark after Lemma 3.1, B_1 is nonsingular, since $d_1=1$. We now apply lemma 2.7 to conclude that f_3^{-1} is an isomorphism at every accessible non-intersection point of $M_1^{(3)}$, $M_2^{(3)}$ and M_3 except on $f_3[D_4]$. By this same lemma, we conclude that D_4 is the blowing up of $f_3[D_4]$. By lemma 2.4,

$$w_4=w_i+k'-1.$$

Since $w_i\leq 5$, and $w_4\geq 7$, we conclude that k', the codimension $f_3[D_4]$, is 3, indicating that $f_3[D_4]$ is an isolated point. However this contradicts the connectedness of $f_3(D_4)$ which contains $M_2^{(3)}\cap M_3$.
Case B: Since $s=2$, the component $f_k^{-1}[M_k]$ of the pinch locus must lie in D_4, with

$$2=s\geq e_4s_4+dl.$$

We conclude that $e_4=l=d=s_4=1$, whence $B_1=f_1[D_2]$ must have degree 1, and be isomorphic to P^1. Since B_1 is nonsingular, and $f_2\ [D_3]\subset M_2\cap M_1^{(2)}$, we find that by lemma 2.7, f_2^{-1} is an isomorphism except on $f_2[D_4]$ and $M_2\cap M_1^{(2)}$. Since $f_2(D_4)\supset M_2\cap M_1^{(2)}$ and is connected, but must have generic point in a single component of K_{h_2} so that $s_4=1$ will hold, we conclude that $f_2[D_4]$ is a curve. By lemma 2.7, D_4 is generically isomorphic to the blowing up of that curve, and thus has canonical pair $(5, 1)$. In that case, however, the excess of D_4 is 2, in contradiction to lemma 2.11.

Case C: This case is more difficult.

$\bar{s}=(1, 1, 2, 3)$. Applying lemma 2.11 again, since the excesses of D_1, D_2 and D_3 are 0, 1, and 2 respectively, we see that A_0 must lie in D_4. Thus from lemma 2.9, $s>\sum_{A_0\subset D_i} s_i)+1$, so we have $s_4\leq 2$ and $A_0\not\subset D_3\cap D_4$. Since the total excess along a "bad" curve lying in a single component must be at least 3, and along an intersecting curve must be at least 4, D_4 must have an excess of at least 3. Since S_f is a point, $s_4\geq 1$, so $w_4-3s_4\geq 3$ implies that $w_4\geq 6$.

C. 1: $s_4=1$. Consider lemma 2.7 applied to the simple factorization sequence Y_0, Y_1 dominated by the morphism $f: X\rightarrow Y$. Every point y_1 of M_1 is a singleton accessible point, and M_1 has generic order 1. Thus, except possibly at a finite number of points, either f_1^{-1} is an isomorphism at y, with $f_1^{-1}(y_1)\in D_1$, or else there is a component D_i of K_f such that $y_1\in f_1[D_i]$ and D_i is generically isomorphic to the blowing up of $f_1[D_i]$. Since only one component, D_2, has the canonical pair $(4, 1)$ appropriate to the blowing up of a curve, we see that $B_1=f_1[D_2]$ is the only curve in M_1 on which f_1^{-1} is not an isomorphism. Since $f_1(D_4)\supset f_1(A_0)=B_1$, then by the Zariski connectedness theorem, the strict image $f_1[D_4]$ is connected to B_1. Since it cannot be connected by a curve intersecting B_1, and B_1 is irreducible, the only two possibilities for $f_1[D_4]$ are a point of B_1 or all of B_1. We thus divide into subcases

C. 1. a: $f_1[D_4]$ is point P_1. Consider $f_1^{-1}[P_1]$. It cannot be surface, for then it would have canonical pair $(5, 1)$, with excess 2, which is not possible for D_4. Thus $f_1^{-1}[P_1]$ has dimension no greater than 1. We now apply the Danilov lemma [3], modified as in lemma 1.8, to conclude that P_1 has an etale neighborhood in which B_1 has a smooth branch, such that along the fiber over P^1 in the blowing-up of this branch we have an isomorphism of the exceptional divisor M_2 with D_2. Let us assume that the locally factorizable morphism b_2 was chosen so that it factored through such a blowing-up. Then $f_2[D_4]$ would be a point on that fiber. It could not be in the intersection, since $s_4=1$. Thus it would be an isolated point on the fiber. Over the general point of B_1, f_2^{-1} must be an isomorphism on $M_2-M_1^{(2)}$, by lemma 2.7, since there are no components other than D_1 and D_2 with canonical pair $(w', s')\leq(5, 1)$. Thus $f_2[D_4]$ is an isolated point. Since $M_1^{(2)}\cap M_1\subset f_2(A_0)\subset f_2(D_4)$, this contradicts the connectedness of the image under a birational correspondence.

C. 1. b: Suppose $f_1[D_4]=B_1$. The same argument from lemma 2.7 used above shows that f_2^{-1} is an isomorphism on the generic fiber of $M_2-M_1^{(2)}$. Thus $f_2[D_4]$ could lie only in $M_2\cap M_1^{(2)}$. However, if so, by lemma 2.3 $s_4\geq s_1+s_2=2$, in contradiction to our assumption that $s_4=1$.

C. 2: $s_4=2$. From lemma 2.9(v), $s\geq s_4+dl$, whence, since $s=3$ and $s_4=2$, we get $l=1$ and $d=d_1=1$. As remarked after lemma 3.1, the degree of B_1 as a curve in $M_1\cong P^2$ is bounded by d_1. We conclude that deg $B_1=1$, i.e. that $B_1\cong P^1$ and is thus non-singular. Since every component of K_f has its image at least connected to B_1, and there is only one component D_2 with the canonical pair $(4, 1)$ appropriate to the blowing up of a curve, we conclude, by applying 2.7 to Y_0, Y_1, that f_1 is an isomorphism outside of B_1. We now go one step further and apply 2.7 to Y_0, Y_1, Y_2, where we can assume that $b_2: Y_2\rightarrow Y_1$ is simply the blowing up of B_1. Since B_1 is nonsingular, the set of bad points in lemma 2.7 is empty, and every point of $M_2-M_1^{(2)}$ is accessible of order one. Since (2) cannot hold because there are no components in K_f with the canonical pairs $(5, 1)$ or $(6, 1)$ appropriate to the blowing-up of a curve or point we conclude that at every point y_2 of $M_2-M_1^{(2)}$, f_2^{-1} is an isomorphism. Thus $f_2[D_4]\subset M_2\cap M_1^{(2)}=B_2$.

Y_3 is obtained by blowing-up B_2. Let us show that f_3^{-1} is an isomorphism at every point of $M_3-M_1^{(3)}-M_2^{(3)}$. Let y_3 be any such point, and let Γ_3 be a transversal test curve through y_3. Let H be a generic hyperplane through $y_0\in Y$. Then $s_{h_3}(M_3, H)=2$, so $\Gamma_3\cdot h_3^*(H)=2$. Letting y_1, Γ_1 be the images of y_3, Γ_3 respectively in Y_1, we get $\Gamma_1\cdot h_1^*(H)=2$. Let $\Gamma=f_1^{-1}[\Gamma_1]\subset X$, and let x be the closure point.

We first show that the singleton points of D_i are isomorphic to the singleton points of $M_i^{(2)}$, for $i=1,2$. Suppose $\bar{x}$ is a singleton point of D_1 or D_2. Let $\bar{\Gamma}$ be a transversal test curve through $\bar{x}$. Since $s_f(D_i, H)=1$, for $i=1,2$, and f_1 is well defined at $\bar{x}$, so that $\bar{x}\in f^{-1}[H]$, we have $\bar{\Gamma}\cdot f^*(H)=1$, whence by the projection formula $1=f(\bar{\Gamma})\cdot H=h_1^{-1}[f(\bar{\Gamma}]\cdot h_1^*(H)=h_2^{-1}[f(\bar{\Gamma})]\cdot h_2^*(H)$. We conclude that $h_2^{-1}[f(\bar{\Gamma})]$ intersects K_{h_2} at a point of first order, i.e., at a point of $M_1^{(2)}\cup M_2-B_2$. Since f_2^{-1} is an isomorphism at these points, we find that $D_1-(D_2\cup D_3\cup D_4)\cong M_1^{(2)}-M_2\cong M_1^{(3)}-M_3$ and $D_2-(D_1\cup D_3\cup D_4)\cong M_2-M_1^{(2)}\cong M_2^{(3)}-M_3$.

Returning to our original point x, since $y_3\notin M_1^{(3)}\cup M_2^{(3)}$, we find that $x\in D_3\cup D_4$. We can thus apply lemma 1.1 of [9], which says that since $\sum_{x\in D_i}s_f(D_i, H)\geq\Gamma\cdot f^*(H)=2$, we actually have equality, f_1 is well-defined at

x, and x is a singleton point of D_3 or D_4.

Now let H_1 be a generic hypersurface in Y_1 containing B_1. Since Γ_3 is transversal to M_3 at a generic point, its image Γ_2 is transversal to $M_1^{(2)}$ and M_2, and Γ_1 is tangent to M_1, and transversal to B_1. We thus have $\Gamma_1 \cdot H_1 = 1$. Since $f_1[D_i] \subset B_1$ for $i=1,2$, we have $s_{f_1}(D_i, H_1) \geq 1$. Thus we can again apply lemma 1.1 of [9], to conclude that f_2 is well-defined at x. Since we again have $f_2[D_i] \subset B_2$ for $i=1,2$, we can repeat this with a generic hypersurface H_2 through B_2, and conclude that f_3 is well-defined at x. If x is a singleton point of D_3, then since D_3, being generically isomorphic to M_3, does not collapse, we have an isomorphism. If $x \in D_4$, $y_3 \in f_3[D_4]$.

We have thus shown that f_3^{-1} is an isomorphism except on $f_3[D_4]$ and possibly on $M_i^{(3} \cap M_3$, for $i=1,2$. According to our original hypotheses, $f_3(A_0) = M_i^{(3)} \cap M_3$ for either $i=1$ or $i=2$. Since $f_3(D_4)$ must be connected, and the generic point $f_3[D_4]$ cannot have order greater than $s_4=2$, we see that $f_3[D_4]$ must be a curve intersecting $M_i^{(3)} \cap M_3$ properly. Blowing it up, and applying lemma 1.1 one last time, for the same curve Γ_3, we conclude that D_4 is generically ismorphic to the blowing-up of $f_3[D_4]$. It must, therefore, have canonical pair

$$\begin{aligned}(w_4, s_4) &= (w_3, s_3) + (k'-k, d)\\ &= (7, 2) + (2-1, 0)\\ &= (8, 2)\end{aligned}$$

by lemma 2.4. However, if so, the excess $w_4 - 3s_4 = 8-6 = 2$, which is too small.

We have thus eliminated all but the last case:

Case D: This case is considerably more difficult than the previous ones, for here, instead of reaching a contradiction, we must show that the morphism $f: X \to Y$ is one particular morphism. We therefore preface the proof with an outline which we hope will serve for most readers as a satisfactory substitute for the actual detailed proof.

(a) We show that in case D the pinch locus contains an irreducible curve A_0 whose general point is a singleton point of a component D_4 with canonical pair $(6, 1)$.

(b) We determine the nature of the various components of K_f: D_1 is generically isomorphic to the blowing-up M_1 of the point $y \in Y$, D_2 is generically isomorphic to the blowing-up M_2 of a curve $B_1 \simeq P^1$ in M_1, and D_3 with canonical pair $(5, 1)$ is generically isomorphic to the blowing-up M_3 of a curve B_2 in M_2. (See Fig. 5 for the two alternative possibilities

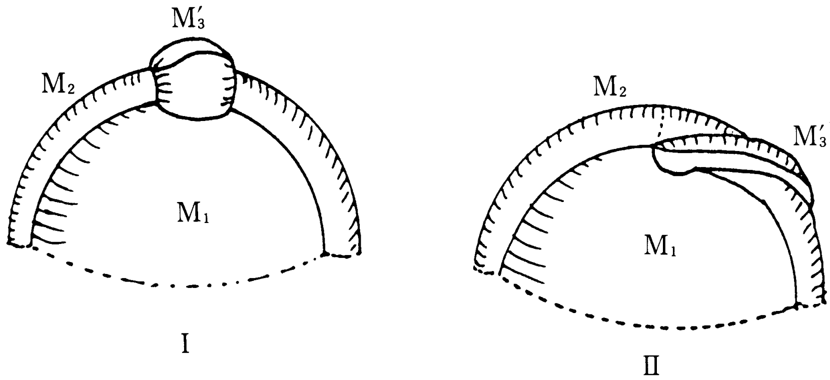

Fig. 5

for M_3.)

(c) *We build a quasi-factorization sequence* Y_0, Y_1, Y_2, Y'_3, Y'_4, show that $D_1 \cap D_2 = \emptyset$, and determine where the induced correspondence f'_4 is an isomorphism, thereby confining the accessible points is the image of the pinch locus to intersections of accessible components.

(d) We prove that the only singleton points of the pinch locus $P_y(f)$ are contained in the irreducible curve A_0.

(e) The strict image of D_3 in Y_2 is fiber, so that the strict image of D_3 in Y_1 is a point y_1.

(f) We then conclude that $D_2 \cap D_3 = \emptyset$.

(g) By blowing-up a singleton point x of A_0 and considering its multiplicity in the strict preimage of two transversal generic hyperplanes, we show that x is a smooth point of A_0.

(h) In a similar manner, we demonstrate that A_0 intersects D_1 and D_2 transversally.

(i) We construct a strong factorization sequence for f by blowing up A_0 to get a component $N_1 \cong P^1 \times P^1$, then contracting N_1 along its second fibration to get a directly factorizable toroidal morphism.

We now carry out (a)-(i).

(a) A_0 lies in D_1 with canonical pair $(6, 1)$:

$\bar{s} = (1, 1, 2)$. We have $u_{h3}(M_3, H) = (7, 2)$, and it is generically isomorphic to the result of a single blowing-up in X, of a locus A_0 lying in a single component D_4 of K_1. We must have $w_4 < 7$, $s_4 < 2$, by lemma 2.9(i, v) on the one hand, and on the other hand, by lemma 2.11, D_4 must have excess at least 3. Thus $(w_4, s_4) = (6, 1)$. Also, applying lemma 2.9, $d_1 = 1$, so B_1 is a P^1 of degree one, and we may take b_2 to be the blowing up of B_1. Similarly, b_3 may be taken as the blowing-up of the smooth intersection.

(b) D_3 has canonical pair $(5, 1)$:

We first establish the canonical pair of the remaining component D_3. Consider a general point $y_1 \in f_1[D_4]$. f_1^{-1} is not an isomorphism at y_1. Since the set $\hat{C}$ of possible bad points given in lemma 2.7 is empty for the case of a single blowing-up of B_1, we conclude from lemma 2.7 that the closure point x of a generic test curve through y_1, i.e., $x \in f_1^{-1}[y_1]$, must lie in a component D_i which is generically isomorphic to the blowing-up of $f_1[D_i]$. Based on this fact, we wish to prove that $(w_3, s_3) = (5, 1)$ and $f_1[D_3] \subset B_1$.

We have two possibilities : $f_1[D_i]$ is a point or a curve. In the first case $d_i = D_3$ and $(w_3, s_3) = (5, 1)$. In this case f_1^{-1} is an isomorphism except on $B_1 = f_1[D_2]$ and $y_1 = f_1[D_3] = f_1[D_4]$, by lemma 1.2 of [9] or by another application of 2.7 to a general point of M_1. Since $B_1 = f_1(A_0) \subset f_1(D_4)$ and $f_1(D_4)$ is connected, we conclude that $y_1 \in B_1$, as desired.

Now consider the case that $f_1[D_i]$ is a curve, so that $(w_i, s_i) = (4, 1)$. Since $\dim f_1^{-1}[y_1] \leq 1$, we can apply lemma 1.8 to find a local center $\tilde{B}$ at the Henselization $\tilde{y}_1 \in \tilde{Y}_1$, such that after base extension, $\tilde{f}_1$ factors through the blowing up of $\tilde{b}_2$ of $\tilde{B}$. Let b_2 be a quasi-blowing up with $\tilde{B}$ as local center at y_1. Let y_2 be a general point of $f_2[D_4]$ such that $h_{21}(y_2) = y_1$. Since $s_4 = 1$, the generic point of $f_2[D_4]$ must lie in a single first order component of K_{h_2}. Thus $y_2 \in f_1^{-1}[y_1]$, and y_2 is accessible. We apply lemma 2.7 to the accessible first order points of M_2. Case (3) occurs neither at y_1, where the component containing $f_1^{-1}[y_1]$ maps to the center of the local blowing-up, nor at y_2, since the set of bad points is finite and y_2 is general. Thus in a neighborhood of the fiber at y_2, f_2^{-1} is an isomorphism except on the strict images of components other than D_1 and D_i. If $f_2[D_4]$ were an isolated point, this would contradict the connectedness of $f_2(D_4)$, since $f_2(A_0) \subset M_2 \cap M_1^{(2)}$. Thus y_2 must lie on some curve in M_2 on which f_2^{-1} is not well-defined. We conclude from lemma 2.7 that there exists a component of K_f generically isomorphic to the blowing up of this curve, hence of canonical pair $(5, 1)$. The only possibility is D_3. We conclude that $D_i = D_2$, $f_1[D_2] = B_1$, and thus that $f_1[D_3] \subset h_{21}(f_2[D_3]) \subset h_{21}(M_2) = B_1$, as desired. Note that we have also shown that $f_2[D_4] \subset f_2[D_3]$.

We are now close to our goal. We have four components, D_1, D_2, D_3, D_4, with canonical pairs $(3, 1)$, $(4, 1)$, $(5, 1)$ and $(6, 1)$ respectively. D_1 is generically isomorphic to the blowing up of y. D_2 is generically isomorphic to the exceptional divisor M_2 which results from blowing up a curve $B_1 \cong P^1$ in M_1. D_3 is generically isomorphic to the blowing up of a curve in M_2. We have seen two possibilities for this curve. It can either be the fiber of M_2 over a point y_1, in which case D_3 is generically isomorphic to the blowing up of y_1, or else it can be a section of B_1 in M_2. See Fig. 5.

Let H_1 be a smooth hypersurface through y such that $h_1^{-1}[H_1]$ contains the linear subspace B_1 of M_1 in Y_1. Assume H_1 is a generic hypersurface with this property. In particular we may assume that if $h_2[D_4]$ is a point on M_2, it is not contained in $h_1^{-1}[H_1]$. We now blow up $M_2\cap M_1^{(2)}$ to obtain M_3. Since $s_{h_2}(M_1^{(2)}, H_1)=s_{h_1}(M_1, H_1)=1, s_{h_2}(M_2, H_1)=s_b{}^1(M_1, H_1)+s_{b_2}(M_2, h_1^{-1}[H_1]=2$, and $M_2\cap M_1^{(2)}$ is not contained in $h_2^{-1}[H_1]$, we get $s_{h_3}(M_3, H_1)=s_{h_2}(M_2, H_1)+s_{h_2}(M_1^{(2)}, H_1)=2+1=3$. On the other hand, D_3 is generically isomorphic to the blowing up of a curve $f_2[D_3]$ in M_2 which is not contained in $h_2^{-1}[H_1]$. Thus $s_f(D_3, H_1)=2$. Similarly, if we blow up $f_2[D_3]$ in M_2 to get a space Y'_3, D_4 is generically isomorphic to the blowing up of a curve in the exceptional divisor M'_3, whence $s_f(D_4, H_1)=2$. We have shown that M_3, the blowing up of $M_2\cap M_1^{(2)}$, is generically isomorphic to the blowing up of the curve A_0 in D_4. Thus

$$3=s_{h_3}(M_3, H_1)=s_f(D_4, H_1)+s_{f_3^{-1}}(M_3, f^{-1}[H_1]).$$

We conclude that $f_3^{-1}[M_3]=A_0$ must be contained in $f^{-1}[H_1]$, since we must have $s_{f_3^{-1}}(M_3, f^{-1}[H_1])=1$. f_1 is not well-defined on $A_0\subset f^{-1}[H_1]$. We wish to use this fact to show that only the first of the two cases in Fig. 5 is possible, and that $D_1\cap D_2=\emptyset$ and $D_2\cap D_3=\emptyset$.

(c) Construction of the quasi-factorization sequence:

We build up a factorization sequence Y_0, Y_1, Y_2, Y'_3, Y'_4, and determine step by step where the corresponding induced morphisms from X are isomorphisms.

<u>Step 1</u>: $b_1: Y_1\to Y_0$ is the blowing-up of the point y_0. f_1^{-1} is an isomorphism except on the strict images of other components, all of which are contained in B_1. Furthermore, for every singleton point of D_1, f_1 is well-defined by lemma 2.11. Since no component through the point collapses, f_1 is an isomorphism there. Thus

$$\begin{aligned} D_1-(D_2\cup D_3\cup D_4)&\overset{f_1}{\rightrightarrows} M_1-B_1\\ &\rightrightarrows M_1^{(2)}-M_2\end{aligned}$$

<u>Step 2</u>: $b_2: Y_2\to Y_1$ is the blowing up of B_1, M_2 is generically isomorphic to D_2. At every point of $M_2-M_1^{(2)}$ we can apply lemma 2.7. Since the strict images of D_2, D_3 and D_4 are all contained in B_1, the set on which f_1^{-1} is not an isomorphism is nonsingular, and thus the set $\hat{C}$ of possible bad points is empty. We conclude that f_2^{-1} is an isomorphism at every

point of $M_2 - M_1^{(2)}$ except on $f_2[D_3]$ (which contains $f_2[D_4]$). For any singleton point x in D_2, we consider a transversal test curve Γ. Since $S_f(D_2, H)=1$ for a general hypersurface $H \subset Y$, we have $\Gamma \cdot f^*(H)=1$. Letting $y_2 = f_2[\Gamma] \cap h_2^{-1}(y)$, we have $1=\deg(\Gamma \cdot f^*(H)) = \deg(f_*(\Gamma) \cdot H) = \deg f_2[\Gamma] \cdot h_2^*(H)$. y_2 must be a point of $M_2 - M_1^{(2)}$ since $h_2^*(H)$ has order 1 there. We now apply lemma 2.6 to conclude that f_2 is well-defined at x. Since there is no collapsing component, it is thus an isomorphism. Thus

$$D_2 - (D_1 \cap D_3 \cup D_4) \overset{f_2}{\cong} M_2 - (M_1^{(2)} \cap M_2) - f_2[D_3].$$

We note, furthermore, that $D_1 \cap D_2$ must be empty. On $D_1 \cap D_2 - (D_3 \cup D_4)$, f_1 is well-defined, by lemma 2.11(ii). On $D_2 - (D_3 \cup D_4)$, f_2 is well-defined, again by lemma 2.11(ii), and is actually an isomorphism, since there are no collapsing components. If $D_1 \cap D_2 \neq \phi$, its image must be in $f_2[D_1] \cap f_2[D_2] = M_1^{(2)} \cap M_2$. However f_2^{-1} is not an isomorphism on $M_1^{(2)} \cap M_2 = f_2(A_0)$. Thus $D_1 \cap D_2$ must be empty.

Step 3: $b_3' : Y_3' \to Y_2$ is the quasi-blowing up of $f_2[D_3]$. If y_2 is a first order point of $f_3[D_3]$, then by lemma 2.7, $f_2^{-1}[y_2] \subset D_3 \cap D_4$. If in D_4, then $f_2^{-1}[y_2] = D_4$ and $y_2 = f_2[D_4]$. If in D_3, then by lemma 1.8, f_2 factors through the blowing up of a smooth branch of $B_3' = f_2[D_3]$ at y_3, and we can assume that b_2 factors through this blowing up too. We conclude from lemma 2.7 that f_3^{-1} is an isomorphism at every accessible point of $M_3' - M_2^{(3)\prime} - f_3[D_4]$. Let x be a point of $D_3 - (D_1 \cup D_2 \cup D_4)$, and let Γ-be a transversal curve at x. Since f_1 is well-defined at x, we can take H to be a generic hyperplane through $y \in Y$, and we will get $\deg \Gamma \cdot f^*(H) = s_f(D_3, H)=1$. Lift to Y_3 getting a point y_3' and a curve Γ_3. b_3' is so constructed that the only first order components in Y_3 are the accessible components $M_1^{(3)}$, $M_2^{(3)}$ and M_3. Let Q_3' be the union of the non-accessible components. Since $\Gamma \cdot h_3^*(H)=1$, y_3 must lie in a first order component, necessarily D_3, since f_3 gives isomorphisms

$$D_1 - (D_2 \cup D_3 \cup D_4) \overset{f_3}{\cong} M_1^{(3)\prime} - M_2^{(3)\prime} - M_3' - Q_3'$$
$$D_2 - (D_1 \cup D_2 \cup D_4) \overset{f_3}{\cong} M_2^{(3)\prime} - M_1^{(3)\prime} - M_3' - Q_3'$$

We conclude from lemma 2.6 that f_3 is well-defined at x. Since there are no collapsing components

$$D_3 - (D_1 \cup D_2 \cup D_4) \cong M_3' - M_2^{(3)\prime} - M^{(3)\prime}{}_1 - f_3[D_4] - Q_3'$$

The key point will be to prove that $D_2 \cap D_3$ is empty. Let x be a general point of $D_2 \cap D_3 - (D_1 \cup D_4)$. f_1 is well-defined there by lemma 2.11 (ii). f_2 is then well-defined there by lemma 2.3(ii), (iii) of [9] applied to D_2, generically isomorphic to the blowing up of B_1 and thus having excess 0, and D_3, which has excess 1 with respect to a coordinate system in Y_1 for which B_1 is one of the coordinate axes. Finally, f_3 is well-defined at x, by lemma 2.2 of [9], and it is an isomorphism since no components at x collapse. Thus $D_2 \cap D_3$ must be isomorphic to M_3 $M_2^{(3)}$. We will return to this point after Step 4.

Step 4: $b'_4: Y'_4 \to Y'_3$ is the quasi-blowing up of $B'_3 = f_3[D_4]$. Applying lemma 2.7 to the sequence Y_0, Y_1, Y_2, Y'_3 and to any first order point Y'_3 of $f_3[D_4]$, we see as in step 3 that $f_3'^{-1}[y'_3]$ must be a curve in D_4, and thus we can choose our quasi-blowing-up to factor through a smooth branch of B'_3 at each first order point y'_3. We thus obtain, as in step 3, that the only first order components of $K_{h'_4}$ are the accessible components $M_1^{(4)\prime}$, $M_2^{(4)\prime}$, $M_3^{(4)\prime}$ and M'_4. Let Q'_4 be the union of the non-accessible components.

Let x be any singleton point of D_4 at which f_1 is well defined. If H_1 is a generic hypersurface in Y such that $h_1^{-1}[H_1]$ contains B_1, then we have already shown that $s_{f_1}(D_4, h_1^{-1}[H_1]) = 1$. If $x \notin f^{-1}[H_1] \cap f^{-1}[H'_1]$ for generic H_1, H'_1 with $h_1^{-1}[H_1]$, $h_1^{-1}[H'_1] \supset B_1$, then f_2 will also be well defined at x. We apply lemma 2.2(ii) of [9] to f_2. If $f_2(x)$ is a singular point of B'_2, we make an etale base extension to separate branches. The multiplicity of D_4 in the canonical divisor of f_2 is $r = w_{f_2}(D_4) - 1 = 3 - 1 = 2$, the multiplicity b of D_1 in the lifting of a generic hypersurface is 1, and the codimension c' of the blowing-up h_{23} is 2. Thus $r = bc'$, so f'_3 is well-defined. A further application of lemma 2.2(i) of [9] to f_3 shows that f'_4 is well-defined, and must in fact be an isomorphism, since the unique component D_4 through x does not collapse. We thus have

$$\begin{aligned} D_4 - (D_1 \cup D_2 \cup D_3) - P_y(f) - f^{-1}[H_1] \cap f^{-1}[H'_1] \\ \xrightarrow{\sim} M'_4 - M_1^{(4)\prime} - M_2^{(4)\prime} - M_3^{(4)\prime} - Q'_4. \end{aligned}$$

Let x' be a point of $D_4 \cap f^{-1}[H_1] \cap f^{-1}[H'_1]$. Since x' has order at least 2 with respect to the lifting of the generic hyperplane $h_1^{-1}[H_1]$ through B_1, the image $f_1[\Gamma']$ of a transversal test curve at x' must intersect B_1 with multiplicity at least 2. Thus

$$f_1[\Gamma'] \cdot M_1 \geq 2.$$

Thus for a generic hyperplane H through $y \in Y$, we have $\Gamma' \cdot f^*(H) = f_1[\Gamma'] \cdot M_1 \geq 2$. Thus either $x' \in P_y(f)$, so that $x' \in f^{-1}[H]$, or else x' is not a singleton point of D_4. We thus have

$$D_4-(D_1\cup D_2\cup D_3)-P_y(f)\simeq M_4'-M_1^{(4)\prime}-M_2^{(4)\prime}-M_3^{(4)\prime}-Q_4'$$

(d) All singleton points of $P_y(f)$ lie in A_0:

We wish to show that the only singleton points of the pinch locus are in A_0. We already showed, by considerations of excess from lemma 2.11, that the pinch locus is contained in D_4, and that it cannot have a component in $D_1\cap D_4$. Suppose A_0' was a component contained solely in D_4. The quasi blowing-up N_1' of A_0' has canonical pair $(6,1)+(1,d)$, and since the excess must be positive, we must have $d=1$, giving $(7,2)$. Blowing up the images of N_1' we could construct a factorization sequence Y_0, $Y_1,\ldots,Y_k$ with M_k generically isomorphic to N_1'. Apply lemma 2.9 with $f=h_1$ and $p=h_{k1}$. Choosing a generic hyperplane H through $y\in Y$ such that $h_1^{-1}[H]$ does not contain the strict image of N_1' in M_1, we find that all the d_i in lemma 2.9(i, ii) will be zero, since B_{i-1} will not be contained in $h_{(i-1)0}^{-1}[H]$. The only exceptional divisor of h_1 is M_1 with $(w_1,s_1)=(3,1)$. Applying lemma 2.9(i, ii) to f_1, we have

$$(7,2)=e_1(3,1)+\sum_{i=1}^{l}c_i(k_i'-k_i,0).$$

Thus $e_1=2$, $k_1=1$, $k_1'=2$, and M_2' is the blowing-up of the curve B_1: (It cannot be another curve since f_1^{-1} is an isomorphism except on B_1). The image of N_1' must be in M_2', so applying 2.9 with $f=h_2$, $p=h_{32}$, we have only the possibility

$$(7,2)=(4,1)+(3,1)+(k_2'-2,0).$$

We conclude that $k_2'=2$, i.e., that $f_2[N_1']=M_2\cap M_1^{(2)}$. Since N_1' then maps into M_3', which has the same canonical pair $(7,2)$ we conclude that it is generically isomorphic to M_3'. Since the blowing-up of A_0 is generically isomorphic to this same divisor, we conclude that $A_0=A_0'$.

(e) The strict image of D_3 in Y_1 is a point y_1:

We now wish to use this information about f_4' to eliminate the possibility that in the quasi-factorization sequence defined in (c), $f_2[D_3]=B_2'$ is a section of B_1 of degree ≥ 1. Suppose it were. Then $M_3'\cap M_2^{(3)}$ would be a section of B_1. Consider $G=f_4'^{-1}[M_3'^{(4)}\cap M_2^{(4)}]$. G must be an irreducible curve. If the $f^{-1}[H]$ were separated along G, they would have to intersect the components of K_f containing G in curves along which f_1 would be well-defined, which would move as H moves. The canonical pairs of these curves in the surface $f^{-1}[H]$ would have to be the same as in $h_4^{-1}[H]$, since, as we have already shown, f_4' is an isomorphism at the generic point of each component. Since in Y_4' they sweep out $M_2^{(4)}$ and

$M_3^{(4)}$, in X they sweep out D_2 and D_3. We conclude that if G were not contained in the pinch locus, then $G \subset D_2 \cap D_3$, which would therefore be non-empty, and f'_4 would be an isomorphism there.

If G were contained in the pinch locus, then since the blowing-up of $M_3'^{(4)} \cap M_2^{(4)}$ has canonical pair $(9, 2)$, and $f^{-1}[H] \supset G$ for generic H in Y containing Y, we would conclude that G is not contained in an intersection, but rather $G \subset A_0$. However, since the multiplicity of $f^{-1}[H]$ along A_0 is one, we could not have $h_4'^{-1}[H]$ intersecting $f'_4(A_0)$ at two different points, one in $M_1'^{(4)} \cup M_2'^{(4)}$ and one in $M_2'^{(4)} \cap M_3'^{(4)}$. We conclude that G would not be contained in the pinch locus, and thus that $D_2 \cap D_3$ would be non-empty, and f'_4 would be an isomorphism at the generic point of $D_2 \cap D_3$. Composing with h_{43}, we would also have that f'_3 is an isomorphism at the generic point of $D_2 \cap D_4$. We would then have $D_2 \cap D_3 - (D_1 \cap D_4) \cong M_2'^{(4)} \cap M_3'^{(4)} - M_1'^{(4)} - M'_4 - Q_4$.

Since $f'_3(D_4)$ would have to be connected, since we have shown that $f_3'^{-1}$ is an isomorphism except on $f'_3[D_4]$ and $M_1'^{(3)} \cap M_2'^{(3)}$, and since $M_1'^{(3)} \cap M_2'^{(3)} \subset f'_3(A_0)$, we would conclude that $B'_3 = f_3[D_4]$ must intersect $M_1'^{(3)} \cap M_2'^{(3)}$. Since the multiplicity of D_4 is one, $f'_3(D_4)$ could not be contained in an intersection. In order for a fiber of M'_3 to intersect $M_1'^{(3)}$, it would have to be the blowing-up of a point of $f_2[D_3] \cap M_1^{(2)}$. In this case the fiber would lie entirely in $M'_3 \cap M_1'^{(3)}$. We conclude that $f'_3[D_4]$ could not be a fiber, and would therefore have to be a section of $f_2[D_3]$. (see Fig. 5, II). We can therefore repeat the argument made above, replacing G by $G' = f_4'^{-1}[M'_4 \cap M_3^{(4)}]$ and replacing $(9, 2)$ by $(11, 2)$. We would conclude as there that $f_4'^{-1}$ would be an isomorphism on $M_3'^{(4)} \cap M'_4$. We would thus have $f_4'^{-1}$ an isomorphism except on $M_1'^{(4)} \cap M_2'^{(4)}$ and possibly on $M'_4 \cap (M_1'^{(4)} \cup M_2'^{(4)})$.

Taking H_1 to be a generic hyperplane with the property that $B_1 \subset h_1^{-1}[H_1]$, we now have the contradiction we desired. $h_4'^{-1}[H_1]$, if it intersects M'_4 at all, would cut M'_4 at a generic fiber, at which it does not intersect $M_1'^{(4)}$ or $M_2'^{(4)}$, and at which there are no non-accessible components. Thus $f_4'^{-1}$ would be an isomorphism everywhere along $h_4^{-1}[H]$. It would have to be isomorphic to $f^{-1}[H_1]$, whence f'_4 would be well-defined everywhere along $f^{-1}[H_1]$. Composing with h'_{41}, we find that f_1 would be well-defined everywhere on $f^{-1}[H_1]$. However, this contradicts the fact we proved in (b), that $f^{-1}[H_1]$ contains A_0, along which f_1 is not well-defined. This was the desired contradiction, so we may finally conclude that $f_2[D_3]$ was not a section of B_1, but rather $f_2[D_3]$ is the fiber in M_2 over a point y_1 in M_1.

(f) $D_2 \cap D_3 = \emptyset$:

$f_2[D_4]$ is either a point or all of $f_2[D_3]$. In the first case $f_3[D_4]$ is a fiber of M'_3, and since it must be connected to $f_3(A_0)$ we find that $f_3'^{-1}$ is not an isomorphism on $M'_3 \cap M_2^{(3)}$, whence $D_2 \cap D_3 = \emptyset$.

We need to show that $D_2 \cap D_3 = \emptyset$ would be empty even if we had $f_2[D_4] = f_2[D_3]$. If $D_2 \cap D_3 \neq \emptyset$ then as we showed in Step 3, of (b), it is isomorphic to $M_2^{(3)} \cap M'_3$. Let H_1 be a generic hyperplane in Y containing y such that $B_1 \subset h_1^{-1}[H_1]$. We have shown that $f_4'^{-1}$ cannot be an isomorphism on all of $h_4'^{-1}[H_1]$. We conclude that it is not well-defined at the generic point of $B'_4 = M'_4 \cap M_3'^{(4)}$, which is the only curve intersected by $h_4^{-1}[H_1]$ on which it is not known or assumed to be an isomorphism. Blowing up this intersection, B'_4 gives canonical pair $(11, 2)$.

Let $G' = f_4'^{-1}[B'_4]$. We are presuming that $G' \subset f^{-1}[H_1] \cap f^{-1}[H'_1]$ for generic H_1, H'_1 such that $h_1^{-1}[H_1]$, $h_1^{-1}[H'_1] \supset B_1$. Thus f_2 would not be well-defined along G'. Calculating canonical B_1-pairs with respect to h'_{41} : $Y'_4 \to Y_1$, we find that

$$
\begin{aligned}
u_{f1}(D_2 \cdot H_1) &= (2, 1) \\
u_{f1}(D_3 \cdot H_1) &= (3, 1) \\
u_{f1}(D_4 \cdot H_1) &= (4, 1)
\end{aligned}
$$

The canonical B_1-pair of the blowing-up of B'_5 is $(3, 1) + (4, 1) = (7, 2)$. Since f_2 is well-defined except on D_4, G' would have to be contained in D_4. By Step 4 of (c), since f_4 would not be an isomorphism on G', we would have $G^1 \subset D_4 \cap (D_1 \cup D_2 \cup D_3) \cup P_y(f)$.

We have already shown in (d) that all singleton points of $P_y(f)$ lie in A_0. On purely combinatorial grounds, we see that G' cannot be contained in $D_4 \cap D_3$ or $D_4 \cap D_2$, for if we let b'_5 be the blowing-up of B'_5, we have

$$s_{h51}(N'_5, h_1^{-1}[H_1]) = 2.$$

On the other hand, $G' \subset f_1^{-1}[h_1^{-1}[H_1]] = f^{-1}[H_1]$, and if $G' \subset D_4 \cap D_i$, $i = 2, 3$, then since N'_5 maps into G', the additivity formula would require

$$
\begin{aligned}
s_{h51}(N'_5, h_1^{-1}[H_1]) &\geq s_{f_1^{-1}}(D_4, h_1^{-1}[H_1]) + s_{f_1^{-1}}(D_i, h_1^{-1}[H_1]) + d \\
&\geq 1 + 1 + 1 = 3.
\end{aligned}
$$

Thus we must have $G' \subset A_0 \cup (D_4 \cap D_1)$. $G' \subseteq D_4 \cap D_1$ would be combinatorially possible since $s_{f_1^{-1}}(D_1, h_1^{-1}[H_1]) = 0$.

We first show that G' does not lie in A_0.

If G' is in A_0, then f_5^{-1} factors through the blowing-up of A_0, and its generic point would be a singleton point of the exceptional divisor N_1. However, it must also be in the fiber over $y_1 = f_2[D_4]$, which is the intersection of N_1 and the lifting $D_4^{(1)}$ of D_4. This gives a contradiction.

The other possibility is $G' \subset D_4 \cap D_1$. Since $f^{-1}[H_1]$ would not intersect D_1 at any singleton point, $f^{-1}[H_1]$ and $f^{-1}[H'_1]$ could not separate at this point, so $f^{-1}[H_1] \cap f^{-1}[H'_1]$ would have a curve component in this intersection. Furthermore a single blowing-up would suffice to separate these surfaces. Let $a'_1 : X'_1 \to X$ be the blowing-up of this component of $D_1 \cap D_4$. $f'_{12} : X'_1 \to Y_2$ would map the exceptional divisor N'_1 to the fiber over y_1 in M_2. Applying lemma 2.9(i , iii) to $f'_{13} : X'_1 \to Y'_3$, we would get

$$u_{f'_{10}}(N'_1, H) = (9, 2) = e_2(4, 1) + e_3(5, 1) + c_i(k'_i - k_i, 0), \ e_3 \geq 1.$$

We obtain $e_2 = e_3 = 1$.

If we now compare $f^{-1}[H_1]$ with $h'_3[H_1]$, we find that $h_3'^{-1}[H_1] \cap f'_3[D_4]$ maps to $G' \cap f^{-1}[H_1] \subset D_1 \cap D_4$, whereas the calculation just made of the blowing-up of this component of $D_1 \cap D_4$ shows that $f'_3(G' \cap f^{-1}[H_1]) \subset h_3'^{-1}[H_1] \cap M_2'^{(3)} \cap M'_3$. This would be a contradiction, since these two points are distinct on the connected tree $h_3'^{-1}[H_1]k_{h_3}$. We conclude that $f_4'^{-1}$ could not fail to be an isomorphism on $M'_4 \cap M_3'^{(4)}$, and thus that the only place where it could fail to be an isomorphism on $h_4'^{-1}[H_1]$ would be in $M_2'^{(4)} \cap M_3'^{(4)}$. Since $f_4'^{-1}$ is not an isomorphism there, we would conclude that $D_2 \cap D_3 = \emptyset$, as shown in Step 3 of (c).

We thus have four components D_1, D_2, D_3, D_4 with canonical pairs (3, 1), (4, 1), (5, 1) and (6, 1). $B_1 = f_1[D_2]$ is smooth of degree 1, and there is a curve A_0 in D_4 along which f_1 is not well-defined, whose blowing-up has canonical pair (7, 2). f_1 is well-defined except on A_0 and possibly $D_4 \cap (D_1 \cup D_3)$. $D_1 \cap D_2 = \emptyset$ and $D_2 \cap D_3 = \emptyset$.

(g) A_0 is smooth at each of its singleton points

We want to show that A_0 is smooth and transversal to D_1 and D_2. We let Y_0, Y_1, Y_2, Y_3 be the factorization sequence obtained by blowing up y, B_1, and then $B_2 = M_2 \cap M_1^{(2)}$. M_3 has a fibration which is induced by f_3^{-1} mapping M_3 onto A_0. Except for two special fibers $B'_3 = M_3 \cap M_1^{(3)}$ and $B''_3 = M_3 \cap M_2^{(3)}$, all the other fibers consist of singleton points. Because f_1 fails to be well-defined at every point of A_0, each fiber contains a section B_3 of B_1. Let Y_4 be the space obtained by a quasi-blowing up of one of these sections B_3, which will have canonical pair $(8, 2) = (7, 2) + (2-1, 0)$, by lemma 2.4. Applying lemma 2.9, the image of M_4 in X must lie in a single component, with $s_1 = 1$, and

$$4 = 8 - 2 \cdot 2 \leq e_1(w_1 - 2s_1)$$

The only possibility is D_4, with $e_1 = 1$. By lemma 2.4 we then have

$$(8, 2) = (6, 1) + (k_1' - 1, d_1)$$

So $k_1' = 3$ and $d_1 = 1$. Thus M_4 is generically isomorphic to the blowing up of a single point x of A_0 lying only in D_4, and $f^{-1}[H]$ has multiplicity one at x. Since a single blowing up separates generic hyperplanes H_1, H_2 whose liftings $h_1^{-1}[H_1]$ and $h_1^{-1}[H_2]$ to Y_1 intersect B_1 at different points, we conclude that $A_0 = f^{-1}[H_1] \cap f^{-1}[H_2]$, as the transversal intersection of smooth surfaces, is nonsingular at x. It remains to check $A_0 \cap D_1$ and $A_0 \cap D_2$.

(h) A_0 intersects D_1 and D_2 transversally:

We now make a similar analysis for the two special sections B_3' and B_3''. We begin with $B_3' = M_3 \cap M_1^{(3)}$. Blowing up B_3' to get $b_4' : Y_4' \to Y_3$, we have an exceptional divisor M_4' with canonical pair $(7, 2) + (3, 1) = (10, 3)$. We want to locate $f_4'^{-1}[M_4']$. From the formulas of 2.9.

$$3 \leq \sum e_i s_i + dl, \quad 10 \leq \sum e_i w_i.$$

If $l = 1$, then each $e_i \leq 1$, since the multiplicity of $s_p(E_i, N_1) = 1$. Furthermore, $x = f_4'^{-1}[M_4]$ is in D_4, since we proved early in our consideration of case D that the entire pinch locus is in D_4. Thus the only possible combinations of components are $(3, 1) + (6, 1) + (1, 1)$, $(4, 1) + (6, 1) + (0, 0)$, or $(6, 1) + (2, 1) + (2, 1)$.

The last possibility involves as an intermediate stage a component with canonical pair $(8, 2)$ which has excess 2, too small to contain a singleton point in the pinch locus. We want to show that the first possibility is the only one which can hold, so we must eliminate the second possibility, that $f_4'^{-1}[M_4']$ is $D_2 \cap D_4$.

We have already shown that $D_1 \cap D_2 = \phi$ and $D_2 \cap D_3 = \phi$. Except for A_0, we have already shown that components of the pinch locus all lie in $D_4 \cap D_2$ and $D_4 \cap D_3$. For generic H, we consider $f^{-1}[H]$, which is generically isomorphic to $h_3^{-1}[H]$. $h_3^{-1}[H] \cap K_{h_3}$ is a union of three curves, C_1' in $M_1^{(3)}$, C_2' in $M_2^{(3)}$, and C_3' in M_3, which lies between C_1' and C_2'. (See Figure 6.)

Since f_3^{-1} is an isomorphism at the generic point of $M_1^{(3)}$ and $M_2^{(3)}$, we have curves $C_1 \subset f^{-1}[H] \cap D_1$ and $C_2 \subset f^{-1}[H] \cap D_2$, which are isomorphic to C_1' and C_2' respectively. The restriction of f_3 to $f_3^{-1}[H]$ will map A_0 to C_3', since the blowing up of A_0 is generically isomorphic to M_3. Thus in the correspondence $f_3 : f^{-1}[H] \dashrightarrow h_3^{-1}[H]$, there are no components of $h_3^{-1}[H]$ which collapse under f_3^{-1}. We conclude that $\bar{f}_3 = f_3|_{f^{-1}[H]}$ is well-defined. If we let $P_1' = C_1' \cap C_3'$ and $P_2' = C_2' \cap C_3'$, we find that the pinch locus is the union of A_0 and the preimages of P_1' and P_2'. Since $D_1 \cap D_2$ is

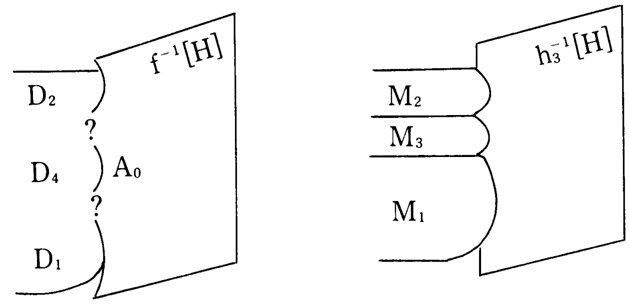

Fig. 6

empty the only possible component of the pinch locus which could intersect C_1 would be a component of $D_4 \cap D_3$. Since $D_3 \cap D_2$ is also empty, this could not be followed by a component of $D_4 \cap D_2$, but only by A_0. Since $f_3^{-1}[B_3]=f_4'^{-1}[M_4']$ must be contained in $f_1^{-1}(B_1) \cap D_1$, we conclude that it cannot be a curve in $D_2 \cap D_4$, and we are left with the possibility that we wanted, that $f_4'^{-1}[M_4]$ is a point P_1 in $D_1 \cap D_4$.

We have $(10,3)=(3,1)+(6,1)+(1,1)$, so M_4' is generically isomorphic to the blowing up of the point. We conclude that for generic H, $f^{-1}[H]$ is not tangent to either D_1 or D_4 at the point. Since it has degree 1 and is nonsingular, we conclude that A_0 is nonsingular and transversal to D_1 at P_1.

We now make a similar analysis at the other end of A_0. Let $C_2'= h_3^{-1}[H] \cap M_2^{(3)}$ and let P_2' be the point where it intersects C_3'. Letting $\bar{f}_3 : f^{-1}[H] \to h_3^{-1}[H]$ be the morphism of surfaces induced by $f_1 : X \to Y_3$, we consider the preimage $\bar{f}_3^{-1}(P_2')$ which is a tree of curves contained in the pinch locus. Because $D_2 \cap D_3 = \emptyset$, if $\bar{f}_3^{-1}(P_2')$ were not a point, it could only be a single component of $D_2 \cap D_3$. We wish to show that it is indeed a point.

Let $b_4'' : Y_4'' \to Y_3$ be a blowing up of $B_3''=M_2^{(3)} \cap M_3$. $h_4''^{-1}[H] \cap M_4$ is just the blowing up of the point P_2'. $f_4''^{-1}[M_4'']$ is thus contained in $\bar{f}_3^{-1}(P_2')$. The canonical pair of M_4'' is just the sum of the canonical pairs of $M_2^{(3)}$ and M_3.

$$(11,3)=(4,1)+(7,2)$$

We have shown that the image must be contained in D_2 and D_4. By 2.9 (ii) we have $3=s=\sum e_i s_i + \sum c_i d_i$. We conclude that there is only one blowing up, and $e_2=e_4=1$. We then have

$$(11,3)=(4,1)+(6,1)+(k'-2,1).$$

We conclude that M_4'' is generically isomorphic to the blowing-up of a point in $D_2 \cap D_4$, at which $f^{-1}[H]$ has multiplicity 1. As before we see that for generic H, $f^{-1}[H]$ cannot be tangent to D_2 or D_4. We conclude that A_0 intersects D_2 transversally at this point, and that there are no more components to the pinch locus.

(i) We construct a strong factorization for f:

We now blow up A_0. Since that is the only component of the pinch locus and the resulting space is generically isomorphic to M_3, so that the liftings $f^{-1}[H]$ of generic hypersurfaces are separated, we conclude that $f_{11}: X_1 \to Y_1$ is well-defined. For any point of B except $f_1[D_3]$, we can choose a hyperplane H such that $h_1^{-1}[H]$ passes through the point. $f^{-1}[H]$ will be A_0 which in $f^{-1}[H]$ will be a P^1 with self intersection -1. The blowing up to X_1 will not change the configuration of exceptional curves in $f^{-1}[H]$, since we are blowing up a curve in a surface. The fiber of N_1 over $f_1[D_3]$ is $N_1 \cap D_4$, also isomorphic to A_0. Thus N_1 has irreducible fibers, the generic fiber being a P^1 of selfintersection -1. We conclude that N_1 is contractible. After contracting it we are left with three components collapsing to a non-singular curve. By the main theorem of [9], this is locally factorizable. In the particular case, the appropriate factorization is the one given by blowing up $P=f_1[D_3]$ to get $M'_2 \sim D_3$, then blowing up $f'_2[D_4]$, and finally $f'_3[D_2]$. This concludes the proof.

§ 4: Three collapsing surfaces

In analyzing morphisms collapsing four surfaces to a point, we encountered two cases, those which do not factor through the blowing up of the point, and those which do. In the previous chapter we analyzed those which do not. We now wish to show that those which do are locally factorizable. After factoring through the blowing up the resulting morphism $f_1: X \to Y_1$ collapses three normally crossing surfaces to a set of higher codimension. It suffices, therefore, to prove the following:

PROPOSITION 2: *Let $\bar{f}: \bar{X} \to \bar{Y}$ be a proper birational morphism of three dimensional algebraic spaces, collapsing three or fewer normally crossing surfaces. Then $\bar{f}$ is locally factorizable.*

PROOF: Over isolated points of $S_{\bar{f}}$ this is just the main theorem of Crauder [1], in the three surface case. We may thus assume that $S_{\bar{f}}$ contains a curve. By lemma 1.6, if $\bar{f}$ were not locally factorizable then there would be a morphism $f: X \to Y$ occurring in a local factorization tree of $\bar{f}$, with a point obstruction at a point $y \in Y$.

It thus suffices to show that for any f in such a tree there is an etale covering such that f factors through some blowing-up in a neighborhood of each point y. We may presume that we have passed to an arbitrarily fine neighborhood of y, and that $b_1 : Y_1 \to Y$ is the blowing up of the point. By lemma 1.8, we may presume that $f_1^{-1}[M_1]$ is a surface, but f_1 is not well defined. We must show that for a properly chosen scheme Y, there is a subscheme $B \subset Y$ such that f factors through the blowing-up $b_1' : Y_1' \to Y$ of B.

Let $D_1 = f_1^{-1}[M_1]$. Let $\alpha = \beta^-$ be a pair of adjacent vertices in the partial factorization tree leading to f. Let D_j be a component of K_f. $\pi_\beta : X_\beta \to X_\alpha$ is an etale morphism. We define $D_j^\alpha = \pi_\beta(D_j^\beta)$. By the construction of the local factorization, π_β is one-to-one over any point y_α which is the image of a surface in X_α. If $f_\beta(D_j^\beta)$ is a point y_β, then $f_\alpha(D_j^\alpha)$ is also a point $y_\alpha = e_\beta(y_\beta)$. Thus $(\pi_\beta)^{-1}(D_j^\alpha) = D_j^\beta$.We conclude that if S_f is a point, each component of K_f corresponds one-to-one to a component of $S_{\bar{f}}$. Thus there would be at most three components in K_f, and the morphism would be locally factorizable by [1]. Henceforward we may assume that S_f is a curve.

By lemma 1.2 of [9], there must be a component of K_f which is generically isomorphic to the blowing-up of any component of S_f. Thus there is at least one component D_{21} with canonical pair $(2, 0)$. We denote all other components of K_f whose image in $\bar{X}$ is the same component D_2 by $D_{22}, \dots, D_{2j_2}$, and note that $D_{2j} \cap D_{2j'} = \phi$ for $j \neq j'$, since components of $K_{\bar{f}}$ have no self-intersections.

There is at most one other class of divisors $D_{31}, \dots, D_{3j_3}$ *in* K_f, all mapping to the same divisor $\bar{D}_3$ in $K_{\bar{f}}$. At least one of the images $\bar{f}(\bar{D}_2)$, $\bar{f}(\bar{D}_3)$ is a curve in $S_{\bar{f}}$. We presume D_{21} to be chosen so that if it is only one of them, $\bar{f}(\bar{D}_2)$ is the one, with canonical $\bar{y}$-pair $(2, 0)$, and that if both map to the same curve B_0, then $(\bar{D}_2)$ is the component generically isomorphic to the blowing up of B_0, which exists by 1.2 of [9]. In that case $\bar{f}(\bar{D}_3)$ is generically isomorphic to the blowing up of a section, therefore has canonical pair $(3, 0)$. By the additivity formula, the weights of components can only drop as we proceed out the branches of a local factorization tree, and the canonical y-pair of a surface D_{ij} whose image is a curve will always have second component 0 because $f(D_{ij})$ is not contained in the generic hyperplane H through y, whence $s_f(D_{ij}, H) = 0$. Since the weight is always at least the codimension of the image, we see that the components D_{2j} all have canonical pair $(2, 0)$. If there is a second divisor class $\{D_{3i}\}$ for which each $f(D_{3i})$ is a curve, D_{3i} either has canonical pair $(2, 0)$ or, if $f(D_{3i}) = f(D_{2j})$ for some j, it has the pair $(3, 0)$.

EXAMPLE: Divisor class with different canonical pairs. Let $C \subset P^3 = Y$ be an ordinary node contained in a hyperplane $G \xrightarrow{\sim} P^2$. Let $\bar{y}$ be the singular point of C, and let $e_1 : Y_1 \to Y$ be an etale neighborhood of $\bar{y}$ in which C splits into two irreducible normally crossing branches C_1 and C_2. Let $\bar{f} : X \to Y$ be the locally factorizable morphism which is obtained in Y_1 by five blowings-up with the following centers: (i) C_1, (ii) the intersection of the preimage of C_2 with the fiber over $\bar{y}$, (iii) the curve which is the intersection of the first exceptional divisor with the strict preimage of G, (iv) the strict preimage of C_2, (v) the intersection of the exceptional divisor over C_2 with the strict preimage of G. In $\bar{f}$ itself we have a divisor D_1 collapsing to $\bar{y}$, a divisor class $\{D_{21}, D_{22}\}$ in which both divisors have canonical pair $(2, 0)$ and a divisor class $\{D_{31}, D_{32}\}$ in which both divisors have canonical pair.

Now let $f_1 : X_1 \to Y_1'$ be the first node in the local factorization tree. Let y_1 be the center of the second blowing-up. In K_{f_1} we have a divisor D_1', a divisor class $\{D_{22}'\}$ with canonical pair $(2, 0)$ and a divisor class $\{D_{31}', D_{32}'\}$ in which the first divisor has canonical pair $(2, 0)$ and the second has canonical pair $(3, 0)$. (See Fig. 7.)

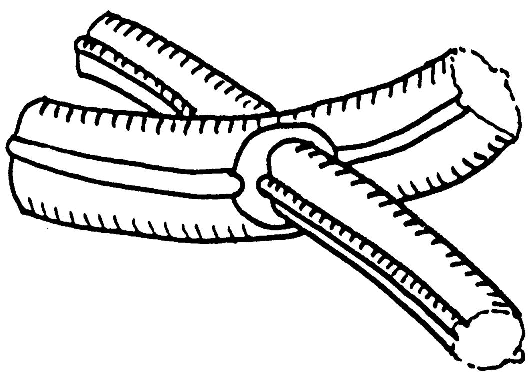

Fig. 7

Let Δ be an irreducible curve in M_1 along which f_1^{-1} is not an isomorphism. Let Y_2 be the space obtained by quasi blowing-up with center $\Delta_1 = \Delta$ and accessible component M_2. Consider $f_2^{-1}[M_2]$. We claim that it cannot be a surface. If it were a surface D_3, then it would also have a point image, and thus be the unique preimage of some component $\bar{D}_3$ in $\bar{X}$. There can only be one remaining class of components, all of canonical pair $(2, 0)$. None of the components has an excess of 3, and none of the intersections has an excess of 4, since the excesses of D_1, D_{2j}, D_3 are

0,2,1 respectively. f_1 would then be well defined, a contradiction. Thus $f_2^{-1}[M_2]$ is not a surface.

Since M_2 is the blowing up of a curve on a surface with canonical y-pair $(3, 1)$, the canonical y-pair of M_2 must be $(4, 1)$. Applying lemma 2.9(v) with $(w, s)=(4, 1)$, and (w_1, s_i) the canonical y-pairs of components of K_f containing $f_2^{-1}[M_2]$, we have

$$s \leq \sum s_i + dl,$$

where l is the number of blowings up in the quasi-factorization sequence obtained by blowing-up the image of M_2 until a component generically isomorphic to M_2 is obtained. Since $s=1$, we must have $s_i=0$ for all i, and $d=l=1$. Lemma 2.8 then gives

$$(4, 1)=\sum(w_i, 0)+(k'-k, 1),$$

with each $w_i=2, 3$. $f_2^{-1}[M_2]$ is in the pinch locus and thus by lemma 2.11 it cannot lie only in a component with canonical pair $(2, 0)$ and excess $2-0=2$. Thus the only possibilities are a single component with pair $(3, 0)$ or an intersection $(2, 0)$, $(2, 0)$. In the first case $k=1$ and $k'-k=1$, so the codimension k' of $f_2^{-1}[M_2]$ is 2, and in the second case $k=2$ and $k'-k=0$, so again $f_2^{-1}[M_2]$ is a curve, with codimension 2. In both cases $d=1$ implies that B_1 is of degree 1, and is thus isomorphic to P^1.

There may be several bad curves on M_1. We want to analyze the various possibilities, and show that in every case there is some smooth curve L_i in Y such that f factors through the blowing-up of L_i. Each bad curve Δ_i in M_1 corresponds to a unique bad curve C_i in X, with the blowing-up of Δ_i generically isomorphic to the blowing-up of A_i, and having canonical y-pair $(4, 1)$.

For a given bad curve C_1 in X, we want to show that $f^{-1}(y)$ has multiplicity 1 along C_1. We construct a quasi-factorization sequence Y, Y_1, Y_2 by blowing up first y and then Δ_1. The accessible component $M_2 \subset Y_2$ is thus, as we showed above, generically isomorphic to the blowing up of C_1. Let y_2 be a general point of M_2 and $t_2 \in O_{Y_2,y_2}$ be a local parameter for the divisor M_2. We first show that $h_2^{-1}(g)$ has multiplicity 1 on M_2. More precisely, we want to show that the ideal $h_2^{-1}(I_y)O_{Y_2,y_2}$ is the principal ideal (t_2). Since h_2 factors through the blowing-up h_1 of y, $h_2^{-1}(I_y)O_{Y_2,y_2}$ must be invertible. Since M_2 is the only exceptional divisor of h_2 containing y_2, this ideal must be generated by some power t_2^r of the local parameter t_2. The lifting of an arbitrary generator of I_y must therefore be divisible by t_2^r, which translates in our combinatorial notation into the statement that $s_{h_2}(M_2, H) \geq r$ for every hypersurface H through y.

Since the canonical y-pair of M_2 is $(4,1)$, we have $s_{h_2}(M_2, H)=1$ for generic H, so $r=1$, and thus $h_2^{-1}(I_y)O_{Y_2,y_2}=(t_2)$. Let x be the image of y_2 in X. h_2 factors locally through X, and $h_2^{-1}(I_y)O_{Y_2,y_2}$ is the lifting of $f^{-1}(I_y)O_{X,x}$, whence this latter ideal must also have multiplicity one. Translating back from ideals to subvarieties, this is what we mean by saying that $f^{-1}(y)$ has multiplicity 1 at x. For y_2 a general point of M_2, x is a general point of C_1, so we have $f^{-1}(y)$ of multiplicity 1 along C_1.

Passing to the Henselization $\tilde{Y}$ of Y at y, then for each component of K_f containing general point x of C_1, we can choose a transversal curve Z_i through x contained in that component which does not intersect $f^{-1}(y)$ at any other points. By Nakayama's lemma, the image $L_i=f(Z_i)$ must be nonsingular (see Danilov's argument in the proof of lemma 1.8). The L_i will be the images of the components containing the Z_i. Let V be a closed hypersurface containing Z_1 and Z_2. Let $H_1 \subset \tilde{Y}$ be a generic hyperplane through L_1, and let $\bar{f}$ be the restriction of $\tilde{f}: X\times\tilde{Y}\to\tilde{Y}$ to V. $\bar{f}$: $V\to\tilde{Y}$ is also proper, so by the projection formula we will get

$$\begin{aligned}\deg\, Z_2\cdot\bar{f}^*(H_1) &= \deg\, \bar{f}(Z_2)\cdot H_1\\ &= \deg\, L_2\cdot H_1.\end{aligned}$$

Z_2 can only intersect $\bar{f}^*(H_1)$ on $f^{-1}(y)$, since L_2 intersects H_1 only at y, $\tilde{Y}$ being local. Thus if D_{j1} is the component containing Z_1, $\deg\, Z_2\cdot \bar{f}^*(H_1)=\deg\, Z_2\cdot\tilde{f}^*(H_1)=\deg\, Z_2\cdot s_{\tilde{f}}(D_{j1}, H_1)D_{j1}=s_{\tilde{f}}(D_{j1}, H_1)=1$, since all the components mapping to L have L_1 pairs with $(2,1)$ or $(3,1)$, being the result of one or two blowings-up of L_1. $Z_2\cdot D_{j1}=1$ because Z_2 is transversal to $C_1\subset D_{i1}$. We conclude that L_1 and L_2 are transversal.

We can repeat this analysis for each bad curve C_i, continuing to work after base extension by the Henselization. We now assume that our base scheme Y was chosen sufficiently fine that all the L_i are smooth curves in Y.

Suppose C_1 is contained in a single component D_{ii}, let Z_3 be a transversal curve at a point of C_1, and let H be a generic hyperplane through L_1 in Y, $1=\deg\, Z_3\cdot f^*(H)=\deg\, f(Z_3)\cdot H$. We conclude that $f(Z_3)$ is nonsingular and transversal to H, therefore to L_1. Let H_1 be a smooth hypersurface in Y containing L_1 and $f(Z_3)$. Then $H_1'=f^{-1}[H_1]$ is a smooth hypersurface transversal to C_1 at $Z_3\cap C_1$. H_1' thus intersects the general fiber of any component containing C_1, and thus H_1 contains L_1. If C_1 is contained in two components, then their images, as we proved above, are transversal, and we choose H_1 to be a smooth hypersurface containing both.

We now consider the two possible cases:

Case 1: For some curve Δ, $f^{-1}[M_2]$ is a curve on a component D_{31} of order $(3, 0)$. For such a component to exist, there must also be a component D_{21} such that D_{31} is generically the blowing up of a section of the image C_1 of D_{21} in Y. For generic H we found above that $f^{-1}[H]$ has order 1 along the bad curve, so Δ has degree 1, and thus is isomorphic to P^1. In fact, the additivity analysis in the previous paragraph shows that every bad curve in M_1 is a P^1. We wish to show that if there is more than one bad curve, all intersect at a single point P, which will be the intersection of M_1 with the strict image of D_{21}, and that f will factor through the blowing up of $L_1=f(D_{21})$.

Let us suppose that there is a second bad curve C_2. Let Δ_1, $\Delta_2 \subset M_1$ be the bad curves in Y_1 corresponding to C_1 and C_2, whose blowings-up, with canonical pair $(4, 1)$, are generically isomorphic to the blowing up of C_1 and C_2 respectively. C_1 is contained in a single component D_{31}, and we can find a hyperplane H_1 in Y containing L_1, by taking the image in Y of a hypersurface transversal to C_1 at general point. Since that means that after blowing up C_1 to get X_1, $f_{10}^{-1}[H_1]$ would contain a fiber of N_1 over C_1, we conclude that $h_1^{-1}[H_1]$ contains the image Δ_1 of such a fiber. Since $\Delta_1 \neq \Delta_2$, and $h_1^{-1}[H_1]$ is nonsingular since H_1 is nonsingular, this means that $h_1^{-1}[H_1]$ does not contain Δ_2, since $h_1^{-1}[H_1] \cap M_1 \rightleftarrows$ cannot contain any points not in Δ_1.

Now consider the factorization sequences corresponding to C_2 and Δ_2. We let $a': X' \to X$ be the blowing up of C_2 with exceptional divisor N'_1, and we let $b'_2: Y'_2 \to X_1$ be the blowing up of Δ_2, with exceptional divisor M'_2, generically isomorphic to N'_1. Since $h_1^{-1}[H_1] \not\supset \Delta_2$, we have $1 = s_{h_1}(M_1, H_1) = s_{h_2}(M'_2, H_1) = s_{f_{10}}(N'_1, H_1)$. By lemma 2.8

$$s_{f_{10}}(N'_1, H_1) = \sum_{C_2 \subset E_i} s_f(E_i, H_1) + s_{a'_1}(N'_1, f^{-1}[H_1]).$$

$f^{-1}[H_1]$ intersects $f^{-1}(y)$ only on C_1, so $C_2 \not\subset f^{-1}[H_1]$, whence $s_{a'_1}(N^1_1, f^{-1}[H_1])=0$. We conclude that C_2 in contained in exactly one component E_1 with $s_f(E_i, H_1)=1$. Let $L=f(E_i)$. Since $f^{-1}(H_1)$ is connected, $f^{-1}[H_1]$ must intersect $f^{-1}(L)$ in a section of L. This section must intersect $f^{-1}(y)$. However, the only component of K_f containing the unique intersection point of $f^{-1}[H_1]$ and $f^{-1}(y)$ is D_{31}. Thus $f^{-1}[H_1] \cap f^{-1}(L)$ contains a point of the generic fiber of D_{31}. We conclude that $L = L_1$.

Letting H_2 be a smooth hypersurface in Y whose strict preimage $f^{-1}[H_2]$ in X intersects $f^{-1}(y)$ only on C_2, we know that H_2 contains the image L_1 of the unique component containing C_2. Since $h_1^{-1}[H_2]$ is a P^2, equal to Δ_2, we see that $P=h_1^{-1}[L_1]\cap M_1$ must lie in $\Delta_2=h_1^{-1}[H_2]\cap M_1$. This proves the claim that $P\in\Delta_1\cap\Delta_2$.

Let $h_1' : Y_1'\to Y$ be the blowing-up of $L_1=f(D_{21})$. We will now show that $f_1' : X\to Y_1'$ is well-defined everywhere. We begin by showing that D_{21} intersects $f^{-1}[y]=D_1$. We have a section $D_{21}\cap D_{31}$ of L_1, which must intersect $f^{-1}(y)$. $f^{-1}(y)$ is the union of D_1 and isolated curves, all belonging to the pinch locus, and thus not contained in D_{21}. Since D_{21} and D_{31} are representatives of the only divisor classes with curve image, and components of the same divisor class cannot intersect, the point of intersection $D_{21}\cap D_{32}\cap f^{-1}(y)$ must be a point of D_1. Thus $D_{21}\cap D_1$ is a non-empty curve C. Let H_1, H_2, H_3 be coordinate hypersurfaces at y with $L_1=H_1\cap H_2$. Since the excesses of D_{21} and D_1 in these coordinates are 0, both the map f_1' to the blowing up of L_1 and the map f_1 to the blowing up of y, are well-defined at all double points of C, by lemma 2.3 of [9]. Since f_1 is well-defined there, $f_1[C]=f_1[D_{21}]\cap f_1[D_1]=h_1^{-1}[L_1]\cap M_1=P$. Letting Y_2' be obtained by blowing up $f_1[D_2]$, and X_1' by blowing up $D_{21}\cap D_1$, we get $f_2'[N_1]\subset M_1^{(2)}\cap M_2'$. Since $(5,1)=(3,1)+(2,0)$, the image is the whole intersection, and N_1' is generically isomorphic to the blowing-up of $M_1^2\cap M_2'$. Taking a generic test curve through this intersection, its closure point x then lies in $D_{21}\cap D_3$. Applying lemma 1.2 of [9] to f_1' at x, and regarding Y_2' as the blowing up of $h_1'^{-1}(y)$, we get f_2' well defined at x. Since f_2' is a quasifactor for D_{21}, D_3 we conclude that it is an isomorphism at x, by 1.3 of [9]. Since $f_2'^{-1}$ is then an isomorphism except on the bad curves in M_1, $f_2'^{-1}$ is an isomorphism except on their strict transforms, each of which is fiber over a point in Y_1'. Thus $f_1'^{-1}$ is an isomorphism on the generic point of $h_1'^{-1}(y)$. Thus by lemma 1.4 of [9], f_1' is well-defined. This was what we needed to show.

Case 2: All bad curves in M_1 are of the $(2,0)+(2,0)$ type. We want to show that all the bad curves intersect at a single point P. Let D_{21} and D_{31} be components containing a bad curve. Let $L_2=f(D_{21})$ and $L_3=f(D_{31})$. Let $P_2=f_1^{-1}[L_2]\cap M_1$ and $P_3=f_1^{-1}[L_3]\cap M_1$. We may presume that L_2 and L_3 are smooth, and transversal at y as we showed above.

Let A_0 be the bad curve C_1, and blow up to get $a_1 : X_1\to X_0$. By the normal crossings of K_1, A_0 must be be smooth, and by the connectedness of $f^{-1}(y)$, it must intersect D_1. Let H_1' be a plane intersecting $f^{-1}(y)$ at a single point of A_0 with multiplicity 1. Take H_1 so that $H_1'=f^{-1}[H_1]$. If A_0' is another bad curve, and a_1' is the blowing up, then E_1, E_2 are the components of K_f containing A_0'.

$$\begin{aligned}(4,1)=u_{f_{10}}(N_1', H_1)=(4, s_f(E_1, H_1)+s_f(E_2, H_1)\\ +s_{a'_1}(N_1', f^{-1}[H_1])).\end{aligned}$$

Since $A_0'\not\subset f^{-1}[H_1]$, we must have $s_f(E_1, H_1)=1$ for some i. Thus A_0 and A_0' share a common component, D_{21}. Let D_{32} be the second component containing A_0'. Every other bad curve A_0'' in X must be in $D_{21}\cup D_{3i}$ for $i=1$, 2, by applying the previous argument with A_0'' in place of A_0 or of A_0'. Since $D_{31}\cap D_{32}=\emptyset$, $A_0''\subset D_{21}$. Thus every bad curve in M_1 passes through P. We know that $D_{21}\cap D_1$ is non-empty, at $A_0\cap D_1$, and conclude as in the previous case that if Y_2' is the blowing up of $f_1[D_{21}]$, then f_2' is an isomorphism except over a finite number of fibers of Y_2' over Y_1'. Thus by lemma 1.4 of [9], f' is well defined, as we wished to show.

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