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The configurations of the M-curves of degree (4, 4) in $RP^1\times RP^1$ and periods of real $K3$ surfaces

Dedicated to Professor Haruo Suzuki on his 60th birthday

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Abstract. For M-curves of degree (4, 4) in $RP^1\times RP^1$ whose components are all contractible, it is known that three configuration types are possible. We prove that all these configuration types are realized by some M-curves of degree (4, 4) by means of the existence of locally universal families of real $K3$ surfaces and the local surjectivity of period mappings defined over those families.

0. Introduction.

We consider the zero set RA of a real homogeneous polynomial F $(\neq 0)$ of degree (d, r) in $RP^1\times RP^1$, where d and r are integers (≥ 1). We assume that the zero set A of F in $CP^1\times CP^1$ is nonsingular. (In what follows, we write $P^1\times P^1$ for $CP^1\times CP^1$.) Then A is a connected complex 1-dimensional manifold. But RA is a possibly disconnected real 1-dimensional manifold (a disjoint union of finitely many copies of S^1) or the empty set. It is known that the number of the connected components of RA does not exceed $(d-1)(r-1)+1$(see [5]). We remark that the number $(d-1)(r-1)$ is the genus of the nonsingular curve A. We say RA is an *M-curve* of degree (d, r) if it has precisely $(d-1)(r-1)+1$ connected components.

In this paper we make clear the "configurations" of the M-curves of degree (4, 4) in $RP^1\times RP^1$, where we consider only the curves whose components (embedded S^1) are all contractible in $RP^1\times RP^1$. We define the meaning of the "configurations" as follows. In our cases, each component of RA, which is called an *oval*, divides $RP^1\times RP^1$ into two connected components. One of those is homeomorphic to an open disk and called the *interior* of the oval. The other is called the *exterior* of that. As a consequence of [5], every M-curve of degree (4, 4) lies in one of the following three cases (cf. Figure 1).

(1) Each of certain 9 ovals lies in the exteriors of the others, and the interior of one of those contains one oval. $\left(\text{Notation}: \frac{1}{1}8\right)$

(2) Each of certain 5 ovals lies in the exteriors of the others, and the interior of one of those contains 5 ovals. Each of the latter 5 ovals lies in the exteriors of the others. $\left(\text{Notation}: \frac{5}{1}4\right)$

(3) An oval contains 9 ovals in its interior and each of the 9 ovals lies in the exteriors of the others. $\left(\text{Notation}: \frac{9}{1}\right)$

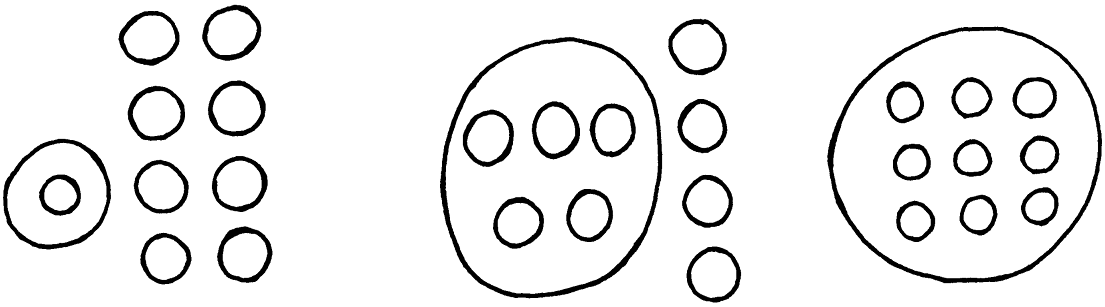

Figure 1.

We call the above three cases the *configurations* of types $\frac{1}{1}8$, $\frac{5}{1}4$, and $\frac{9}{1}$ respectively. We can easily construct curves of degree $(4, 4)$ of configuration type $\frac{1}{1}8$ by the " Harnack's method ", which is well known in the studies of Hilbert's 16th problem (see [**2**]). Here we omit the statement of this method. In this paper we prove that there exist curves of degree $(4, 4)$ of configuration types $\frac{5}{1}4$ and $\frac{9}{1}$ (Corollary 8 in §4). For this, it is sufficient to show the existence of 2-*sheeted coverings* (for the definition, see [**11**]) Y of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ branched along nonsingular real curves of degree $(4, 4)$ whose *real parts* (see below) are homeomorphic to $\Sigma_6 \amalg 5S^2$ and $\Sigma_2 \amalg 9S^2$ respectively (see [**5**, §3]), where Σ_g denotes a sphere with g handles and kS^2 denotes the disjoint union of k copies of S^2. Notice that the complex conjugation of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ is lifted into two anti-holomorphic involutions T^+ and T^- on Y. In the above statement, we call fixed point sets of these involutions real parts of Y.

It is well known that every 2-sheeted covering Y of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ branched along a nonsingular curve of degree $(4, 4)$ is a $K3$ surface. The topological types of real parts of real projective $K3$ surfaces are investigated in Nikulin [8]. Let h be the homology class of the preimage in Y of a hyperplane section of $\boldsymbol{P}^1 \times \boldsymbol{P}^1 (\subset \boldsymbol{P}^3)$. Then h is *primitive* (for the definition, see [8]) in $H_2(Y, \boldsymbol{Z})$ and we have $h^2=4$. Hence the triple $(H_2(Y), T^{\pm}_{*}, h)$ is a *polarized integral involution* (see [8]) with invariants $\delta_L=0$, $l_{(+)}=3$, $l_{(-)}=19$, $n=4$, $t_{(+)}=1$ and $t_{(-)}$ (for the notations, see [8]). Since we assume that $\boldsymbol{R}A$ is an M-curve whose components are all contractible in $\boldsymbol{R}P^1 \times \boldsymbol{R}P^1$, we moreover have $a=0$ (see also [8]) for either T^+ or T^- because of a consequence of [**5**, § 3]. Hence, by [8, Theorem 3.10.6], the real part of Y with respect to T^+ or T^- is homeomorphic to $\Sigma_g \amalg kS^2$, where $g=(21-t_{(-)})/2$ and $k=(1+t_{(-)})/2$. Furthermore, by [8, Theorem 3.4.3], a polarized integral involution with the above invariants exists if and only if $t_{(-)}=1, 9$ or 17. By [8, Theorem 3.10.1], the isomorphism classes of polarized integral involutions with the above invariants are in bijective correspondence with the *coarse projective equivalence classes* (see [8, §3, 10°]) of real projective $K3$ surfaces for which homology classes h of hyperplane sections (or those preimages) are primitive and $h^2=4$. Therefore, we see that there exist real projective $K3$ surfaces with $h^2=4$ (h: primitive) whose real parts are homeomorphic to $\Sigma_6 \amalg 5S^2$ or $\Sigma_2 \amalg 9S^2$. But these $K3$ surfaces are not necessarily 2-sheeted coverings of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ branched along nonsingular real curves of degree (4, 4). We must make a closer investigation of [8, Theorem 3.10.1].

We first prepare a sufficient condition for $K3$ surfaces (not necessarily algebraic) with antiholomorphic involutions, which are called *real* $K3$ *surfaces*, to be 2-sheeted coverings of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ branched along nonsingular real curves of degree (4, 4)(Lemma 2 in §2). In [**3**] it is proved that for every real $K3$ surface, there exists an " equivariant " locally universal Kähler family of its complex structures (Lemma (Kharlamov) in §1). For the real projective $K3$ surfaces (X, t) with $h^2=4$ (h: primitive) whose real parts are homeomorphic to $\Sigma_6 \amalg 5S^2$ or $\Sigma_2 \amalg 9S^2$ stated above, $L_\varphi = \mathrm{Ker}(1+t^*)$ are isomorphic to $U \oplus U \oplus (-E_8)$ and $U \oplus U$ respectively (see [8]), where U and E_8 are even unimodular lattices with $\mathrm{rank}\, U=2$, $\mathrm{sign}\, U=0$, and $\mathrm{rank} E_8 = \mathrm{sign} E_8 = 8$. We show that if for a real $K3$ surface (X, t), L_φ has $U \oplus U$ as its sublattice, then there exist real $K3$ surfaces which satisfy the conditions of Lemma 2 arbitrarily closely to the surface (X, t) in the equivariant family stated above (the proof of Theorem 6 in §4). Before this, we prepare Lemma 3 and its Corollary 4, which are finer versions of Tjurina's lemma concerning integer vector sequences ([**10**, Chap. IX, §5]).

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1. Real $K3$ surfaces and equivariant families of their complex structures.

We say a compact connected Kähler surface X is a *K3 surface* if the first Betti number of X vanishes and there exists a nowhere vanishing holomorphic 2-form ω_X on X. The following are known (cf. [**10**, Chap. IX]).

(1) $H^2(X, \boldsymbol{Z})$ is free of rank 22.

(2) The intersection form $H^2(X, \boldsymbol{Z}) \times H^2(X, \boldsymbol{Z}) \to \boldsymbol{Z}$ is isomorphic to $U \oplus U \oplus U \oplus (-E_8) \oplus (-E_8)$.

(3) $\omega_X \wedge \omega_X = 0$, $\omega_X \wedge \bar{\omega}_X > 0$, $\dim_C H^0(X, \Omega^2) = 1$. We set

$$\mathrm{Pic} X = (\omega_X)^{\perp} \cap H^2(X, \boldsymbol{Z}) = H^{1,1}(X) \cap H^2(X, \boldsymbol{Z}).$$

Since $h^1(X, \mathcal{O}_X) = \frac{1}{2} b_1(X) = 0$, we can regard $\mathrm{Pic} X$ as the group of isomorphism classes of complex line bundles on X. We denote by $Q(\,,\,)$ the intersection form of X. We set $P(X, \boldsymbol{C}) = \boldsymbol{P}(H^2(X, \boldsymbol{C}))$ and $K_{20} = \{\lambda \in P(X, \boldsymbol{C}) | Q(\lambda, \lambda) = 0\}$. Then we see that $H^{2,0}(X) = [\omega_X]$ is contained in K_{20}.

(4) There exists an effectively parametrized and locally universal family (V, M, π) of complex structures of X, where M is complex 20-dimensional. Here, by a family (V, M, π) of complex structures of X, we mean a C^∞-fibre bundle $\pi : V \to M$ with the fibre X, where V and M are connected complex manifolds, π is a holomorphic map onto M.

(5) For every family (V, M, π) of complex structures of a $K3$ surface $X = \pi^{-1}(m)$, there exists a contractible neighborhood U such that for any $\alpha \in U$, $V(\alpha) = \pi^{-1}(\alpha)$ are $K3$ surfaces and $(\pi^{-1}(U), U, \pi)$ is a C^∞-trivial bundle. Let $i_\alpha : V(\alpha) \to \pi^{-1}(U)$ be the inclusion map. Then $i_\alpha^* : H^2(\pi^{-1}(U), \boldsymbol{Z}) \to H^2(V(\alpha), \boldsymbol{Z})$ is an isomorphism. We define $\tau : U \to P(X, \boldsymbol{C})$ by $\tau(\alpha) = i_m^* \circ i_\alpha^{*-1}(H^{2,0}(V(\alpha)))$. This is called the *period mapping*. From [**10**, Chap. IX, Theorem 2], if (V, M, π) is effectively parametrized, then τ is a holomorphic embedding on a neighbourhood U' of m in U.

Furthermore, Kharlamov [**3**] shows the following.

LEMMA (KHARLAMOV [**3**]). *Let (X, t) be a real $K3$ surface, namely, X is a $K3$ surface and t is an antiholomorphic involution on it. Then there exist a locally universal family (V, M, π) of complex structures of X*

and antiholomorphic involutions t_V on V and t_M on M which satisfy the following conditions.

(i) *Each fibre $V(\alpha)$ is a $K3$ surface and $V(m)=X$.*

(ii) *M is contractible, and (V, M, π) is a C^∞-trivial bundle.*

(iii) *τ(see (5) above) is a holomorphic embedding on M and $\tau(M)$ is a neighborhood of $\tau(m)$ in K_{20}.*

(iv) *$t_V|_X=t$, $\pi\circ t_V=t_M\circ\pi$, $\tau\circ t_M=\overline{t^*\circ\tau}$, where $\overline{}$ is the natural complex conjugation on $P(X, \boldsymbol{C})$.*

REMARK. We can restrict t_V on $V(\alpha)$ for any $\alpha\in\mathrm{Fix}\ t_M$. We set $t_\alpha=t_V|_{V(\alpha)}$. Then $(V(\alpha), t_\alpha)$ are real $K3$ surfaces.

2. A sufficient condition for real $K3$ sufaces to be 2-sheeted coverings of $\boldsymbol{P}^1\times\boldsymbol{P}^1$ branched along real curves of degree (4, 4).

We prepare the following lemmas in order to catch 2-sheeted coverings (in the sense of [**11**, §1]) of $\boldsymbol{P}^1\times\boldsymbol{P}^1$ branched along (real) curves in the family of (real) $K3$ surfaces given in § 1.

LEMMA 1. *Let X be a $K3$ surface with rank* Pic$X=2$. *If there exist primitive elements c_1 and c_2 in* Pic$X$ *such that $c_1^2=c_2^2=0$ and $c_1\cdot c_2=2$, then X can be a 2-sheeted branched covering of $\boldsymbol{P}^1\times\boldsymbol{P}^1$, and the branch locus is a nonsingular curve of degree* (4, 4).

PROOF. We choose an element b such that b and c_1 generate the free $\boldsymbol{Z}$-module Pic$X$. Then $c_2=mc_1+nb$ for some integers m and n. Since $2=c_1\cdot c_2=n(c_1\cdot b)$, we have $n=\pm 1$ or ± 2. We show that $D^2\geq 0$ for any irreducible curve D on the surface X. In case $n=\pm 1$, we have Pic$X=\boldsymbol{Z}(c_1, c_2)$. Let D be an irreducible curve on X and $[D]$ be the linearly equivalence class of the divisor D. Then $[D]=kc_1+lc_2$ for some integers k and l, and we have $D^2=4kl$. Since $D^2\geq -2$, we have $D^2\geq 0$. In case $n=\pm 2$, since c_2 is primitive, we see that m is odd. Since $(2b)^2=(\pm c_2\mp mc_1)^2=-4m$, we have $b^2=-m$. Let D be an irreduible curve on X. Then we have $[D]=kc_1+lb$ for some integers k and l. Since $D^2=k^2c_1^2+2klc_1\cdot b+l^2b^2=\pm 2kl-l^2m$ and D^2 is even, we see that l is even. Hence $[D]$ is contained in $\boldsymbol{Z}(c_1, c_2)$. Therefore we see that $D^2\geq 0$ as in the case $n=\pm 1$.

Now let $F_i(i=1, 2)$ be a complex line bundle whose first Chern class is c_i. By the Riemann-Roch theorem, $h^0(F_i)+h^0(-F_i)\geq 2$. Since F_i is not trivial, we may assume that $h^0(-F_i)=0$ and $h^0(F_i)\geq 2$ replacing c_i by $-c_i$ if necessary. We will verify that $c_1\cdot c_2=2$ later on. Let C_i be the divisor of a global holomorphic section of F_i on X. We show that the complete linear system $|C_i|$ has no fixed components. If Γ is the fixed part of $|C_i|$, and D is an irreducible component of Γ, then we choose an effective divisor E such that $\Gamma+E$ is a member of $|C_i|$. We may assume that all irreducible components of E are distinct from D. In our cases, since $D^2\geq 0$, we have dim $|D|\geq 1$ by the Riemann-Roch theorem. Hence D is movable. This contradicts the assumption that Γ is the fixed part. Hence $|C_i|$ has no fixed components. Therefore, by [**6**, Proposition 1 ii)], each element of $|C_1|$ can be written as $E_1+\cdots+E_k$ with $E_i\in|C_1'|$, C_1' being nonsingular elliptic. (For $|C_2|$, we have the same results.) Hence we have $C_1\sim kC_1'$ (linearly equivalent). Since $[C_1']\in\boldsymbol{Z}(c_1, c_2)$, we have $[C_1']=sc_1+tc_2$ for some integers s and t. Then, since $c_1=k(sc_1+tc_2)$, we see that $k=1$. Hence we have $C_1\sim C_1'$. Thus we may consider C_1 and C_2 to be nonsingular elliptic curves. Hence we have $C_1\cdot C_2=2$. We set $C=C_1+C_2$. The complete linear system $|C|$ also has no fixed components. Hence, by [**6**, Proposition 1 i)], $|C|$ has no base points and contains an irreducible nonsingular curve C'. Since $C'^2=4$ (>0), the surface X is algebraic by [**4**, Theorem 3.3]. Thus we see that there exist elliptic curves C_1 and C_2 on the algebraic $K3$ surface X such that $C_1\cdot C_2=2$. Then the system $|C_i|$ $(i=1, 2)$ defines a morphism $\Phi_{|C_i|}: X\to\boldsymbol{P}^1$. We can define a holomorphic mapping $\Phi: X\to\boldsymbol{P}^1\times\boldsymbol{P}^1$ by the formula $\Phi(x)=(\Phi_{|C_1|}(x), \Phi_{|C_2|}(x))$ for any $x\in X$. Since $\Phi_{|C_1|}$ and $\Phi_{|C_2|}$ are surjective and $C_1\cdot C_2=2$, we see that Φ is surjective. Let $S:\boldsymbol{P}^1\times\boldsymbol{P}^1\to\boldsymbol{P}^3$ be the Segre embedding. This embedding gives a biholomorphic mapping onto a nonsingular quadric Q in $\boldsymbol{P}^3$. Then the composition $S\circ\Phi: X\to\boldsymbol{P}^3$ is nothing but a morphism $\Phi_{|C|}$ defined by the system $|C|$. From the well known formula $C^2=\deg\Phi_{|C|}\cdot\deg Q$, we see that the morphism $\Phi_{|C|}$ is of degree 2. Moreover, for any irreducible curve D, the image $\Phi_{|C|}(D)$ is an irreducible curve. In fact, if $\Phi_{|C|}(D)$ is a point P, then $\Phi_{|C|}^{-1}(H)\cdot D=0$ for a hyperplane section H of Q which does not meet the point P. Since $\Phi_{|C|}^{-1}(H)^2=C^2=4$, we have $D^2<0$ by the Hodge index theorem. But $D^2\geq 0$ on our surface X. This is a contradiction. We also see that for any point P in Q, the preimage $\Phi_{|C|}^{-1}(P)$ consists of finitely many points. Let B be the ramification divisor (see, for example, [**1**, p. 668]) of the finite surjective mapping $\Phi_{|C|}: X\to Q$. We use the same notation B for the support of the divisor B. We set $A=\Phi_{|C|}(B)$. Then A also defines a divisor. By the definition of the ramification divisor, $\Phi_{|C|}$ is locally biholomorphic on $X\backslash B$, and in our case, all the points in B are branch points in the sense of [**11**, Definition 1.3]. Let K_X (resp. K_Q) be the canonical divisor of X (resp. Q). Then we have (see, for example, [**7**, Lemma (6.20)])

$$K_X \sim \Phi^*_{|C|}(K_Q) + B.$$

Since we know that $K_X \sim 0$ and $K_Q = (-2)(pt \times \boldsymbol{P}^1 + \boldsymbol{P}^1 \times pt)$ identifying Q with $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ via the Segre embedding S, we have

$$B \sim 2\Phi^*(pt \times \boldsymbol{P}^1 + \boldsymbol{P}^1 \times pt).$$

Hence, in particular, $B \neq \phi$. Recall that the morphism $\Phi_{|C|}$ is of degree 2. Thus we obtain a 2-sheeted branched covering $\Phi : X \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ with branch locus A in the sense of [**11**, §1]. Hence the branch locus A is nonsingular. Moreover, from the proof of [**11**, Theorem 1.2], we have $[B] = \Phi^* F$ for a line bundle F over $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ with $F^{\otimes 2} = [A]$. Since Pic $(\boldsymbol{P}^1 \times \boldsymbol{P}^1) = \boldsymbol{Z}([pt \times \boldsymbol{P}^1], [\boldsymbol{P}^1 \times pt])$, we have $F = k[pt \times \boldsymbol{P}^1] + l[\boldsymbol{P}^1 \times pt]$ for some integers k and l. Since $B \sim 2\Phi^*(pt \times \boldsymbol{P}^1 + \boldsymbol{P}^1 \times pt)$, we have $k = l = 2$ by considering intersection numbers. Hence we have

$$A \sim 4(pt \times \boldsymbol{P}^1 + \boldsymbol{P}^1 \times pt).$$

Thus A is a nonsingular curve of degree (4, 4). Q. E. D.

REMARK. In the above lemma, for every irreducible curve D on the algebraic $K3$ surface X, we see that D^2 is divisible by 4. Hence, if $D^2 > 0$, then $D^2 \geq 4$, namely $p_a(D) \geq 3$. Moreover, for the irreducible curve C' $(\sim C)$, we know that $p_a(C') = 3$. Hence the surface X belongs to the class $\pi = 3$ (see [**10**, Chap. VIII, p. 188] or [**9**, § 1, p. 46]). Hence, by [**10**, Chap. VIII, Theorem 2], $\Phi_{|C|}$ is a birational morphism onto a quartic surface in $\boldsymbol{P}^3$, or a morphism of degree 2 onto a quadric in $\boldsymbol{P}^3$. We see that our surface X lies in the latter case.

LEMMA 2. *Let* (X, t) *be a real* $K3$*surface such that* X *satisfies the conditions of Lemma 1. If moreover,* c_1 *and* c_2 *are contained in* Ker$(1 + t^*)$, *then there exists a holomorphic mapping* Φ *which makes* X *a* 2-*sheeted branched covering of* $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ *and satisfies* $conj \circ \Phi = \Phi \circ t$. *Hence the branch locus is a nonsingular curve defined by a real homogenous polynomial of degree* (4, 4).

PROOF. In the proof of Lemma 1, we define $\Phi = (\Phi_{|C_1|}, \Phi_{|C_2|})$. Let s_1 and s_2 form a basis for the space $H^0(X, \mathcal{O}(C_1))$. Let ξ_0 and ξ_1 be holomorphic functions on X such that $\xi_1(x)s_1(x) = \xi_0(x)s_2(x)$ for any $x (\in X)$. Then $\Phi_{|C_1|}$ is defined to be $[\xi_0 : \xi_1]$. We show that $conj \circ \Phi_{|C_1|} = \Phi_{|C_1|} \circ t$ if we choose an appropriate basis for $H^0(X, \mathcal{O}(C_1))$.

We define the line bundle F_1 to be $[C_1]$. By the assumption, we see the first Chern class $c_1(F_1)$ is contained in Ker$(1 + t^*)$. Hence we have $c_1(F_1) = c_1(t^* \overline{F_1})$, where $\overline{F_1}$ is the conjugate bundle of F_1. Since $H^1(X,$

$\mathcal{O}_X)=0$, the line bundle F_1 and $t^*\overline{F_1}$ are isomorphic. We denote by E_1 and pr_1 the total space and the projection of F_1. Let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an open covering of X, $\varphi_\lambda : pr_1^{-1}(U_\lambda) \to U_\lambda \times \boldsymbol{C}$ be trivializations, and $g_{\lambda\mu} : U_\lambda \cap U_\mu \to \boldsymbol{C}^*$ be transition functions. We may assume that there exists an involution σ on Λ such that $U_{\sigma(\lambda)}=t(U_\lambda)$. Then the transition functions of the line bundle $t^*\overline{F_1}$ are $\overline{g_{\sigma(\lambda)\sigma(\mu)}\circ t} : U_\lambda \cap U_\mu \to \boldsymbol{C}^*$. Since F_1 and $t^*\overline{F_1}$ are isomorphic, there exists a collection of functions $f_\lambda(\in \mathcal{O}^*(U_\lambda))$ such that

$$(1)\quad g_{\lambda\mu}(x)=\frac{f_\lambda(x)}{f_\mu(x)}\overline{g_{\sigma(\lambda)\sigma(\mu)}(t(x))} \text{ for any } x(\in U_\lambda \cap U_\mu),$$

where we may consider that

$$(2)\quad f_{\sigma(\lambda)}=\overline{f_\lambda\circ t}^{-1}.$$

Then we can define an antiholomorphic involution T_1 on E_1 such that $t\circ pr_1 = pr_1\circ T_1$ and the restrictions $(T_1)_x : pr_1^{-1}(x) \to pr_1^{-1}(t(x))$ are antilinear as follows. (It turns out that the line bundle F_1 is a "real vector bundle".) We define T_1 on $pr_1^{-1}(U_\lambda)$ by the following formula.

$$\varphi_{\sigma(\lambda)}\circ T_1\circ \varphi_\lambda^{-1}(x, c)=(t(x), \overline{f_\lambda(x)^{-1}c})$$

By the equality (1), T_1 is well defined over E_1, and by (2), we see that T_1 is an involution. We now define an antilinear involution $\theta_1 : H^0(X, \mathcal{O}(F_1)) \to H^0(X, \mathcal{O}(F_1))$ by $\theta_1(s)=T_1\circ s\circ t$, and choose s_1 and s_2 stated above in Fix θ_1. Then we see that $\Phi_{|C_1|}=[\overline{\xi_0\circ t} : \overline{\xi_1\circ t}]$. Hence $conj\circ\Phi_{|C_1|}=\Phi_{|C_1|}\circ t$. We have the same results for $|C_2|$. Thus we have $conj\circ\Phi=\Phi\circ t$. It follows that $conj(A)=A$, where A is the branch locus. Q. E. D.

3. A lemma concerning integer vector sequences.

LEMMA 3. *For any integer sequence $\alpha_1'(n)$ with $\alpha_1'(n)\to\infty$, any positive real number α, any real numbers x_3 and x_4, there exist a subsequence $\alpha_1(n)$ of $\alpha_1'(n)$ and an integer vector sequence $(\beta_1(n), \beta_2(n), \beta_3(n), \beta_4(n))$ which satisfy the following five conditions.*

(1) $\beta_1\beta_2+\beta_3\beta_4=1$

(2) $\lim\limits_{n\to\infty}\dfrac{\beta_3}{\beta_1}=x_3$

(3) $\lim\limits_{n\to\infty}\dfrac{\beta_4}{\beta_1}=x_4$

(4) *β_1 and β_4 are odd.*

(5) $\lim\limits_{n\to\infty}\dfrac{\beta_1}{\alpha_1}=\alpha$

PROOF. We first prove in the case x_4 is a rational number. The rational number x_4 can be expanded into a finite simple continued fraction as follows.

$$x_4 = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{r-1} + \cfrac{1}{a_r}}}}$$

In the above, a_1 is an integer,and $a_2, \dots, a_r$ are positive integers. We define $(u_0, v_0), \dots, (u_r, v_r)$ inductively as follows.

$$(u_0, v_0) = (-1, -1)$$
$$(u_j, v_j) = \begin{cases} (v_{j-1}, u_{j-1}) \text{if } a_j \text{ is even or } (u_{j-1}, v_{j-1}) = (-1, 1) \\ (v_{j-1}, -u_{j-1}) \text{ otherwise} \end{cases}$$

In the case $r \geq 2$, we define b_i $(2 \leq i \leq r)$ as follows.

$$b_i = a_i + \cfrac{1}{a_{i+1} + \cfrac{1}{a_{i+2} + \ddots + \cfrac{1}{a_{r-1} + \cfrac{1}{a_r}}}}$$

Remark that every b_i is positive. We set $\alpha' = \frac{\alpha}{b_2 \times \cdots \times b_r}$. In the case $r = 1$, we set $\alpha' = \alpha$. Now we choose and fix a subsequence $\alpha_1(n)$ of $\alpha_1'(n)$ such that $\frac{\alpha_1(n)}{n} \to \infty$. Let $\tilde{\beta}_1(n)$ be the closest integer to $\alpha_1(n)\alpha'$. Since $\alpha_1(n) \to \infty$, we have $\lim \frac{\tilde{\beta}_1}{\alpha_1} = \alpha'$ and $\frac{\tilde{\beta}_1}{2n} = \frac{\tilde{\beta}_1}{\alpha_1}\frac{\alpha_1}{2n} \to \infty$. We set $\beta_1(n) = \left[\frac{\tilde{\beta}_1(n)}{2n}\right]$ or $\left[\frac{\tilde{\beta}_1(n)}{2n}\right] + 1$, where we take $\beta_1(n)$ to be odd (resp. even) if $v_r = -1$ (resp. 1). We have $\beta_1(n) \to \infty$. We set $x_3' = (-1)^r x_3$. In the case $(u_r, v_r) = (1, -1)$, let β_3 be the closest integer to $\beta_1 x_3'$ that is relatively prime to β_1. Since β_1 is odd, β_1 and $2\beta_3$ are relatively prime, and hence, there exist integers u and v such that $u\beta_1 + 2v\beta_3 = 1$ and $|u| < |2\beta_3|$, $|v| < |\beta_1|$. We set $\beta_2 = u$ and $\beta_4 = 2v$. In the case $(u_r, v_r) = (-1, 1)$, let β_3 be as above. Then there exist integers u and v such that $u\beta_1 + v\beta_3 = 1$ and $|u| < |\beta_3|, |v| < |\beta_1|$. We set $\beta_2 = u$ and $\beta_4 = v$. In the case $(u_r, v_r) = (-1, -1)$, let β_3 be the closest integer to $\beta_1 x_3'$ that is relatively prime to $2\beta_1$.

Then there exist integers u and v such that $2u\beta_1+v\beta_3=1$ and $|u|<|\beta_3|$, $|v|<|2\beta_1|$. We set $\beta_2=2u$ and $\beta_4=v$. The case $(u_r, v_r)=(1,1)$ cannot occur. It follows that β_4 is odd (resp. even) if $u_r=-1$ (resp. 1). In all the cases, we have $\beta_1\beta_2+\beta_3\beta_4=1$, $\lim_{n\to\infty}\frac{\beta_3}{\beta_1}=x_3'$, and $\left|\frac{\beta_4}{\beta_1}\right|<2$. We see that $\frac{\beta_2}{\beta_1}$ are also bounded. We define a new sequence $P(n)=(p_1(n), p_2(n), p_3(n), p_4(n))$ to be

$$(-\beta_4(n)+2n\beta_1(n), -\beta_3(n), 2n\beta_3(n)+\beta_2(n), \beta_1(n)).$$

Then we have $p_1p_2+p_3p_4=1, \lim\frac{p_3}{p_1}=x_3'$ and $\lim\frac{p_4}{p_1}=0$. Since $|\beta_1-\frac{\tilde{\beta}_1}{2n}|\leq 1$, $\lim\frac{\tilde{\beta}_1}{\alpha_1}=\alpha'$, and $\frac{\alpha_1}{n}\to\infty$, we have $\lim\frac{p_1}{\alpha_1}=\alpha'$. Remark that the parity of (p_1, p_2, p_3, p_4) corresponds to $(\beta_4, \beta_3, \beta_2, \beta_1)$.

We now assume that a new sequence $\beta(n)=(\beta_1, \beta_2, \beta_3, \beta_4)$ satisfies the conditions (1), (2), (3) and (5) in the statement of Lemma 3 for a positive real number α, real numbers x_3 and x_4, and a sequence $\alpha_1(n)$ with $\alpha_1(n)\to\infty$. Let k be an arbitrary integer with $k-x_4>0$. We define a new sequence $I_k(\beta(n))=(q_1, q_2, q_3, q_4)$ to be

$$(-\beta_4(n)+k\beta_1(n), -\beta_3(n), k\beta_3(n)+\beta_2(n), \beta_1(n)).$$

Then we see that $q_1q_2+q_3q_4=1$ and $\lim\frac{q_3}{q_1}=x_3$. Hence the properties (1) and (2) are preserved by the transformation I_k. On the other hand, we see that

$$\lim\frac{q_4}{q_1}=\frac{1}{k-x_4}$$

and

$$\lim\frac{q_1}{\alpha_1}=\alpha(k-x_4)(>0).$$

We next define a new sequence $J(\beta(n))$ to be $(\beta_1, \beta_2, -\beta_3, -\beta_4)$. Then the properties (1) and (5) are preserved by the transformation J. But for the properties (2) and (3), the limit values are multiplied by (-1).

The sequence $P(n)$ has the properties (1), (2) (for $x_3=x_3'$), (3) (for $x_4=0$) and (5). In the case $r\geq 2$, we can transform $P(n)$ by I_{a_r}. Then $I_{a_r}(P(n))$ has the properties (3) (for $x_4=\frac{1}{a_r}$) and (5) (for $\alpha=\alpha' a_r=\frac{\alpha}{b_2\times\cdots\times b_{r-1}}$ (>0)). Next we can transform $J\circ I_{a_r}(P(n))$ by $I_{a_{r-1}}$. Then

$I_{a_{r-1}} \circ J \circ I_{a_r}(P(n))$ has the properties (3) (for $x_4 = \frac{1}{a_{r-1} + \frac{1}{a_r}}$) and (5) (for $a = a' a_r (a_{r-1} + \frac{1}{a_r}) = \frac{a}{b_2 \times \cdots \times b_{r-2}}$ (>0)). Thus we obtain the sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = J \circ I_{a_2} \circ J \circ \cdots \circ J \circ I_{a_{r-2}} \circ J \circ I_{a_{r-1}} \circ J \circ I_{a_r}(P(n))$. In the case $r=1$, we set $(\gamma_1, \gamma_2, \gamma_3, \gamma_4) = P(n)$. Then we have (1) $\gamma_1\gamma_2 + \gamma_3\gamma_4 = 1$ (2) $\lim \frac{\gamma_3}{\gamma_1} = -x_3$ (3) $\lim \frac{\gamma_4}{\gamma_1} = a_1 - x_4$ (5) $\lim \frac{\gamma_1}{a_1} = a$. Finally we set $(\beta_1, \beta_2, \beta_3, \beta_4) = (\gamma_1, a_1\gamma_3 + \gamma_2, -\gamma_3, -\gamma_4 + a_1\gamma_1)$. Then this sequence satisfies the condition (1), (2), (3) and (5) of Lemma 3. From the definition of (u_r, v_r), we observe that the condition (4) is also satisfied. Thus Lemma 3 is proved in the case x_4 is a rational number. To complete the proof of the lemma, let x_4 be an arbitrary real number. Let $\{x_4(n)\}$ $(n=1, 2, 3\ldots)$ be a rational number sequence which converges to x_4 satisfying $|x_4(n) - x_4| < \frac{1}{n}$. From the results above, there exist sequences $(\beta_{1,n}, \beta_{2,n}, \beta_{3,n}, \beta_{4,n})$ such that (1) $\beta_{1,n}\beta_{2,n} + \beta_{3,n}\beta_{4,n} = 1$ (2) $\lim_{m\to\infty} \frac{\beta_{3,n}(m)}{\beta_{1,n}(m)} = x_3$ (3) $\lim_{m\to\infty} \frac{\beta_{4,n}(m)}{\beta_{1,n}(m)} = x_4(n)$ (4) $\beta_{1,n}$ and $\beta_{4,n}$ are odd (5) $\lim_{m\to\infty} \frac{\beta_{1,n}(m)}{a_1(m)} = a$. Remark that the subsequence $a_1(m)$ of $a_1'(m)$ does not depend on n. We choose a natural number sequence $m(1) < m(2) < m(3) < \cdots$ such that $\left|\frac{\beta_{3,n}(m(n))}{\beta_{1,n}(m(n))} - x_3\right| < \frac{1}{n}$, $\left|\frac{\beta_{4,n}(m(n))}{\beta_{1,n}(m(n))} - x_4(n)\right| < \frac{1}{n}$ and $\left|\frac{\beta_{1,n}(m(n))}{a_1(m(n))} - a\right| < \frac{1}{n}$. We set $(\beta_1(n), \beta_2(n), \beta_3(n), \beta_4(n)) = (\beta_1(m(n)), \beta_2(m(n)), \beta_3(m(n)), \beta_4(m(n)))$. It is sufficient that we define $a_1(n)$ to be $a_1(m(n))$ newly. This completes the proof of Lemma 3.

COROLLARY 4. *For any integer sequence $a_1'(n)$ with $a_1'(n) \to \infty$, any positive real number a, any real numbers x_3 and x_4, there exist a subsequence $a_1(n)$ of $a_1'(n)$ and an integer vector sequence $(\beta_1(n), \beta_2(n), \beta_3(n), \beta_4(n))$ which satisfy the following five conditions.*

(1) $\beta_1\beta_2 + \beta_3\beta_4 = 2$

(2) $\lim_{n\to\infty} \frac{\beta_3}{\beta_1} = x_3$

(3) $\lim_{n\to\infty} \frac{\beta_4}{\beta_1} = x_4$

(4) *β_1 and β_3 are relatively prime, and so are β_2 and β_4.*

(5) $\lim_{n\to\infty} \frac{\beta_1}{a_1} = a$

PROOF. There exists a sequence $(\beta_1, \beta_2, \beta_3, \beta_4)$ which satisfies the conditions (1), (3), (4), (5) in Lemma 3 and the condition that $\lim_{n\to\infty}\frac{\beta_3}{\beta_1}=\frac{x_3}{2}$. Then, from (1) and (4), β_1 and $2\beta_3$ are relatively prime, and so are $2\beta_2$ and β_4. Thus the new sequence $(\beta_1, 2\beta_2, 2\beta_3, \beta_4)$ is a required one. Q. E. D.

REMARK. Lemma 3 is a finer version of [**10**, Chap. IX, §5, Lemma] for $\pi=2$, and Corollary 4 is for $\pi=3$.

4. The main theorem.

Let (X, t) be a real $K3$ surface. We set $L_\varphi=\mathrm{Ker}(1+t^*)$, and $L^\varphi=\mathrm{Ker}(1-t^*)$ in $H^2(X, \boldsymbol{Z})$. Remark that Fix $\overline{t^*}=((L^\varphi\otimes\boldsymbol{R})\oplus i(L_\varphi\otimes\boldsymbol{R}))/\boldsymbol{R}^*$ in $P(X, \boldsymbol{C})$.

PROPOSITION 5. *If L_φ has $U\oplus U$ as its sublattice, then there exists a pair $\{c_1(n)\}$, $\{c_2(n)\}$ of sequences which consist of primitive elements of $U\oplus U$ and satisfy the conditions that $Q(c_1(n), c_1(n))=Q(c_2(n), c_2(n))=0$, $Q(c_1(n), c_2(n))=2$, the sequence of the subspaces $L_n=\{\lambda\in P(X, \boldsymbol{C})|Q(\lambda, c_1(n))=Q(\lambda, c_2(n))=0\}$ of codimension 2 converges to a subspace $L=\{\lambda\in P(X, \boldsymbol{C})|Q(\lambda, \xi_1)=Q(\lambda, \xi_2)=0\}$ of codimension 2, where ξ_1 and ξ_2 are elements of $(U\oplus U)\otimes\boldsymbol{R}$, and L intersects K_{20} transversely at $H^{2,0}(X)$ in $P(X, \boldsymbol{C})$.*

Hence the sequence of the subspaces $L_n\cap(\mathrm{Fix}\,\overline{t^*})$ of real codimension 2 converges to the subspace $L\cap(\mathrm{Fix}\,\overline{t^*})$ of real codimension 2, and $L\cap(\mathrm{Fix}\,\overline{t^*})$ intersects $K_{20}\cap(\mathrm{Fix}\,\overline{t^*})$ transversely at $H^{2,0}(X)$ in Fix $\overline{t^*}$.

PROOF. For our sublattice of L_φ which is isomorphic to $U\oplus U$, we use the same notation $U\oplus U$. Since $U\oplus U$ is unimodular, we have $H^2(X, \boldsymbol{Z})=(U\oplus U)\oplus(U\oplus U)^\perp$. Let e_1, e_2, e_3, e_4 form a basis for $U\oplus U$ and represent the intersection form Q by the matrix

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}.$$

We set $s=\mathrm{rank}L_\varphi$ and let $e_5, \dots, e_s$ form a basis for $L_\varphi\cap(U\oplus U)^\perp$. Then $e_1, \dots, e_s$ form a basis for L_φ. Remark that $(L_\varphi\otimes\boldsymbol{Q})\oplus(L^\varphi\otimes\boldsymbol{Q})=H^2(X, \boldsymbol{Q})$, $L_\varphi=(L^\varphi)^\perp$ and $L^\varphi=(L_\varphi)^\perp$ in $H^2(X, \boldsymbol{Z})$. Let $e_{s+1}, \dots, e_{22}$ form a basis for L^φ. Then $e_1, \dots, e_{22}$ form a basis for $H^2(X, \boldsymbol{Q})$. Since $H^{2,0}(X)=\overline{t^*}(H^{2,0}(X))$, we can take ω_X so that $\omega_X=\overline{t^*\omega_X}$. Then we have $\omega_X=(\sum_{j=s+1}^{22}\lambda_j e_j)+i(\sum_{j=1}^{s}\lambda_j e_j)$ for some real numbers $\lambda_j (1\le j\le 22)$. We set

$\omega_+ = \sum_{j=s+1}^{22} \lambda_j e_j$ and $\omega_- = \sum_{j=1}^{s} \lambda_j e_j$. Since $\omega_X \wedge \omega_X = 0$ and $\omega_X \wedge \bar{\omega}_X > 0$ (recall §1), we have $\omega_+{}^2 = \omega_-{}^2 > 0$. Moreover, we set $\omega'_- = \sum_{j=5}^{s} \lambda_j e_j$. Then $\omega_-{}^2 = 2(\lambda_1\lambda_2 + \lambda_3\lambda_4) + \omega'_-{}^2$. Remark that $\omega_+ \in L^\varphi \otimes \boldsymbol{R}$, $U \oplus U \subset L_\varphi$, where sign $(U \oplus U) = (2, 2)$, and $\omega'_- \in (L_\varphi \cap (U \oplus U)^\perp) \otimes \boldsymbol{R}$. Since sign $(H^2(X, \boldsymbol{Z}), Q) = (3, 19)$, we have $\omega'_-{}^2 \leq 0$. Therefore we obtain $\lambda_1\lambda_2 + \lambda_3\lambda_4 > 0$.

We may assume that $\lambda_4 \neq 0$ replacing (e_1, e_2, e_3, e_4) by (e_3, e_4, e_1, e_2) if necessary. We set

$$x_3 = \frac{\lambda_1}{\lambda_4},\ x_4 = \lambda_1 x_3 + \lambda_4,\ y_4 = (1 + x_3{}^2)(\lambda_2 x_3 + \lambda_3),$$
$$\xi_1 = e_2 - x_3 e_3,\ \xi_2 = x_3 x_4 (1 + x_3{}^2) e_1 - x_3 y_4 e_2 - y_4 e_3 + x_4 (1 + x_3{}^2) e_4.$$

We define $L = \{\lambda \in P(X, \boldsymbol{C}) | Q(\lambda, \xi_1) = Q(\lambda, \xi_2) = 0\}$. The subspace L meets $H^{2,0}(X)$ because $Q(\omega_X, \xi_1) = i(\lambda_1 - \frac{\lambda_1}{\lambda_4}\lambda_4) = 0$ and $Q(\omega_X, \xi_2) = i(x_3 x_4(1 + x_3{}^2)\lambda_2 - x_3 y_4 \lambda_1 - y_4 \lambda_4 + x_4(1 + x_3{}^2)\lambda_3) = i((1 + x_3{}^2)(\lambda_2 x_3 + \lambda_3) x_4 + (-\lambda_1 x_3 - \lambda_4) y_4) = i(y_4 x_4 - x_4 y_4) = 0$. We show that L intersects K_{20} at $H^{2,0}(X)$ transversely. We identify $P(X, \boldsymbol{C})$ with $\boldsymbol{P}^{21} = \{[X_1 : \ldots : X_{22}]\}$ taking a basis $ie_1, \ldots, ie_s, e_{s+1}, \ldots, e_{22}$. Then K_{20} is identified with the subset defined by an integral homogeneous polynomial of degree 2 of the form $f(X_1, \ldots, X_{22}) = -2(X_1 X_2 + X_3 X_4) + f_1(X_5, \ldots, X_{22})$. Hence the tangent space of K_{20} at $H^{2,0}(X)$ is identified with the subspace defined by a real linear form of the form $h(X_1, \ldots, X_{22}) = \lambda_2 X_1 + \lambda_1 X_2 + \lambda_4 X_3 + \lambda_3 X_4 + h_1(X_5, \ldots, X_{22})$. Let H denote this space. L intersects H transversely at $H^{2,0}(X)$ in $\boldsymbol{P}^{21}$. If not, then H contains L. In particular, $(H \cap \boldsymbol{RP}^3 \times \{0\}) \supset (L \cap \boldsymbol{RP}^3 \times \{0\})$, where

$$H \cap \boldsymbol{RP}^3 \times \{0\} = \{\lambda_2 X_1 + \lambda_1 X_2 + \lambda_4 X_3 + \lambda_3 X_4 = 0\} \times \{0\}$$

and

$$L \cap \boldsymbol{RP}^3 \times \{0\}$$
$$= \{X_1 - x_3 X_4 = -x_3 y_4 X_1 + x_3 x_4 (1 + x_3{}^2) X_2 + x_4 (1 + x_3{}^2) X_3 - y_4 X_4 = 0\} \times \{0\}.$$

But the following matrix is of rank 3.

$$\begin{pmatrix} \lambda_2 & 1 & -x_3 y_4 \\ \lambda_1 & 0 & x_3 x_4 (1 + x_3{}^2) \\ \lambda_4 & 0 & x_4 (1 + x_3{}^2) \\ \lambda_3 & -x_3 & -y_4 \end{pmatrix}$$

In fact, the determinant of the following matrix is equal to $\dfrac{2(\lambda_1{}^2 + \lambda_4{}^2)^2 (\lambda_1\lambda_2 + \lambda_3\lambda_4)\lambda_1}{\lambda_4{}^5}$.

$$\begin{pmatrix} \lambda_2 & 1 & -x_3 y_4 \\ \lambda_1 & 0 & x_3 x_4 (1 + x_3{}^2) \\ \lambda_3 & -x_3 & -y_4 \end{pmatrix}$$

Hence, the above matrix is of rank 3 if $\lambda_1 \neq 0$. And if $\lambda_1 = 0$, then the above matrix is as follows.

$$\begin{pmatrix} \lambda_2 & 1 & 0 \\ 0 & 0 & 0 \\ \lambda_4 & 0 & \lambda_4 \\ \lambda_3 & 0 & -\lambda_3 \end{pmatrix}$$

This matrix is of rank 3 if $\lambda_1 = 0$. Thus we have a contradiction. Therefore L intersects K_{20} at $H^{2,0}(X)$ transversely.

We now show that there exists a pair $\{c_1(n)\}$, $\{c_2(n)\}$ of sequences for which the sequence $\{\lambda \in P(X, \boldsymbol{C}) | Q(\lambda, c_1(n)) = Q(\lambda, c_2(n)) = 0\}$ converges to the above L and the properties in the statement of Proposition 5 hold. By Corollary 4 in § 3, there exists an integer vector sequence $(\alpha_{13}, \beta_{24}, -\alpha_{24}, \beta_{13})$ such that

(1) $\alpha_{13}\beta_{24} - \alpha_{24}\beta_{13} = 2$,

(2) $\lim \dfrac{-\alpha_{24}}{\alpha_{13}} = x_3$,

(3) $\lim \dfrac{\beta_{13}}{\alpha_{13}} = x_4$,

(4) α_{13} and $-\alpha_{24}$ are relatively prime, and so are β_{24} and β_{13}, and

(5) $\alpha_{13} \to \infty$.

By Lemma 3, replacing the above sequence by an appropriate subsequence if necessary, we can find an another integer vector sequence $(\alpha_{14}, \beta_{23}, -\alpha_{23}, \beta_{14})$ such that

(1′) $\alpha_{14}\beta_{23} - \alpha_{23}\beta_{14} = 1$,

(2′) $\lim \dfrac{-\alpha_{23}}{\alpha_{14}} = 0$,

(3′) $\lim \dfrac{\beta_{14}}{\alpha_{14}} = y_4$, and

(4′) $\lim \dfrac{\alpha_{14}}{\alpha_{13}} = \dfrac{1}{\sqrt{2}}$.

We set

$$\alpha_1 = \alpha_{13}\alpha_{14},\ \alpha_2 = \alpha_{23}\alpha_{24},\ \alpha_3 = -\alpha_{13}\alpha_{23},\ \alpha_4 = \alpha_{14}\alpha_{24},$$
$$\beta_1 = \beta_{13}\beta_{14},\ \beta_2 = \beta_{23}\beta_{24},\ \beta_3 = -\beta_{13}\beta_{23},\ \beta_4 = \beta_{14}\beta_{24}.$$

Then we have

$$\alpha_1\alpha_2 + \alpha_3\alpha_4 = \beta_1\beta_2 + \beta_3\beta_4 = 0$$

and

$$\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_4+\alpha_4\beta_3=(\alpha_{13}\beta_{24}-\alpha_{24}\beta_{13})(\alpha_{14}\beta_{23}-\alpha_{23}\beta_{14})=2.$$

From (4) and (1′) above, we see that $\alpha_1, \alpha_2, \alpha_3$ and α_4 are relatively prime. So are $\beta_1, \beta_2, \beta_3$ and β_4. Hence, if we set $c_1=\alpha_1e_2+\alpha_2e_1+\alpha_3e_4+\alpha_4e_3$ and $c_2=\beta_1e_2+\beta_2e_1+\beta_3e_4+\beta_4e_3$, then $Q(c_1(n), c_1(n))=Q(c_2(n), c_2(n))=0$, $Q(c_1(n), c_2(n))=2$, and moreover, $c_1(n)$ and $c_2(n)$ are primitive elements in $U\oplus U$ (hence in $H^2(X, \boldsymbol{Z})$).

Finally we show that the sequence $L_n=\{Q(\lambda, c_1(n))=Q(\lambda, c_2(n))=0\}$ converges to L. We first observe that

$$\lim\frac{\alpha_2}{\alpha_1}=\lim\frac{\alpha_{24}}{\alpha_{13}}\lim\frac{\alpha_{23}}{\alpha_{14}}=(-x_3)\cdot 0=0,$$
$$\lim\frac{\alpha_3}{\alpha_1}=\lim\frac{-\alpha_{23}}{\alpha_{14}}=0,$$
$$\lim\frac{\alpha_4}{\alpha_1}=\lim\frac{\alpha_{24}}{\alpha_{13}}=-x_3,$$
$$\lim\frac{\beta_2}{\beta_1}=\lim\frac{\beta_{24}}{\beta_{13}}\lim\frac{\beta_{23}}{\beta_{14}}=(-x_3)\cdot 0=0,$$
$$\lim\frac{\beta_3}{\beta_1}=\lim\frac{-\beta_{23}}{\beta_{14}}=0,$$

and

$$\lim\frac{\beta_4}{\beta_1}=\lim\frac{\beta_{24}}{\beta_{13}}=-x_3.$$

Hence both $[\alpha_1:\alpha_2:\alpha_3:\alpha_4]$ and $[\beta_1:\beta_2:\beta_3:\beta_4]$ converge to $[1:0:0:-x_3]$. Thus both $\{Q(\lambda, c_1(n))=0\}$ and $\{Q(\lambda, c_2(n))=0\}$ converge to $\{Q(\lambda, \xi_1)=0\}$. In order to know the limit subspace of $\{L_n\}$, we set

$$B_j=\Big(\sum_{i=1}^{4}\alpha_i^2\Big)\beta_j-\Big(\sum_{i=1}^{4}\alpha_i\beta_i\Big)\alpha_j\ (j=1, 2, 3, 4).$$

Remark that (B_1, B_2, B_3, B_4) are orthogonal to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in $\boldsymbol{R}^4$ with respect to the Euclidean inner product. We set

$$\tilde{c}_2=B_1e_2+B_2e_1+B_3e_4+B_4e_3.$$

Then we see $L_n=\{Q(\lambda, c_1(n))=Q(\lambda, \tilde{c}_2(n))=0\}$. We now consider the limit hyperplane of the sequence $\{Q(\lambda, \tilde{c}_2(n))=0\}$. Since

$$B_1=\alpha_2(-2\alpha_{23}\beta_{14}-\alpha_{13}\beta_{24})+\alpha_3\alpha_{13}\beta_{13}-2\alpha_4\alpha_{14}\beta_{14},$$
$$B_2=\alpha_1(2\alpha_{23}\beta_{14}+\alpha_{13}\beta_{24})-2\alpha_3\alpha_{23}\beta_{23}+\alpha_4\alpha_{24}\beta_{24},$$
$$B_3=\alpha_4(2\alpha_{14}\beta_{23}-\alpha_{13}\beta_{24})-\alpha_1\alpha_{13}\beta_{13}-2\alpha_2\alpha_{23}\beta_{23}$$

and

$$B_4=\alpha_3(-2\alpha_{14}\beta_{23}+\alpha_{13}\beta_{24})+2\alpha_1\alpha_{14}\beta_{14}-\alpha_2\alpha_{24}\beta_{24}\,;$$

we have

$$\lim\frac{B_1}{\alpha_1^2}=\sqrt{2}\,x_3y_4,$$

$$\lim\frac{B_2}{\alpha_1^2}=-\sqrt{2}\,x_3x_4(1+x_3^2),$$

$$\lim\frac{B_3}{\alpha_1^2}=-\sqrt{2}\,x_4(1+x_3^2)$$

and

$$\lim\frac{B_4}{\alpha_1^2}=\sqrt{2}\,y_4.$$

Hence

$[B_1:B_2:B_3:B_4]$ converges to $[-x_3y_4:x_3x_4(1+x_3^2):x_4(1+x_3^2):-y_4]$. Namely, $\{Q(\lambda,\tilde{c}_2(n))=0\}$ converges to $\{Q(\lambda,\xi_2)=0\}$. Therefore L_n converges to L. With respect to the identification $P(X, \boldsymbol{C})\simeq\boldsymbol{P}^{21}$ stated above, $\boldsymbol{RP}^{21}$ corresponds to Fix $\overline{t^*}=(i(L_\varphi\otimes\boldsymbol{R})\oplus(L^\varphi\otimes\boldsymbol{R}))/\boldsymbol{R}^*$. Hence the latter assertion of the proposition follows. Q. E. D.

We next consider a family (V, M, π) of complex structures of X with antiholomorphic involutions t_V and t_M, and the period mapping $\tau: M\to P(X, \boldsymbol{C})$ as stated in Kharlamov's lemma (recall §1).

THEOREM 6. *Let (X, t) be a real $K3$ surface. If L_φ has $U\oplus U$ as its sublattice, there exist points α in* Fix t_M *for which real $K3$ surfaces $(V(\alpha), t_\alpha)$ can be 2-sheeted coverings of $\boldsymbol{P}^1\times\boldsymbol{P}^1$ (Let Φ_α denote the covering maps.) branched along nonsingular curves defined by real homogeneous polynomials of degree $(4,4)$ and satisfy $conj\circ\Phi_\alpha=\Phi_\alpha\circ t_\alpha$ arbitrarily closely to m.*

PROOF. We set $(U\oplus U)_\alpha=i_\alpha^*\circ i_m^{*-1}(U\oplus U)$ for any α in M. The isomorphisms $i_\alpha^*\circ i_m^{*-1}: H^2(X, \boldsymbol{Z})\to H^2(V(\alpha), \boldsymbol{Z})$ preserve the intersection forms. Let Q_α denote the intersection form on $V(\alpha)$. Recall that we set $t_\alpha=t_V|_{V(\alpha)}$ for every α in Fix t_M. We set $L_\alpha=\mathrm{Ker}(1+t_\alpha^*)$ in $H^2(V(\alpha),\boldsymbol{Z})$. Since $L_\alpha=i_\alpha^*\circ i_m^{*-1}(L_\varphi)$, we have $(U\oplus U)_\alpha\subset L_\alpha$. Let $\{L_n\}$ be a sequence obtained by Proposition 5. Then for a sufficiently large natural number N, $L_n\cap\boldsymbol{RP}^{21}$ intersects $\tau(\mathrm{Fix}\ t_M)=K_{20}\cap\boldsymbol{RP}^{21}$ transversely at $H^{2,0}(X)$ in $\boldsymbol{RP}^{21}=(i(L_\varphi\otimes\boldsymbol{R})\oplus(L^\varphi\otimes\boldsymbol{R}))/\boldsymbol{R}^*$ (recall the proof of Proposition 5) for any $n\geq N$. Hence $L_n\cap\tau(\mathrm{Fix}\ t_M)$ is nonempty and real 18 dimensional. We set

$$\hat{E}=\{\tau(\alpha)\in\tau(M)|\text{rank Pic}\,V(\alpha)\geq 3\}.$$

From the results in [**10**, Chap. IX, § 4, p. 215], rank Pic $V(\alpha)\geq 3$ if and only if $Q(\tau(\alpha), c_j^\alpha)=0$ for elements $c_j^\alpha (j=1,2,3)$ in $H^2(X, \boldsymbol{Z})$ which are linearly independent over $\boldsymbol{C}$ (hence, over $\boldsymbol{R}$). Hence $L_n \cap \tau(\text{Fix } t_M) \cap \hat{E}$ can be covered by countably many real 17 dimensional submanifolds. Hence $(L_n \cap \tau(\text{Fix } t_M)) \backslash \hat{E}$ is dense in $L_n \cap \tau(\text{Fix } t_M)$, and for every $\tau(\alpha) \in (L_n \cap \tau(\text{Fix } t_M)) \backslash \hat{E}$, we have $\alpha \in \text{Fix } t_M$ and rank $\text{Pic } V(\alpha)=2$. We set $c_{j\alpha}(n)= i_\alpha^* \circ i_m^{*-1}(c_j(n))$ for $j(=1,2)$. Then $Q_\alpha(c_{1\alpha}, c_{1\alpha})=Q_\alpha(c_{2\alpha}, c_{2\alpha})=0$ and $Q_\alpha(c_{1\alpha}, c_{2\alpha})=2$. Since $Q(i_m^* \circ i_\alpha^{*-1}(H^{2,0}(V(\alpha))), c_j)=0$, we have $Q_\alpha(H^{2,0}(V(\alpha))), c_{j\alpha})=0$, that is, $c_{j\alpha} \in \text{Pic } V(\alpha)=(H^{2,0}(V(\alpha))^\perp) \cap H^2(V(\alpha), \boldsymbol{Z})$. We see that $c_{1\alpha}$ and $c_{2\alpha}$ are primitive elements in $(U \oplus U)_\alpha$, hence in $H^2(V(\alpha), \boldsymbol{Z})$. Recall that $(U \oplus U)_\alpha \subset L_\alpha = \text{Ker}(1+t_\alpha^*)$. Hence $(V(\alpha), t_\alpha)$ satisfies the conditions of Lemma 2. Since $(L_n \cap \tau(\text{Fix } t_M)) \backslash \hat{E}$ is dense in $L_n \cap \tau(\text{Fix } t_M)$ and n $(\geq N)$ is an arbitrary number, we can choose such $\alpha \in \text{Fix } t_M$ arbitrarily closely to m. This completes the proof of Theorem 6.

COROLLARY 7. *Let* (X, t) *be a real* $K3$ *surface. If* L_φ *has* $U \oplus U$ *as its sublattice, then there exists a 2-sheeted covering* $\Phi: Y \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ *branched along a nonsingular real curve of degree* (4, 4) *and an antiholomorphic involution* T *on* Y *such that* $conj \circ \Phi = \Phi \circ T$ *and* Fix$T$ *is diffeomorphic to* Fix$t$.

PROOF. We can consider the restriction $\boldsymbol{R}\pi: \text{Fix } t_V \to \text{Fix } t_M$ of the family (V, M, π). Although Fix t_M is possibly disconnected, we may consider that α of Theorem 6 and m are contained in the same connected component of Fix t_M. Since $\boldsymbol{R}\pi$ is a proper submersion onto Fix t_M, $\boldsymbol{R}\pi^{-1}(\alpha)$ is diffeomorphic to $\boldsymbol{R}\pi^{-1}(m)$, where $\boldsymbol{R}\pi^{-1}(\alpha)=\text{Fix } t_\alpha$ and $\boldsymbol{R}\pi^{-1}(m)=\text{Fix } t$. It is sufficient to set $Y=V(\alpha)$ *and* $T=t_\alpha$. Q. E. D.

COROLLARY 8. *Three possible configuration types* $\frac{1}{1}8$, $\frac{5}{1}4$ *and* $\frac{9}{1}$ *are all realized by some real curves of degree* (4, 4).

PROOF. As stated in §0, there exist real projective $K3$ surfaces (X, t) with $h^2=4$ (h: primitive) whose real parts are homeomorphic to $\Sigma_{10} \amalg S^2$, $\Sigma_6 \amalg 5S^2$ and $\Sigma_2 \amalg 9S^2$ respectively. Moreover, for such real $K3$ surfaces, L_φ are isomorphic to $U \oplus U \oplus (-E_8) \oplus (-E_8)$, $U \oplus U \oplus (-E_8)$ and $U \oplus U$ respectively (see [**8**]). Hence L_φ have $U \oplus U$ as sublattices. By Corollary 7 and [**5**, §3] (recall §0), we obtain our required results. Q. E. D.

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