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Boundedness of minimizers

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Abstract. We find conditions guaranteeing that solutions to typical problems of the calculus of variations are bounded.
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§ 0 Introduction

Consider a variational problem of the following type:

$$(0.0) \qquad \begin{cases} \int_G f(u, Du)\,dx = \text{minimum}, \\ \text{under the condition}: u=g \text{ on } \partial G. \end{cases}$$

Here G, f, g are given; G is an open subset of euclidean space $\boldsymbol{R}^n$; n, the dimension, is greater than 1; competing functions u are assumed to have scalar values. D stands for gradient and $dx = dx_1 \ldots dx_n$, the Lebesgue n-dimensional measure.

We address ourselves to the following question. Suppose a minimizer exists and g, the boundary datum is bounded. Is such a minimizer bounded?

An approach to this question is using the Euler equation of problem (0.0) and developing maximum principles for weak solutions to nonlinear partial differential equations of elliptic type — one aim of an earlier paper [Tal]. This approach requires some smoothness of integrand f and essentially involves the first order derivatives of f.

In the present paper we merely assume a condition on the growth of integrand $f(u, \xi)$ with respect to the last variable ξ. In its simplest form, such a condition reads:

$$f(u, \xi) \geq A(|\xi|)$$

for every scalar u and any vector ξ in $\boldsymbol{R}^n$. More generally, we assume

$$(0.1) \qquad f(u, \xi) \geq A(|\xi|) - A(\lambda |u|).$$

Here A is some Young function (see section 1(i)) and λ is some nonnegative constant; $|\xi| = (\xi_1^2 + \ldots + \xi_n^2)^{1/2}$, the length of ξ.

Let B be any increasing function such that

$$f(u, 0) \leq B(|u|)$$

for every u — let

$$B(r) = \sup\left\{f(u, 0) : -r \leq u \leq r\right\} \tag{0.2}$$

for instance. Here we pay attention to the behavior of $f(u, \xi)$ in a situation where u grows without the growth of u being balanced by a simultaneous growth of ξ: the opposite of a situation where an unbounded minimizer and its gradient may possibly be involved.

An answer to the question in hand can be derived from a balance between the growth of $A(r)$ as $r \longrightarrow +\infty$ and the growth of $B(r)$. Roughly speaking, we show that solutions to problem (0.0) are bounded if A prevails over B. Of course, we must assume the boundary datum is bounded and the set of competing functions is suitably modeled. Precise statements are in section 1 below.

Similar results were previously obtained under the additional hypothesis that functions A and B are powers. See [S], [HS], [LU]. We stress that neither A nor B need be a power for our results to hold — yet our aim is not a great display of generality, but simply avoiding unnecessary hypotheses. Related results are in [Sch].

§ 1 Statement of results

List of basic hypotheses:

(i) A Young function A and a nonnegative constant λ exist such that inequality (0.1) holds for every scalar u and every vector in $\boldsymbol{R}^n$. Recall that a function is called Young if is defined in $[0, \infty[$, takes nonnegative real values and possibly the value $+\infty$, vanishes at 0 — maybe in a neighborhood of 0 too, but does not vanish identically — is increasing and convex. See [ON], for example.

(ii) g, the boundary datum, is bounded. Moreover, g can be continued in the whole of $\boldsymbol{R}^n$ by a bounded weakly differentiable function — still denoted g — whose gradient Dg satisfies: $\int_{\boldsymbol{R}^n} A(|Dg|)\,dx < \infty$.

(iii) Competing functions u have the following properties: u is weakly differentiable and $\int_G A(|Du|)\,dx < \infty$; u fits the boundary data in a generalized sense — i.e. the continuation of u-g in the whole of $\boldsymbol{R}^n$, that is zero outside G, is weakly differentiable. Furthermore, the set of competing functions is invariant under truncations of function values which do not affect anything belonging to the range of the boundary datum.

(iv) $m(G)$, the n-dimensional measure of region G, is finite.

THEOREM 0. *If $A(r)$ grows as $r \to +\infty$ so fast that*

$$\int^{\infty}\left[\frac{r}{A(r)}\right]^{1/(n-1)} dr < \infty \tag{1.0}$$

then any competing function is bounded — irrespective of whether it minimizes or not.

THEOREM 1. *If A grows faster than B so that*

$$\int^{\infty}\frac{dr}{[A^{-1}(B(r))]^{1-1/n}} = \infty \tag{1.1}$$

then any minimizer is bounded.

THEOREM 2. *Assume*

$$\sup_{t>1}\liminf_{r\to+\infty}\frac{ln\dfrac{A(rt)}{A(r)}}{lnt} > 1 \tag{1.2}$$

and call the left hand side p. If

$$\int^{\infty}\frac{dr}{[A^{-1}(B(r))]^{1-k/n}} = \infty \tag{1.3}$$

for some k less than p, then any minimizer is bounded.

REMARKS:

(i) A^{-1} denotes the inverse function of A throughout. In the case A is not strictly increasing, A^{-1} is conveniently defined by:

$$A^{-1}(r) = \sup\{t \geq 0 : A(t) \leq r\}$$

for every nonnegative r — so that A^{-1} is continuous to the right and $A^{-1}(A(r)) \geq r \geq A(A^{-1}(r))$ for every nonnegative r.

B is defined by formula (0.2). More generally, B could be defined as the smallest increasing function such that

$$f(w, 0) - f(u, \xi) + A(|\xi|) - A(\lambda|u|) \leq B(|w|)$$

whenever $|u| \geq |w|$ and ξ is in $\boldsymbol{R}^n$.

(ii) Theorem 0 follows immediately from [Ta2]. A noticeable Young function, that satisfies (1.0) is given by

$$A(r) = r^n(ln_+ r)^{n-1+a},$$

where a is any positive parameter — subscript plus signs stand for positive part, as usual.

(iii) Consider the special example where A and B are powers, i.e.

$$A(r)=r^p \qquad B(r)=r^q$$

If

$$p=1 \qquad 0\leq q\leq n/(n-1),$$

then theorem 1 is appropriate. If

$$p>1,\ q\geq 0,\ p>nq/(n+q),$$

then theorem 2 can be applied. Thus, in both cases any minimizer, that belongs to standard Sobolev space $W^{1,p}(G)$ and is bounded on the boundary, is bounded. This result overlaps with [LU, chap. 5, thm 3.2] and [S, thm 6.2].

(iv) Condition (1.2), theorem 2, is fulfilled if A has regular variation and the index of A is greater than 1. Recall that a real-valued function A, defined in a neighborhood of $+\infty$, is said to vary regularly if $\lim_{r\to\infty} A(rt)/A(r)$ exists, is finite and $\neq 0$ for every t from a set of positive measure; the index of a regularly varying function A is the number p such that

$$\lim_{r\to+\infty}\frac{A(rt)}{A(r)}=t^p$$

for every positive t. A function A has regular variation and index p if, and only if, $A(r)$ can be represented in the following form:

$$A(r)=\exp\left[\int^r a(t)\frac{dt}{t}+b(r)\right]$$

for every sufficiently large r, where

$$\lim_{r\to+\infty} a(r)=p$$

and $b(r)$ has a finite limit as $r\longrightarrow+\infty$. For more information on this matter see [BGT].

The most elementary functions of regular variation are powers, of course. The following are examples of Young functions, that have regular variation:

$$A(r)=r^p(ln_+r)^q \quad (p\geq 1,\ q\geq 1)$$
$$\dots \quad =r^p\exp(\sqrt{1+qln_+r}) \quad (p\geq 1,\ q>0)$$

$$\dots \quad = r^3[1+(lnr)^2]^{-1/2}\exp[lnr \ \arctan(lnr)]$$

$$\dots \quad = \left[\frac{2}{3}\sqrt{1+ln_+r}+\frac{1}{3}\sqrt{1+4ln_+r}\right]^{-3/2}\times$$

$$\exp\left[\sqrt{4(ln_+r)^2+5ln_+r+1}-1\right]-1.$$

(v) A tractable condition, that implies (1.2), is the following:

$$(1.4) \qquad \liminf_{r\to+\infty}\frac{rA'(r)}{A(r)}>1.$$

Indeed

$$(1.5) \qquad \liminf_{r\to+\infty}\frac{rA'(r)}{A(r)}\leq p,$$

the left-hand side of (1.2); for

$$ln\frac{A(rt)}{A(r)}=\int_r^{rt}\frac{A'(s)}{A(s)}\,ds,$$

hence

$$(lnt)^{-1}\,ln\frac{A(rt)}{A(r)}\geq\inf\left\{\frac{sA'(s)}{A(s)}:\ s\geq r\right\}$$

provided $t>1$ and $r>0$. Incidentally, notice that a finite-valued Young function A satisfies: $rA'(r)\geq A(r)$ for every positive r.

The following are examples of Young functions, that satisfy condition (1.4), but do not have regular variation:

$$A(r)=r^{4+\sin(ln\sqrt{1+(lnr)^2})}$$

$$\dots \quad = r^2\exp\left[(ln_+r)^2\left(\frac{1}{\sqrt{2}}+\frac{1}{2}\sin(ln(lnr)^2)\right)\right]$$

$$\dots \quad = r^5\exp\left[2\sqrt{ln_+r}\ \sin(\sqrt{ln_+r})\right].$$

(vi) Theorems 0, 1, 2 merge as follows.

As is easy to see, integrals of the form

$$\int_1^{+\infty}s^{-z}a(s)\,ds$$

share properties with the classical Dirichlet series

$$\sum_{k=1}^{+\infty}\frac{a_k}{k^z}.$$

For instance, suppose $a(r)$ is real or complex-valued and locally integrable in $\left[1, \infty\right[$, $\int^{\infty} a(s)\ ds=\infty$; let

$$\rho=\limsup_{s\to+\infty}\frac{ln\left|\int_1^s a(t)\,dt\right|}{ln\ s}.$$

Then ρ is exactly the abscissa of convergence; i. e. the integral in question converges for any z satisfying $Re(z)>\rho$, does not converge if $Re(z)<\rho$ — parallel statements hold in the case $\rho=\pm\infty$.

Consider

$$\int_1^{\infty} s^{-z}\tilde{A}(s)\frac{ds}{s} \tag{1.6}$$

and let

$$\rho=\limsup_{s\to+\infty}\frac{ln\left|\int_1^s \tilde{A}(t)\frac{dt}{t}\right|}{ln\ s}, \tag{1.7}$$

the relevant abscissa of convergence. Here $\tilde{A}$ denotes the Young conjugate of function A; recall that $\tilde{A}$ is a Young function too and

$$\tilde{A}(s)=\sup\{rs-A(r):\ 0<r<+\infty\} \tag{1.8}$$

for every positive s — see [ON], or [VT] for example. Clearly $\rho\geq 1$, since $\tilde{A}$ grows not less than linearly.

We itemize in the following way.

First case: ρ is small enough, i. e. $\rho<n'$. Here n=dimension and n' stands for $n/(n-1)$—primes are used consistently throughout.

We have

$$\int^{\infty}\frac{\tilde{A}(s)}{s^{n'+1}}\,ds<\infty,$$

by the very definition of abscissa of convergence. According to [Ta2], this property of A ensures that every real-valued function u, that makes $\int_G A(|Du|)\,dx$ converge and is bounded on the boundary of G, is bounded in G. Theorem 0 now follows from Lemma 1 below.

LEMMA 1. *If a Young function A is such that*

$$\int^{\infty}\left[\frac{r}{A(r)}\right]^{q-1} dr<\infty$$

for some q greater than 1, then the Young conjugate $\tilde{A}$ satisfies :

$$\int^{\infty} \frac{\tilde{A}(s)}{s^{q+1}}\, ds < \infty.$$

Thus, theorem 0 is a convenient picture of the case in hand. See [Ta2] for details.

Second case: $\rho = +\infty$. Here the the recipe is theorem 1: thus theorem 1 fits the case where $A(r)$ grows slowly as r grows — just the condition causing ρ to be infinite. Of course, theorem 1 is de facto a recipe when no information on A is available other than convexity.

Note that certainly ρ is infinite if A' — the derivative of A — has a finite limit at $+\infty$. Because if such a limit is l, then $\tilde{A}(s) = +\infty$ for every s greater than l. More generally, ρ must be infinite if

$$\lim_{r \to +\infty} \frac{rA'(r)}{A(r)} = 1.$$

As for the proof, we can suppose $A'(r) \to +\infty$ as $r \to +\infty$; for simplicity, we suppose also that A is strictly convex and continuously differentiable. We have then

$$A(s) = rA'(r) - A(r) \qquad \frac{d\tilde{A}}{ds}(s) = r$$

by (1.8), the very definition of Young conjugate — here r and s are related via the equation:

$$s = A'(r).$$

We may rewrite this way:

$$\left[\frac{s \dfrac{d\tilde{A}}{ds}(s)}{\tilde{A}(s)} - 1 \right] \left[\frac{r \dfrac{dA}{dr}(r)}{A(r)} - 1 \right] = 1,$$

provided r and s are so large that $A(r)$ and $\tilde{A}(s)$ are different from zero. Thus, if $rA'(r)/A(r) \to l$ as $r \to +\infty$, then $sd\tilde{A}(s)/\tilde{A}(s)ds \to l'$ as $s \to +\infty$, where

$$\frac{1}{l} + \frac{1}{l'} = 1.$$

On the other hand, formula (1.7) and Hôpital rule give

$$\rho=\lim_{s\to+\infty}\frac{s}{\tilde{A}(s)}\frac{d\tilde{A}}{ds}(s),$$

provided the limit exists. The conclusion follows.

Apropos examples are the following:

$A(r)=$	$\tilde{A}(s)=$	$A^{-1}(s)=\ldots$ as $s\to+\infty$
rln_+r	s if $0<s<1$, e^{s-1} if $s\geq 1$	$\frac{s}{lns}\left[1+O\left(\frac{ln(lns)}{lns}\right)\right]$
$r(ln_+r)^2$	$2e^{\sqrt{1+s}-1}(\sqrt{1+s}-1)$	$\frac{s}{(lns)^2}\left[1+O\left(\frac{ln(lns)}{lns}\right)\right]$
$r-ln(1+r)$	$ln\frac{1}{1-s}-s$ if $0<s<1$, $+\infty$ if $s\geq 1$	$s+lns+O\left(\frac{lns}{s}\right)$

Third case: ρ is finite — but not too small, i.e. $n'\leq\rho<\infty$. Now our strategy results in theorem 2. Lemma 1 above enables us to test for finite ρ — thus theorem 2 covers cases where suitable information is available about the growth of $A(r)$ as r grows. As a matter of fact, hypothesis (1.2) ensures per se that ρ is finite: we have indeed

$$(1.9)\qquad \rho\leq p'$$

by virtue of lemma 2 below. Notice that hypothesis (1.2) yields case one, i.e. $\rho<n'$, if p is too large, i.e. $p>n$—thus theorem 2 is significant only if $p\leq n$.

LEMMA 2. *Assume condition* (1.2) *is in force and let q be any number such that* $1<q<p$. *Then* (i) *a positive constant exists such that*

$$A(s)\geq \text{Const.}\, A(r)(s/r)^q$$

if $s\geq r$ and r is large enough; (ii) *a large constant exists such that*

$$\tilde{A}(s)\leq \text{Const.}\, \tilde{A}(r)(s/r)^{q'}$$

if $s\geq r$ and r is large enough.

Let us sketch a proof of lemma 2. By inequality (1.2), a number t — greater than 1 — and a positive R exist such that $A(rt)\geq t^qA(r)$ for every r larger than R. Suppose r and s are such that $s\geq r\geq R$. Then $s\geq rt^k$, $A(rt^k)\geq t^{qk}A(r)$ and $t^k>t^{-1}(s/r)$ if k is the integer part of $ln(s/r)/ln\ t$. Hence $A(s)\geq t^{-q}(s/r)^qA(r)$. Assertion (i) is demonstrated. We claim:

$$\text{(iii)}\quad \inf_{t>1}\limsup_{s\to+\infty}\frac{ln\dfrac{\tilde{A}(st)}{\tilde{A}(s)}}{ln\ t}\leq p'.$$

A proof of (iii) parallels that of [BGT, thm 1.8.10] and is omitted here. Property (ii) can be derived from (iii) as property (i) was derived from (1.2).

§ 2 Proofs

The proofs of theorems 1 and 2 start in a standard way. Let u be a minimizer. We have

$$(2.0) \qquad \int_G f(u, Du)\,dx \le \int_G f(v, Dv)\,dx$$

for any competing function v. By virtue of hypotheses, functions v defined by:

$$(2.1) \qquad v(x) = \begin{cases} u(x) & \text{where} -t < u(x) < t \\ t & \text{where } u(x) \ge t \\ -t & \text{where } u(x) \le -t \end{cases}$$

are in competition if interval $]-t, t[$ includes the range of the boundary datum. We deduce

$$(2.2) \qquad \int_{\{x \in G : |u(x)| > t\}} f(u, Du)\,dx \le \int_{\{x \in G : |u(x)| > t\}} f(t\ sgn(u), 0)\,dx,$$

a fortiori

$$(2.3) \qquad \int_{\{x \in G : |u(x)| > t\}} [A(|Du|) - A(\lambda |u|)]\,dx \le \mu(t) B(t)$$

for any t such that

$$(2.4) \qquad t > \sup |g|,$$

the l. u. b. of the boundary datum. Here A and B are related to integrand f by (0.1) and (0.2) respectively;

$$(2.5) \qquad \mu(t) = m\{x \in G : |u(x)| > t\},$$

the distribution of u. In the above derivation we used the chain rule in a version for weakly differentiable functions; see [A, lemma 8.31].

Theorems 1 and 2 are nothing but decoding information comprised in inequalities (2.3) and (2.4). We commence by putting these inequalities in a more convenient form.

Let

$$(2.6) \qquad C_n = \pi^{n/2} / \Gamma(1 + n/2),$$

the measure of the unit ball in $\boldsymbol{R}^n$;

$$(2.7) \qquad \chi_n = nC_n^{1/n},$$

the isoperimetric constant in $\boldsymbol{R}^n$. Lemma 3 below tells us that

$$(2.8) \qquad \int_{\{x\in G:|u(x)|>t\}} \left[A(|Du|) - nA\left(\chi_n \frac{|u|-t}{n\mu(t)^{1/n}}\right)\right] dx \geq 0$$

if t satisfies inequality (2.4) but is less than the essential supremum of $|u|$ — so that the level set $\{x\in G:|u(x)|>t\}$ does not lean on the boundary of G and has a positive measure. On the other hand, the convexity of A gives

$$(2.9) \qquad A(\lambda|u|) \leq \frac{n\lambda}{\chi_n}\mu(t)^{1/n} A\left(\chi_n \frac{|u|-t}{n\mu(t)^{1/n}}\right) + \left(1-\frac{n\lambda}{\chi_n}\mu(t)^{1/n}\right)\times$$
$$A\left(\frac{\lambda t}{1-\frac{n\lambda}{\chi_n}\mu(t)^{1/n}}\right)$$

provided $|u|$ is larger than t and t is so large that

$$(2.10) \qquad \lambda^n\mu(t) < C_n$$

but is smaller than ess sup $|u|$. Combining (2.3), (2.4), (2.8), (2.9) and (2.10) gives

$$(2.11) \qquad \int_{\{x\in G:|u(x)|>t\}} A(|Du|)\,dx \leq \mu(t)\left\{\frac{B(t)}{1-\frac{n\lambda}{\chi_n}\mu(t)^{1/n}} + A\left(\frac{\lambda t}{1-\frac{n\lambda}{\chi_n}\mu(t)^{1/n}}\right)\right\}$$

for any t such that

$$(2.12) \qquad t > L.$$

Here L stands for the greatest lower bound of levels satisfying (2.4) and (2.10). Inequalities (2.11) and (2.12) are the grounds for the next sections.

LEMMA 3. *Let w be a real-valued weakly differentiable function, defined in the whole of euclidean n-dimensional space $\boldsymbol{R}^n$. Assume sprt w, the support of w, has finite measure V and Dw, the gradient of w, is such that $A(|Dw|)$ is integrable over $\boldsymbol{R}^n$. Then the following inequality holds:*

$$\int_{\boldsymbol{R}^n} A\left(\frac{\chi_n}{n}V^{-1n}|w|\right)dx \leq \frac{1}{n}\int_{\boldsymbol{R}^n} A\left(|Dw|\right)dx.$$

Here χ_n is the isoperimetric constant of $\boldsymbol{R}^n$ and A is any Young function.

PROOF. Let w^* be the decreasing rearrangement of w is the sense of Hardy & Littlewood, i. e. the function which is defined in $[0, \infty[$, is non-negative, decreasing and equidistributed with $|w|$. See [K] for example. Clarly sprt $w^* \subseteq [0, V]$, hence

$$w^*(s) = -\int_s^V \frac{dw^*}{dt}(t)\,dt.$$

Then

$$(V^{1/n} - s^{1/n}) A\left(\frac{\chi_n/n}{V^{1/n} - s^{1/n}} w^*(s)\right) \leq \frac{1}{n}\int_s^V A\left(-\chi_n t^{1-1/n} \frac{dw^*}{dt}(t)\right) t^{-1+1/n} dt$$

by Jensen inequality for convex functions. The convexity of A implies' also that the left-hand side of the last inequality is greater than, or equal to

$$V^{1/n} A\left(\frac{\chi_n}{n} V^{-1/n} w^*(s)\right).$$

Here $0 < s < V$. Thus

$$\int_0^V (V/s)^{1/n} A\left(\frac{\chi_n}{n} V^{-1/n} w^*(s)\right) ds \leq \frac{1}{n-1}\int_0^V A\left(-\chi_n s^{1-1/n} \frac{dw^*}{ds}(s)\right) ds.$$

Chebyshev inequality [M, § 2. 5, thm 8] tells us that

$$V\int_0^V a(s)\, b(s)\, ds \geq \int_0^V a(s)\, ds \int_0^V b(s)$$

if a & b are (nonnegative and) both increasing or both decreasing. Consequently,

$$\int_0^V (V/s)^{1/n} A\left(\frac{\chi_n}{n} V^{-1/n} w^*(s)\right) ds \geq \frac{n}{n-1}\int_0^V A\left(\frac{\chi_n}{n} V^{-1/n} w^*(s)\right) ds.$$

We have shown that

$$\int_0^V A\left(\frac{\chi_n}{n} V^{-1/n} w^*(s)\right) ds \leq \frac{1}{n}\int_0^V A\left(-\chi_n s^{1-1/n} \frac{dw^*}{ds}(s)\right) ds.$$

Now

$$\int_0^V A\left(\frac{\chi_n}{n} V^{-1/n} w^*(s)\right) ds = \int_{R^n} A\left(\frac{\chi_n}{n} V^{-1/n} |w|\right) dx,$$

since $|w|$ and w^* are equidistributed; and

$$\int_0^V A\left(-\chi_n s^{1-1/n} \frac{dw^*}{ds}(s)\right)ds = \int_{R^n} A(|Dw^\star|)\,dx,$$

where $w^\star$ is the symmetric rearrangement of w. Recall that

$$w^\star(x) = w^*(C_n |x|^n)$$

and $C_n = (\chi_n/n)^n$, the measure of the unit ball in $\boldsymbol{R}^n$. Pôlya & Szegö principle — or a generalization of it, see [BZ] for instance — yields

$$\int_{R^n} A(|Dw|)\,dx \geq \int_{R^n} A(|Dw^\star|)\,dx.$$

The proof is complete.

§ 3 Proof of theorem 1, continued

We claim

$$\int_{\{x\in G: |u(x)|>t\}} |Du|\,dx \leq \frac{\mu(t)}{1-\frac{n\lambda}{\chi_n}\mu(t)^{1/n}}[A^{-1}(B(t))+\lambda t] \tag{3.0}$$

for any t satisfying (2.12). In fact, inequality (3.0) is trivially true if $t \geq \text{ess sup}\,|u|$. If $\text{ess sup}\,|u| > t > L$, (2.11) and Jensen inequality for convex functions give

$$A\left(\frac{1}{\mu(t)}\int_{\{x\in G: |u(x)|>t\}} |Du|\,dx\right) \leq \frac{B(t)}{1-\frac{n\lambda}{\chi_n}\mu(t)^{1/n}} + A\left(\frac{\lambda t}{1-\frac{n\lambda}{\chi_n}\mu(t)^{1/n}}\right).$$

Now A^{-1}, the inverse function of A, is concave and takes a nonnegative value at 0: thus $A^{-1}(kr) \leq kA^{-1}(r)$ whenever $k \geq 1$ and A^{-1} is subadditive — i.e. $A^{-1}(r+s) \leq A^{-1}(r) + A^{-1}(s)$. Hence (3.0) follows.

Notice that

$$\int_{\{x\in G: |u(x)|>t\}} |Du|\,dx \geq \chi_n \int_t^\infty \mu(t')^{1-1/n}\,dt' \tag{3.1}$$

for any t satisfying (2.4). Inequality (3.1) is an easy consequence of coarea formula [FR] and isoperimetric inequality [DG], as well as a special case of lemma 4, section 4.

Inequalities (3.0) and (3.1) yield

$$\chi_n \int_t^\infty \mu(t')^{1-1/n}\,dt' \leq \frac{\mu(t)}{1-\frac{n\lambda}{\chi_n}\mu(t)^{1/n}}[A^{-1}(B(t))+\lambda t] \tag{3.2}$$

for any t satisfying (2.12).

Inequality (3.2) can be rewritten in this way:

$$\frac{\chi_n}{n}\left[\frac{1-\frac{n\lambda}{\chi_n}\mu(t)^{1/n}}{A^{-1}(B(t))+\lambda t}\right]^{1-1/n} \leq -\frac{d}{dt}\left[\chi_n\int_t^\infty \mu(t')^{1-1/n}dt'\right]^{1/n}. \tag{3.3}$$

Integrating both sides of (3.3) between L and ess sup $|u|$ gives

$$\frac{\chi_n}{n}\int_L^{\text{ess sup}|u|}\left[\frac{1-\frac{n\lambda}{\chi_n}\mu(t)^{1/n}}{A^{-1}(B)t))+\lambda t}\right]^{1-1/n} dt \leq \left[\chi_n\int_L^\infty \mu(t)^{1-1/n}dt\right]^{1/n}. \tag{3.4}$$

Inequalities (3.2) and (3.4) yield

$$\left[\frac{\chi_n}{n}-\lambda\mu(L)^{1/n}\right]\int_L^{\text{ess sup}|u|}\frac{dt}{[A^{-1}(B(t))+\lambda t]^{1-1/n}} \leq \tag{3.5}$$
$$m(G)^{1/n}[A^{-1}(B(L)+\lambda L]^{1/n}.$$

Hypothesis (1.1) and lemma Al tell us that

$$\int_L^\infty \frac{dt}{[A^{-1}(B(t))+\lambda t]^{1-1/n}}=\infty. \tag{3.6}$$

Clearly, (3.5) and (3.6) imply

$$\text{ess sup } |u|<\infty, \tag{3.7}$$

Q. E. D.

Note incidentally that $L=\sup|g|$ in the case $\lambda=0$. Because condition (2.10) is empty in such a case. Thus, if $\lambda=0$ inequality (3.5) reads

$$\int_{\sup|g|}^{\text{ess sup}|u|}\frac{dt}{[A^{-1}(B(t))]^{1-1/n}} \leq \left[\frac{1}{C_n}m(G)A^{-1}(B(\sup|g|))\right]^{1/n},$$

an actual a priori bound for the essential supremum of $|u|$.

§ 4 Proof of theorem 2, continued

So far the only property of A, that has played a role, is convexity. Now we assume an extra hypothesis: for the time being, let us suppose ρ is finite — as in section 1, remark (vi), ρ stands for the abscissa of convergence of integral (1.6).

Let E_k be the Young function whose conjugate obeys:

$$\tilde{E}_k(r)=k'\int_r^\infty \tilde{A}(t)\left(\frac{r}{t}\right)^{k'}\frac{dt}{t} \tag{4.0a}$$

for every nonnegative r. Here $k'=k/(k-1)$; k—a parameter to be specified later — is larger than 1 but so small that $k'>\rho$, i. e.

(4.1) $\quad 1<k<\rho'$.

The following properties hold:
(i) $\widetilde{E}_k$ is nonnegative, increasing, convex and vanishes at 0 — thus is actually a Young function.
(ii) $\widetilde{E}_k(r)\geq\widetilde{A}(kr)$ — hence $E_k(r)\leq A(r/k)$.
(iii) $\widetilde{E}_k(r)\longrightarrow\widetilde{A}(r)$ as $k\longrightarrow 1$.
(iv) $\widetilde{E}_h(r)\geq\widetilde{E}_k(r)$ if $1<h<k$; more precisely,

$$\widetilde{E}_k(r)-\widetilde{E}_h(r)\geq\frac{k-h}{h(k-1)}[\widetilde{E}_h(kr)-\widetilde{E}_h(r)].$$

Note the alternative formulas:

$$\widetilde{E}_k(r)=\int_1^\infty\widetilde{A}(rt)\frac{k'}{t^{1+k'}}\,dt \tag{4.0b}$$

$$=\int_0^1\widetilde{A}(rt^{-1+1/k})\,dt, \tag{4.0c}$$

which are easily derived from (4.0a) via changes of variables. Property (i) follows at once from (4.0b) or (4.0c). Property (ii) follows from (4.0a) and Jensen inequality for convex functions. Formula (4.0a) tells us that $\widetilde{E}_k$ is the convolution of $\widetilde{A}$ against a kernel — the set of positive real numbers and dt/t being the relevant topological group and Haar measure. Such a kernel obeys

$$J_k(r)\geq 0,\ \int_0^\infty J_k(r)\frac{dr}{r}=1$$

for every k greater than 1, and

$$\left(\int_0^{1/\delta}+\int_\delta^\infty\right)J_k(r)\frac{dr}{r}\longrightarrow 0 \text{ as } k\longrightarrow 1$$

whenever $\delta>1$ — in other words, we have an approximation of Dirac mass. Property (iii) follows. By the way, the convolution in hand is nothing but the inverse of the differential operator $1-(r/k')d/dr$ coupled with an appropriate boundary condition. In other words, $\widetilde{E}_k$ *is characterized by*:

$$\left(\frac{r}{k'}\frac{d}{dr}-1\right)\widetilde{E}_k(r)+\widetilde{A}(r)=0,$$
$$r^{-k'}\widetilde{E}_k(r)\longrightarrow 0 \text{ as } r\longrightarrow\infty.$$

Formulas (4.0) yield

$$\tilde{E}_k(r)-\tilde{E}_h(r)=\frac{k-h}{h(k-1)}\left[k'\int_r^\infty \tilde{E}_h(t)\left(\frac{r}{t}\right)^{k'}\frac{dt}{t}-\tilde{E}_h(r)\right],$$

an equation reflecting algebraic properties of the family of operators $k \longrightarrow 1-(r/k')d/dr$ and their inverses. Property (iv) follows.

We claim that

$$(4.2)\qquad E_k\left(\chi_n\mu(t)^{-1/k}\int_t^\infty \mu(s)^{1/k-1/n}ds\right)\le\frac{1}{\mu(t)}\int_{\{x\in G:|u(x)|>t\}}A(|Du|)\,dx$$

if t satisfies inequality (2.4) but is less than ess sup $|u|$. Indeed, Young's inequality — $ab\le A(a)+\tilde{A}(b)$ — gives

$$r\left(\chi_n\mu(t)^{-1k}\int_t^\infty \mu(s)^{1/k-1/n}ds\right)\le$$
$$\frac{1}{\mu(t)}\int_t^\infty \tilde{A}\left(r\frac{\mu(t)^{1-1/k}}{\mu(s)^{1-1/k}}\right)[-\mu'(s)]ds+$$
$$\frac{1}{\mu(t)}\int_t^\infty A\left(\chi_n\frac{\mu(s)^{1-1/n}}{-\mu'(s)}\right)[-\mu'(s)]ds.$$

The middle term above

$$\le\frac{1}{\mu(t)}\int_t^\infty \tilde{A}\left(r\frac{\mu(t)^{1-1/k}}{\mu(s)^{1-1/k}}\right)[-d\mu(s)],$$

since distribution function μ decreases monotonically. The last quantity

$$=\tilde{E}_k(r),$$

as formulas (4.0) show. Here r is any positive number. Therefore

$$E_k\left(\chi_n\mu(t)^{-1/k}\int_t^\infty \mu(s)^{1/k-1/n}ds\right)\le\frac{1}{\mu(t)}\int_t^\infty A\left(\chi_n\frac{\mu(s)^{1-1/n}}{-\mu'(s)}\right)[-\mu'(s)]ds,$$

by the very definition of Young conjugate. Inequality (4.2) follows via lemma 4 below.

Inequalities (2.11) and (4.2) give

$$(4.3)\qquad E_k\left(\chi_n\mu(t)^{-1/k}\int_t^\infty \mu(s)^{1/k-1/n}ds\right)\le\frac{B(t)}{1-\frac{n\mu}{\chi_n}\mu(t)^{1/n}}+$$
$$A\left(\frac{\lambda t}{1-\frac{n\lambda}{\chi_n}\mu(t)^{1/n}}\right)$$

for any t satisfying inequality (2.12).

Denote the right-hand side of (4.3) by $R(t)$. Then we have

$$(4.4) \qquad \chi_n \mu(t)^{-1/k} \int_t^\infty \mu(s)^{1/k-1/n} ds \leq E_k^{-1}(R(t)).$$

Consequently,

$$(4.5) \qquad [E_k^{-1}(R(t))]^{-1+k/n} \leq -\frac{n}{k\chi_n} \frac{d}{dt}\left[\chi_n \int_t^\infty \mu(s)^{1/k-1/n} ds\right]^{k/n},$$

provided (2.12) holds and

$$k \leq n.$$

Integrating both sides of (4.5), then estimating the resulting right-hand side via (4.4) gives

$$(4.7) \qquad \int_L^{\text{ess sup}|u|} \frac{dt}{[E_k^{-1}(R(t))]^{1-k/n}} \leq \frac{n}{k\chi_n} m(G)^{1/n} [E_k^{-1}(R(L))]^{k/n}.$$

The right-hand side of((4.7) is a finite quantity for every k satisfying inequality (4.1) — recall that L was defined as the g.l.b. of levels t satisfying inequalities (2.4) and (2.10). Then (4.7) implies

$$(4.8) \qquad \text{ess sup } |u| < \infty,$$

the goal, if we are able to show that

$$(4.9) \qquad \int^\infty \frac{dt}{[E_k^{-1}(R(t))]^{1-k/n}} = \infty$$

for some k satisfying (4.1) and (4.6).

Actually, (4.9) is a consequence of (1.3). Note however that properties (ii) (iii) (iv) of E_k go in the wrong direction: this is the reason why hypothesis (1.2) comes in. Call therefore (1.2) into play and let

$$(4.10) \qquad 1 < k < p.$$

Lemma 2, section 1, tells us that abscissa of convergence $\rho \leq p'$. Then (4.1) is true. As already pointed out in remark (vi), section 1, we can assume $p \leq n$ without loss of generality. Then (4.6) is true too. Importantly, lemma 2 gives also a reverse of property (ii), namely:

$$(4.11) \qquad \tilde{E}_k(r) = O(\tilde{A}(r)) \text{ as } r \longrightarrow \infty.$$

Consequently,

$$(4.12) \qquad E_k^{-1}(r) = O(A^{-1}(r)) \text{ as } r \longrightarrow \infty.$$

It is now clear that (1.3) and (4.12) yield (4.9). The derivation is immediate in the case λ is zero — when $R(t)$ is the same as $B(t)$ — otherwise involves lemma A1 — details are as in section 3.

The proof is complete.

LEMMA 4. *Let w be a real-valued weakly differentiable function, defined in $\boldsymbol{R}^n$. Assume the support of w has a finite measure and $A(|Du|)$ is integrable over $\boldsymbol{R}^n$. Then*

$$\int_{\boldsymbol{R}^n} A(|Dw|)\,dx \geq \int_0^\infty A\left(\chi_n \frac{\mu(t)^{1-1/n}}{-\mu'(t)}\right)(-\mu'(t))\,dt.$$

Here μ is the distribution function of w; χ_n is the isoperimetric constant of $\boldsymbol{R}^n$; A is any Young function.

PROOF. Let t be positive. Jensen inequality for convex functions yields

$$\frac{1}{h}\int_{\{x\in\boldsymbol{R}^n:\, t<|w(x)|\leq t+h\}} A(|Dw|)\,dx \geq$$

$$A\left(\frac{\int_{\{x\in\boldsymbol{R}^n:\, t<|w(x)|\leq t+h\}} |Dw|\,dx}{\mu(t)-\mu(t+h)}\right)\frac{\mu(t)-\mu(t+h)}{h}.$$

Fleming-Rishel coarea formula [FR] tells us that

$$\int_{\{x\in\boldsymbol{R}^n:\, t<|w(x)|\leq t+h\}} |Dw|\,dx$$

equals

$$\int_t^{t+h} \text{perimeter of } \{x\in\boldsymbol{R}^n : |w(x)|>t'\}\,dt',$$

therefore is greater than (or equal to)

$$\chi_n \int_t^{t+h} \mu(t')^{1-1/n}\,dt',$$

thanks to the isoperimetric inequality [DG]. In the above formulas h is any positive parameter.

Letting h tend to zero gives

$$-\frac{d}{dt}\int_{\{x\in\boldsymbol{R}^n:\, |w(x)|>t\}} A(|Dw|)\,dx \geq A\left(\chi_n \frac{\mu(t)^{1-1/n}}{-\mu'(t)}\right)(-\mu'(t))$$

for almost every positive t. Indeed the involved functions are decreasing — hence almost everywhere differentiable.

In particular, we have shown that $-\mu'(t)>0$ for almost every positive t, at least in the case A grows more than linearly — i. e. $A(r)/r \longrightarrow \infty$ for $r \longrightarrow \infty$.

Recall that $\varphi(0)-\varphi(\infty)\geq -\int_0^\infty \varphi'(t)\,dt$, if φ is a decreasing function. Thus, integrating both sides of the above inequality concludes the proof.

APPENDIX

LEMMA A1. *Let w be an increasing strictly positive function of a real variable t, defined in a neighborhood of $+\infty$. Let a be nonnegative and large enough; then*

$$\int_a^\infty \frac{dt}{w(t)+t} \geq ln\left(1+\frac{\int_a^\infty \frac{dt}{w(t)}}{1+\frac{a}{w(a)}}\right).$$

In particular, if $\int^\infty \frac{dt}{w(t)}$ *diverges, then* $\int^\infty \frac{dt}{w(t)+t}$ *diverges too.*

PROOF. Replace w by $1/w$ and use lemma A2 below.

LEMMA A2. *Assume w is positive and decreasing in $[a, b[$; assume $a \geq 0$. Then*

$$\text{(A 1)} \qquad \int_a^b \frac{w(t)}{1+tw(t)}\,dt \geq ln\left(1+\frac{\int_a^b w(t)\,dt}{1+aw(a)}\right).$$

Inequality (A1) is sharp: equality holds in (A1) if w is constant near point a and vanishes elsewhere.

PROOF. The proof consists in showing that the variational problem:

$$\text{(A 2)} \qquad \begin{cases} \int_a^b \frac{w(t)}{1+tw(t)}dt = \text{minimum}, \\ \text{under the conditions:} \\ w \text{ is positive and decreasing,} \\ \int_a^b w(t)\,dt = A, \\ w(t) \leq B \text{ for } a \leq t < b \end{cases}$$

has the following solution:

$$\text{(A 1)} \qquad w(t) = \begin{cases} B & \text{if } a<t<a+A/B \\ 0 & \text{otherwise.} \end{cases}$$

Here A and B are positive constants, such that $B \geq A/(b-a)$.

Let us discretize this problem, i. e. restrict the competing functions to run in a class of piecewise constant functions. Thus the problem becomes to render

$$\text{(A 4)} \qquad \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \frac{w_i}{1+tw_i} dt$$

a minimum under the following constraints:

$$\text{(A 5)} \qquad \begin{array}{l} (t_1-t_0)w_1+(t_2-t_1)w_2+\ldots+(t_N-t_{N-1})w_N=A, \\ B \geq w_1 \geq w_2 \geq \ldots \geq w_N \geq 0. \end{array}$$

Here $w_1, w_2, \ldots, w_N$ are real — the competing variables; knots $t_0, t_1, \ldots, t_N$ are given by

$$t_i=a+i\Delta t,$$

mesh size Δt and number N are given by

$$\Delta t=A/(hB),$$
$$N = \text{integer part of } (b-a)/\Delta t,$$

h is a large integer — the discretization parameter. Note that $a=t_0$, $b=t_N+O(1/h)$ as $h \longrightarrow +\infty$ and

$$t_h, \text{ knot no. } h, =a+A/B.$$

Two remarks are basic at this stage. First, (A4) is a concave function of $w_1, w_2, \ldots, w_N$. Indeed, its second order derivatives are

$$-2\delta_{ij}\int_{t_{i-1}}^{t_i} t(1+tw_i)^{-3}dt.$$

Second, the set of points $(w_1, w_2, \ldots, w_N)$ satisfying (A5) is a simplex.

Thus the minimum in question is attained at one of the vertices. Note that the vertices of (A5) are just the $(N-k+1)$ points in $\boldsymbol{R}^N$ whose coordinates are displayed in the following tables:

index	1	…	i	$i+1$	…	N
value	$A/(t_i-t_0)$	…	$A/(t_i-t_0)$	0	…	0

$$i=h, \ldots, N.$$

The values of (A2) at the vertices above are

$$ln\left(1+\frac{A}{1+aA/(t_i-t_0)}\right) \quad (i=h, \dots, N),$$

an increasing sequence.

Then the sought after minimum and the sought after minimizer respectively are

$$ln\left(1+\frac{A}{1+aB}\right)$$

and the point whose components obey

$$w_1= \dots =w_h=B,\ w_{h+1}= \dots =w_N=0.$$

The lemma follows, owing to an obvious approximation argument.

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