Hokkaido Mathematical Journal Vol. 19 (1990) p. 229-239

Discrete Lyapunov inequality conditions for partial difference equations

Sui-Sun CHENG, Liang Yu HSIEH and Dale Z. T. CHAO
(Received January 17, 1989, Revised March 20, 1989)

1. Introduction.

A classical result of Lyapunov [11] states that if $x(t)$ is a nontrivial solution of the differential system

$$x''(t)+p(t)x(t)=0, \quad a\leq t\leq b$$
$$x(a)=0=x(b)$$

where $p(t)$ is a continuous and nonnegative function defined in $[a, b]$, then

$$(b-a)\int_{a}^{b}p(s)ds>4,$$

and the constant 4 cannot be replaced by a larger number. Generalizations and/or analogous results [1-3, 5, 7, 8, 10, 13, 16, 17] have since then be obtained for various differential and/or discrete systems. Here we are concerned with the following partial difference system

$$(1.1) \qquad \Delta_1^2 u(i-1, j)+\Delta_2^2 u(i, j-1)+p(i, j)u(i, j)=0, \quad (i, j)\in S$$
$$(1.2) \qquad u(i, j)=0, \quad (i, j)\in\partial S$$

where S is a net with exterior boundary ∂S, $p(i, j)\geq 0$ for $(i, j)\in S$, Δ_1 is defined by $\Delta_1 h(i, j)=h(i+1, j)-h(i, j)$ and Δ_2 by $\Delta_2 h(i, j)=h(i, j+1)-h(i, j)$. We shall find Lyapunov type conditions of the form

$$(1.3) \qquad \sum_{(i,j)\in S}p(i, j)\geq\mu(S)$$

which are necessary for the above system to have a nontrivial solution.

The precise definitions of a net S and its exterior boundary ∂S together with others will be given in section two. There we shall also derive and/or quote some preparatory results to be used in the succeeding sections. Among these results are discrete maximum principle, discrete Green's identity and others which are related to the geometrical aspects of nets. In section three, we shall define the concept of a Green's function and derive a general condition of the form (1.3) in terms of the maximum value of this function. In contrast to the simple integral interval of an ordinary difference system such as that discussed by Cheng [3], the domains of definition of partial difference system can be very complicated. We are able, however, to investigate three types of nets in sections four, five and six, and derive the corresponding constants $\mu(S)$.

Discrete systems of the form (1.1)—(1.2) arise in physical problems including random walk and diffusion problems in lattices [19], electrical potential problems of various metallic nets [14], etc. Our results will therefore find applications in such problems.

2. Nets and discrete harmonic functions

The lattice points of the x, y-plane have coordinates (i, j) where i and j take on integral values. Two lattice points are neighbors if their Euclidean distance is one. The lattice points $z_1, z_2, \cdots, z_n$ are said to form a path with terminals z_1 and z_n if z_1 is a neighbor of z_2, z_2 is a neighbor of z_3, etc. A set of lattice points is said to be connected if any two of its points are terminals of a path contained in the set. A component of a set S of lattice points is a nonempty maximal connected subset of S. A connected and finite but nonempty set of lattice points is called a *net.* Let S be a net, a lattice point is an exterior boundary point of S if it does not belong to S but has at least one neighbor in S. The set of all exterior boundary points of S is denoted by ∂S. The degree of a point in a net S is the number of its neighbors in S.

The following is obvious.

LEMMA 2.1. *For any two components of a finite set S of lattice points, none of the exterior boundary points of one component can be a member of the other.*

THEOREM 2.2. *Suppose S is a net. For any $x \in S$, S-$\{x\}$ contains at most four components. Furthermore, the exterior boundary of any component of S-$\{x\}$ is contained in $\partial S \cup \{x\}$.*

PROOF. Let $N(x)$ be the set of neighbors of x. For each y in $N(x)$, let $C(y)$ be a component of S-$\{x\}$ containing y. Since x has at most four neighbors, there are at most four such components. Consider the set

$$J=\{x\} \cup \sum_{y \in N(x)} C(y).$$

We need to show that $J=S$. The fact that $J \subseteq S$ is clear. To see the converse, let $z \in S$ such that $z \neq x$. Since S is connected, there is a path z, $x_1, x_2, \cdots, x_m, x$ contained in S. Since x_m and x are neighbors, $x_m \in N(x)$, so that z belongs to $C(x_m)$. This implies S is a subset of J. Next we note that $x \in \partial C(y)$ for any $y \in N(x)$. Consequently, if u is an exterior boundary point of some component $C(y)$, then u is either equal to x, or u is an exterior boundary point of S. Indeed, if $u \neq x$, then $u \notin \sum_{y \in N(x)} C(y)$ by Lemma 2.1. Thus $u \in \partial S$ since u is an exterior boundary point of $C(y) \subseteq S$ and $u \notin J = S$. Q. E. D.

COROLLARY 2.3. *Suppose S is a net, Let $x \in S$. Then for any component C of S-$\{x\}$, $\partial C \cap \partial S$ is not empty.*

PROOF. If $\partial C \cap \partial S$ is empty, then by Theorem 2.2, ∂C is a subset of $\{x\}$. This is impossible since any finite nonempty set of lattice points (in particular, ∂C) has at least four exterior boundary points. Q. E. D.

In the rest of our discussions, S shall denote a net. A real function $f(i, j)$ defined in $S \cup \partial S$ is said to be discrete harmonic or preharmonic [6] on S if

$$f(i, j) = \frac{1}{4}\{f(i+1, j) + f(i-1, j) + f(i, j+1) + f(i, j-1)\}$$

for all $(i, j) \in S$. Clearly, f is discrete harmonic if and only if

$$Df(i, j) = 0, \quad (i, j) \in S$$

where the discrete Laplacian D (see [6, 9]) is defined by

$$Df(i, j) = f(i+1, j) + f(i-1, j) + f(i, j+1) + f(i, j-1) - 4f(i, j)$$

LEMMA 2.4. *If $f(i, j)$ is discrete harmonic on S, then f is either a constant or it attains its maximum over $S \cup \partial S$ on the boundary only.*

The above Lemma is due to Heilbronn [9, Theorem 1] and is a discrete analog of the maximum principles for harmonic functions. More general discrete maximum principles can be found in Cheng [4].

LEMMA 2.5. (Duffin [6, Lemma 1] *If $f(i, j)$ are real functions on $S \cup \partial S$ such that $f(i, j) = g(i, j) = 0$ for $(i, j) \in \partial S$, then*

$$\sum_S \{f(i, j) Dg(i, j) - g(i, j) Df(i, j)\} = 0.$$

3. Green's functions and Lyapunov's inequalities

By means of the discrete Laplacian defined in Section two, the system (1, 1)—(1. 2) can be written in the form

(3.1) $\quad Du(z) + p(z) u(z) = 0, \quad z \in S$

(3.2) $u(z)=0,\ z\in\partial S.$

As is well known in the theory of differential equations, the concept of a Green's function plays an important role in dealing with differential systems. Here we have an analogous concept for our discrete system. A Green's functin $G(z|w)$ associated with (3.1)—(3.2) is defined as the solution of the system

(3.3) $DG(z|w)=-\delta(z|w),\ z\in S$

(3.4) $G(z|w)=0,\ z\in\partial S$

where w is some fixed but arbitrary point of S and $\delta(z|w)$ is the Dirac delta function defined by $\delta(z|w)=1$ if $w=z$ and $\delta(z|w)=0$ if $w\neq z$.

We can easily show that $G(z|w)$ exists and is nonnegative by iteration (see Duffin [6, p. 242]). Then by means of the maximum principle, the positiveness and uniqueness of $G(z|w)$ can be established by standard arguments (see Protter and Weinberger [15]). Next, by means of the Green's identity (Lemma 2.5) and the substitution property of the Dirac delta function, it is easily shown (see Roach [6, Theorem 9.2]) that $G(z|w)=G(w|z)$ for $w,\ z\in S$.

By means of the Green's function $G(w|z)$, the system (3.1)—(3.2) is equivalent to the following equation

(3.5) $$u(z)=\sum_{w\in S}G(z|w)p(w)u(w),\ z\in S.$$

The proof is standard and is thus omitted. Suppose $u(z)$ is a nontrivial solution of (3.5). Let z^* be a lattice point in S such that $|u(z^*)|=\max_{z\in S}|u(z)|$. By (3.5),

$$|u(z^*)|\leq\sum_{w\in S}G(z^*|w)p(w)|u(z^*)|$$
$$\leq|u(z^*)|\sum_{w\in S}G(z^*|w)p(w)\leq|u(z^*)|\max_{w,z\in S}G(z|w)\sum_{w\in S}p(w),$$

so that

(3.6) $$\sum_{w\in S}p(w)\geq\frac{1}{\max_{w,z\in S}G(z|w)}.$$

LEMMA 3.1. *For any* $z\in S$, $\max_{w\in S\text{-}\{z\}}G(w|z)<G(z|z)$. *In particular,* $\max_{w,z\in S}G(z|w)=\max_{z\in S}G(z|z)$.

PROOF. Let C be an arbitrary component of $S\text{-}\{z\}$ containing a neighbor of z. Since $G(w|z)$ is discrete harmonic on C and $G(w|z)>0$ for $w\in S$, and since $G(w|z)=0$ for $w\in\partial C\cap\partial S$ (which is nonempty by

Corollary 2.3), hence $G(w|z)$ cannot be a constant function over $C\cup\partial C$. According to the maximum principle and Theorem 2.3,

$$\max_{w\in S} G(w|z) < \max_{w\in\partial C} G(w|z) \leq \max_{w\in\partial S\cup\{z\}} G(w|z) = G(z|z)$$

as desired. Q.E.D.

The condition (3.6) is sharp in the sense that we can find a non-negative function $p(z)$ on S and a nontrivial solution of (3.1)—(3.2) such that

$$\sum_{w\in S} p(w) = \frac{1}{\max\limits_{w,z\in S} G(w|z)} = \frac{1}{\max\limits_{z\in S} G(z|z)}.$$

Indeed, let z^* be a point in S such that

$$\max_{z\in S} G(z|z) = G(z^*|z^*).$$

Let $u(z)=G(z|z^*)$ for $z\in S\cup\partial S$ and let $p(z)=-Du(z)/u(z)$ for $z\in S$. Then (3.1)—(3.2) is clearly satisfied and

$$\sum_{z\in S} p(z) = \sum_{z\in S} -\frac{Du(z)}{u(z)} = \sum_{z\in S} \frac{\delta(z|z^*)}{G(z|z^*)} = \frac{1}{G(z^*|z^*)}$$

as required.

We summarize the above discussions as follows.

THEOREM 3.2. Suppose $p(z)$ is a nonnegative function defined on a net S. If (3.1)—(3.2) has a nontrivial solution then

$$(3.7)\qquad \sum_{z\in S} p(z) \geq \mu(S),$$

where $\mu(S)=\{\max\limits_{w,z\in S} G(z|w)\}^{-1}=\{\max\limits_{z\in S} G(z|z)\}^{-1}$, and the inequality is sharp.

Before we turn to the estimation of the constant $\mu(S)$ for various nets, note that the following comparison theorem for the Green's function holds.

THEOREM 3.3. *Let $z_0\in\partial S$ and let $S'=S\cup\{z_0\}$. Let $G'(z|w)$ be the Green's function of the system*

$$DG'(z|w) = -\delta(z|w),\quad z\in S'$$
$$G'(z|w)=0,\quad z\in\partial S'.$$

Then $G(z|w)<G'(z|w)$ for all $z\in S$.

PROOF. Let $h(z)=G(z|w)-G'(z|w)$ for $z\in S\cup\partial S$. Then $Dh(z)=0$ for $z\in S$ and $h(z)=-G'(z|w)$ for $z\in\partial S$. Since $G'(z_0|w)>0$ and since

$G'(z|w)=0$ for any $z\in\partial S\cap\partial S'$, thus $h(z)$ cannot be a constant function over $S\cup\partial S$. By the maximum principle, we have $\max_S h<\max_{\partial S} h\leq 0$ as required. Q. E. D.

4. Maxima of Green's functions on straight nets

A net S said to be straight if there are exactly two points in S with degree 1 and the remaining points in S with degree 2. An example of a straight net is the set $\{(2,2), (2,3), (2,4), (3,4), (4,4), (4,5)\}$. Suppose a straight net S has N points, then it is not difficult to see that these points can be ordered as a chain $z_1, z_2, \cdots, z_N$ such that z_i and z_j are neighbors if and only if $|i-j|=1$. As a consequence, (3.1)—(3.2) can be written as

$$(4.1)\qquad Du(z_i)+p(z_i)u(z_i)=0,\ 1\leq i\leq N$$

$$(4.2)\qquad u(z)=0,\ z\in\partial S$$

and (3.3)—(3.4) can be written as

$$(4.3)\qquad DG(z_i|z_j)=-\delta(z_i|z_j)\ 1\leq i\leq N$$

$$(4.4)\qquad G(z|z_j)=0,\ z\in\partial S,$$

where z_j is some point of S. For convenience's sake, we shall write $G(i, j)$ instead of $G(z_i|z_j)$. We shall find an explicit formula for $G(i, j)$. In order to do this, let $X_{-1}=0$, $X_0=1$ and X_k be defined by the recurrence relation

$$(4.5)\qquad X_k=4X_{k-1}-X_{k-2},$$

for $k=2, 3, \cdots$. We can verify that

$$(4.6)\qquad X_k=\frac{1}{2\sqrt{3}}\,(\alpha^{k+1}-\alpha^{-(k+1)}),\ \alpha=2+\sqrt{3}$$

for $k\geq 0$. By means of (4.6), it is easy to check that x_k is positive and increasing. Also

$$G(i, j)=\begin{cases}X_{i-1}X_{N-j}X_N^{-1} & 1\leq i\leq j\\ X_{N-i}X_{j-1}X_N^{-1} & j\leq i\leq N\end{cases}$$

and

$$\max_{1\leq i\leq N} G(i, j)=\begin{cases}X_{m-1}X_mX_N^{-1} & N=2m\\ X_m^2X_N^{-1} & N=2m+1\end{cases}.$$

The proof of these assertions amounts to direct verification which is straightforward. In view of Theorem 3.2 and the above discussions, we have the following

THEOREM 5. 1. *Let S be a circular net with N points. If* (4. 1)—(4. 2) *has a nontrivial solution, then* (1. 3) *holds where*

$$\mu(S)=\begin{cases} X_N X_{m-1}^{-1} X_m^{-1}, & N=2m \\ X_N X_m^{-2}, & N=2m+1 \end{cases}$$

and X_k is given by (4. 6). *The inequality is sharp.*

The function $\mu(S)$ is a decreasing but bounded function of the size of S as can be seen from the following

THEOREM 4. 2. *Let $M_N=\max_{1\le i\le N} G(i, i)$. Then $\{M_N\}_{N=1}^{\infty}$ is an increasing sequence and has the limit $\sqrt{3}/6$.*

PROOF. We have

$$M_{2k+1}=X_{2k+1}^{-1}X_k^2=\frac{(2\sqrt{3})^{-2}(\alpha^{k+1}-\alpha^{-k-1})^2}{(2\sqrt{3})^{-1}(\alpha^{2k+2}-\alpha^{-2k-2})}$$
$$=\frac{1-2\alpha^{-2k-2}+\alpha^{-4k-4}}{2\sqrt{3}(1-\alpha^{-4k-4})}\to\frac{\sqrt{3}}{6}.$$

Similarly, $M_{2k}\to\sqrt{3}/6$. Next

$$\begin{aligned}
M_{2k+2}-M_{2k+1}&=X_{2k+2}^{-1}X_{k+1}X_k-X_{2k+1}^{-1}X_kX_k\\
&=X_{2k+2}^{-1}X_{2k+1}^{-1}X_k(X_{2k+1}X_{k+1}-X_{2k+2}X_k)\\
&=X_{2k+2}^{-1}X_{2k+1}^{-1}X_k\{X_{2k+1}(4X_k-X_{k-1})-(4X_{2k+1}-X_{2k})X_k\}\\
&=X_{2k+2}^{-1}X_{2k+1}^{-1}X_k(X_{2k}X_k-X_{2k+1}X_{k-1})\\
&=\cdots\\
&=X_{2k+2}^{-1}X_{2k+1}^{-1}X_k(X_{k+1}X_1-X_{k+2}X_0)\\
&=X_{2k+2}^{-1}X_{2k+1}^{-1}X_k^2\\
&>0.
\end{aligned}$$

Similarly, $M_{2k+1}-M_{2k}>0$. The proof is complete.

5. Maxima of Green's functions on circular nets

A net S is said to be circular if every one of its points has degree 2. An example of a circular net is the set $\{(1,1), (1,2), (1,3), (1,4), (2,4), (3,4), (3,3), (4,3), (4,2), (4,1), (3,1), (2,1)\}$. Suppose a circular net S has N points, then it is not difficult to see that these points can be ordered as a "closed" chain $z_1, z_2, \cdots, z_N$ such that z_1 has only two neighbors z_N and z_2, z_k has only tow neighbors z_{k-1} and z_{k+1} where $k=2, 3, \cdots, N-1$, and z_N has only two neighbors z_{N-1} and z_1. Again, (3. 1)—(3. 2) and (3. 3)—(3. 4) can be written as (4. 1)—(4. 2) and (4. 3)—(4. 4) respectively.

We will also write $G(i, j)$ instead of $G(z_i|z_j)$ as in Section 4. We shall need an explicit formula for $G(i, j)$. In order to do this, let Y_k, $-1\leq k\leq N$, be defined by

$$Y_k=\begin{cases} X_k & -1\leq k\leq N-1 \\ 4X_{N-1}-2X_{N-2}-2 & k=N, \end{cases} \tag{5.1}$$

where X_k has been defined in Section 4. We have already shown that $X_k>1$ for $k\geq 2$ and $X_{k+1}-X_k>0$ for $k\geq 0$, thus

$$Y_N=2(X_{N-1}-X_{N-2})+2(X_{N-1}-1)>0.$$

We now assert that

$$G(i, j)=(Y_{|i-j|-1}+Y_{N-1-|i-j|})Y_N^{-1}. \tag{5.2}$$

Again the proof amounts to straight-forward verification and the detail is omitted. Note further that $G(i, i)=Y_{N-1}Y_N^{-1}$. In view of Theorem 3.2, we have the following

THEOREM 5.1. *Let S be a circular net with N points. If* (4.1)—(4.2) *has a nontrivial solution, then* (1.3) *holds where* $\mu(S)=Y_{N-1}^{-1}Y_N$ *and* Y_{N-1}, Y_N *are given by* (5.1). *The inequality is sharp.*

We shall show that the sequence $\{Y_{N-1}Y_N^{-1}\}_{N=0}^{\infty}$ is decreasing and its limit is $\sqrt{3}/6$. For this purpose, we first show that $X_k^2-X_{k-1}X_{k+1}=1$ for $k=0, 1, \cdots, N-1$. Indeed, $X_0^2-X_{-1}X_1=1$ by definition. Assume that out hypothesis holds for $k=m$. Then

$$\begin{aligned} X_{m+1}^2-X_{m+2}X_m&=(4X_m-X_{m-1})X_{m+1}\\ &-X_m(4X_{m+1}-X_m)=X_m^2-X_{m+1}X_{m-1}=1. \end{aligned}$$

THEOREM 5.2. *The sequence* $\{Y_{N-1}Y_N^{-1}\}_0^{\infty}$ *is decreasing and approaches* $\sqrt{3}/6$.

PROOF. We have

$$\begin{aligned} &Y_{N+1}^{-1}Y_N-Y_N^{-1}Y_{N-1}=Y_{N+1}^{-1}Y_N^{-1}\{(4X_{N-1}-2X_{N-2}-2)X_N\\ &\quad-(4X_N-2X_{N-1}-2)X_{N-1}\}=\\ &Y_{N+1}^{-1}Y_N^{-1}\{2X_{N-1}^2-2X_NX_{N-2}+2X_{N-1}-2X_N\}=\\ &\quad-2Y_{N+1}^{-1}Y_N^{-1}(X_N-X_{N-1}-1). \end{aligned}$$

Since $X_N-X_{N-1}\geq 3$ (see (4.6)), thus the last term of the above chain of equalities is negative. Also, by means of (4.6) and (5.1), it is easily verified that $Y_{N-1}Y_N^{-1}$ approaches $\sqrt{3}/6$. The proof is complete.

It is interesting to note that $\mu(A)\geq 6/\sqrt{3}\geq\mu(B)$ for any circular net A and straight net B.

6. Maxima of Green's functions on rectangular nets

A net S is said to be rectangular if S consists of n rows of m lattice points, that is,

(6.1) $$S=\{(i,j)|1\leq i\leq m,\ 1\leq j\leq n \text{ where } i,\ j,\ m,\ n \text{ are integers}\}.$$

Writing z as (i,j) and w as (a,b), the Green's function $G(z|w)$ of the system (3.3)—(3.4) associated with the net (6.1) is given by (see McCrea and Whipple [12])

(6.2) $$\begin{cases} \dfrac{2}{m+1}\sum_{r=1}^{m}\sin\dfrac{ar\pi}{m+1}\sin\dfrac{ir\pi}{m+1}\dfrac{\sinh\ (j\beta_r)}{\sinh\ (\beta_r)}\dfrac{\sinh((n+1-j)\beta_r)}{\sinh((n+1)\beta_r)} & j\leq b \\ \dfrac{2}{m+1}\sum_{r=1}^{m}\sin\dfrac{ar\pi}{m+1}\sin\dfrac{ir\pi}{m+1}\dfrac{\sinh\ (b\beta_r)}{\sinh\ (\beta_r)}\dfrac{\sinh((n+1-j)\beta_r)}{\sinh((n+1)\beta_r)} & j\geq b \end{cases}$$

where β_r, $1\leq r\leq m$, are the roots of the equation

(6.3) $$\cos(r\pi/(m+1))+\cosh(\beta_r)=2.$$

As a consequence,

(6.4) $$G(a,b|a,b)=\frac{2}{m+1}\sum_{r=1}^{m}\left(\sin\frac{ar\pi}{m+1}\right)^2\frac{\sinh(b\beta_r)}{\sinh(\beta_r)}\frac{\sinh((n+1-b)\beta_r)}{\sinh((n+1)\beta_r)}$$

For convenience's sake, we shall write $g(a,b)$ instead of $G(a,b|a,b)$. We need to find the maximum of $g(a,b)$ for $1\leq a\leq m$, $1\leq b\leq n$. Various cases have to be considered. We first consider the case where n is odd. We assert that for any fixed a,

$$\max_{1\leq b\leq n} g(a,b)=g(a,(n+1)/2).$$

Note first that

$$\sinh(b\beta_r)\sinh(n+1-b)\beta_r)=\frac{1}{2}\{\cosh((b+n+1-b)\beta_r)$$
$$-\cosh((b-n-1+b)\beta_r)\}$$
$$=\frac{1}{2}\{\cosh((n+1)\beta_r)-\cosh((n+1-2b)\beta_r)\}.$$

Since $\cosh x$ is an even function and increasing for $x\geq 0$, the minimum of $\sinh b\beta_r \sinh[(n+1-b)\beta_r]$ occurs when $2b=n+1$. In view of (6.4), and the fact that $\sinh(\beta_r)\sinh(n+1)\beta_r)>0$, our assertion is proved. Similarly, when n is even, then for any fixed a,

$$\max_{1\leq b\leq n} g(a,b)=g(a,n/2)=g(a,n/2+1).$$

By symmetry considerations, we see that when m is odd, then for any b,

$$\max_{1\leq a\leq m} g(\mathrm{a}, \mathrm{b})=g((\mathrm{m}+1)/2, b)$$

and when m is even, then for any b,

$$\max_{1\leq a\leq m} g(a, b)=g(m/2, b)=g(m/2+1, b).$$

The following is now clear.

LEMMA 6. 1. *For the rectangular net* (6. 1), *the maximum M of the associated Green's function $G(i, j|a, b)$ is given by*

$$(6.5) \qquad M=\begin{cases} g(m\pm 2, (n+1)/2) & m \text{ is even, } n \text{ is odd} \\ g(m+1)/2, n/2) & m \text{ is odd, } n \text{ is even} \\ g(m/2, n/2) & m \text{ is even, } n \text{ is even} \\ g(m+1)/2, (n+1)/2) & m \text{ is odd, } n \text{ is odd,} \end{cases}$$

where $g(i, j)=G(i, j|i, j)$.

THEOREM 6. 2. *Let S be a rectangular net of the form* (6. 1). *If* (3. 1)—(3. 2) *has a nontrivial solution then* (1. 4) *holds where* $\mu(S)=M^{-1}$, *where M is given by* (6. 5), *and the inequality is sharp.*

7. Final remarks

We have discussed three special types of nets. In general, nets can take on various forms. The question then arises whether explicit Green's functions can be found for other types of nets, and if not, whether $\mu(S)$ can be calculated or estimated. In view of the fact that the Green's function for the rectangular net is implictly given, it is unlikely that explicit Green's function for other types of nets can be found. Even if these functions can be found, their values are not vital since we have found (after a huge amount of computer experimentation) an elementary algorithm for locating the maximal points of the Green's function for an arbitrary net. It is unfortunate that a proof for the validity of our algorithm is not yet known. However, given that the maximal points can befound, it is then straightforward to use standard computer packages to calculate the maxima of these functions (by means of (3. 3)—(3. 4)). As for now, we can rely on Theorem 3. 3 for estimation purposes. Indeed, if the nets S_1 and S_2 are related by $S_1\subseteq S_2$, then by Theorem 3. 3, their corresponding Green's functions G_1 and G_2 are related by $\max G_1\leq \max G_2$ so that $\mu(S_1)\geq\mu(S_2)$. In actual applications, we can take S_1 to be any of the three types of nets discussed before, or any net whose corresponding $\mu(S_1)$ is known and get a lower bound $\mu(S_1)$ for $\sum_{S_2} p$.

References

[1] G. BORG. On a Liapounoff criterion of stability, American J. Math., 71 (1949), 67-70.

[2] L. S. CHEN, A Liapunov inequality and forced oscillations in general nonlinear nth order differential-difference equations, Glasgow Math. J., 18 (1977), 161-166.

[3] S. S. CHENG, A discrete analogue of the inequality of Lyapunov, Hokkaido Math. J. 12 (1983), 105-112.

[4] S. S. CHENG, Maximum principles for solutions of second order partial difference inequalities, Proceedings of Symposium of Functional Analysis and Applications (held at Tsing Hua University, Hsinchu, Taiwan, Republic of China, 1980), 395-401.

[5] J. H. E. COHN, Consecutive zeros of solutions of ordinary second order differential equations, J. London Math. Soc, (2), 5 (1972), 465-468.

[6] R. J. DUFFIN, Discrete potential theory, Buke Math. J., 20 (1953), 233-251.

[7] S. B. ELIASON, a Lyapunov inequality for a certain second order nonlinear differential equation, J. London Math. Soc., (2) 2 (1970), 461-466.

[8] H. GUGGENHEIMER, Concave solutions of a Dirichlet problem, Proc. Amer. Math. Soc., 40 (1973), 501-506.

[9] H. A. HEILBRONN, On discrete hamronic functions, Proc. Cambridge Phil. Soc., 45 (1949), 194-206.

[10] M. K. KWONG, On Lyapunov's inequality for disfocality, J. Math. Anal. Appl., 83 (1981), 486-494.

[11] A. LIAPOUNOFF, " Probleme general de ler Stabilite du Movement ", Princeton University Press, Princeton, N. J.,1947.

[12] W. H. RCCREA and F. J. W. Whipple, Random paths in two and three dimensions, Proc. Roy. Soc. Edinburgh, 60 (1939-1940), 281-298.

[13] Z. NEHARI, some eigenvalue estimates, J. D'Analyse Math., 7 (1959), 79-88.

[14] H. B. PHILLIPS and N. Wiener, Nets and the Dirichlet problem, J. of Math. and Phy., 2 (1923), 105-124.

[15] M. H. PROTTER and H. F. Weinberger, " Maximum Principles in Differential Equations ", Prentice Hall, Englewood Cliff, N. J., 1967.

[16] W. T. REID, A matrix Liapunov inequality, J. Math. Anal. Appl., 32 (1970), 424-434.

[17] W. T. REID, A generalized Liapnnov inequality, J. Diff. Eq., 13 (1973), 182-196.

[18] G. F. ROACH, " Green's Functions ", 2nd edition, Cambridge University press, Cambridge, 1982.

[19] F. SPITZER, " Principles of Random Walk ", Van Nostrand, 1964.

National Tsing Hua University
Department of Mathematics
Hsinchu, Taiwan 300
Republic of China