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On the Gauss-Codazzi equations

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Introduction.

Let (M, g) be an n-dimensional Riemannian manifold. Then it is an interesting and fundamental problem to find the minimum integer m such that (M, g) can be (locally) isometrically immersed into the euclidean space $\boldsymbol{R}^m$. Except the case where (M, g) is a space of constant curvature, a few facts are known about the above integer m.

Related to this problem the Gauss equation brings to us a useful information. Let $\boldsymbol{f}$ be an isometric immersion of (M, g) into the euclidean space $\boldsymbol{R}^m$. Then the second fundamental form of $\boldsymbol{f}$ satisfies the Gauss equation, that is a purely algebraic equation essentially determined by the Riemannian curvature of (M, g) and the codimension $m-n$. In this sense, the Gauss equation may be considered as an obstruction to the existence of isometric immersions. By showing the non-existence of the solutions of the Gauss equation, many authors obtained estimates on the lower bounds of m (see [23], [19], [3] etc.).

In this paper we consider the following problem: Does the existence of solutions of the Gauss equation imply the existence of isometric immersions? As the examples that will be given in this paper show, the above problem is not true in general. There are many higher order obstructions to the existence of isometric immersions. The main purpose of this paper is to formulate two higher order obstructions called the first and second Gauss-Codazzi equations and to show the usefulness of these obstructions.

Let m be an integer with $m \geqq n$. By definition, a differentiable map $\boldsymbol{f}$ of M into $\boldsymbol{R}^m$ is called an *isometric immersion* if the induced metric via $\boldsymbol{f}$ coincides with the Riemannian metric g. In other words, an isometric immersion is regarded as a solution of a system of first order partial differential equations with respect to a differentiable map $\boldsymbol{f}$ of M into $\boldsymbol{R}^m$. We consider this system from the view point of the theory of partial differential equations. Let k be a non-negative integer and $J^k(\boldsymbol{R}^m)$ be the k-th order jet bundle of local differentiable mappings of M into $\boldsymbol{R}^m$. The system of isometric immersions stated above defines a subvaritey P in $J^1(\boldsymbol{R}^m)$. P always forms a submanifold of $J^1(\boldsymbol{R}^m)$ but does not bring any information about the existence of isometric immersions. To obtain useful informations we have to consider the prolongations $P^{(1)}, P^{(2)}, \cdots$ of P.

Let Q be the subvariety in $J^2(\boldsymbol{R}^m)$ composed of all elements of $P^{(1)}$ satisfying the Gauss equation (for the precise definitions see §**A**). In the case where $m \geqq (1/2)n(n+1)$, by considering the property of Q, we obtain a satisfactory result. In fact, it can be shown that in case $m \geqq (1/2)n(n+1)$, Q (precisely an open subset of Q) forms an involutive system in the sense of Cartan-Kähler-Kuranishi (see [9], [14]). Therefore, if (M, g) is real analytic, then (M, g) can be locally isometrically immersed into $\boldsymbol{R}^m$ with $m \geqq (1/2)n(n+1)$ (Theorem of Janet-Cartan).

On the other hand, in case $m < (1/2)n(n+1)$, Q is not necessarily involutive even if Q is not empty. Therefore to obtain further informations about the existence of isometric immersions of (M, g) into $\boldsymbol{R}^m$, we have to consider the prolongations $Q^{(1)}, Q^{(2)}, \cdots$ of Q and the reductions associated with these prolongations.

First we discuss the first order prolongation $Q^{(1)}$ of Q. We show that an element α of Q is prolonged to an element of $Q^{(1)}$ if and only if α satisfies an equation called the *first Gauss-Codazzi equation* (see Proposition 2). The first Gauss-Codazzi equation is obtained by the first order derivative of the Gauss equation and integrability conditions of derivatives of $\boldsymbol{f}$ of order lesser than or equal to 3. Generally, the first Gauss-Codazzi equation is not trivial. We give an example of three dimensional Riemannian manifold (M, g) such that the Gauss equation associated with isometric immersions of (M, g) into $\boldsymbol{R}^4$ admits a unique solution (up to sign) at each point of M but the first Gauss-Codazzi equation does not admit any solution (Example 1). We note that in the special case where (M, g) is locally Riemannian symmetric, the first Gauss-Codazzi equation is always satisfied.

Next we consider the second order prolongation $Q^{(2)}$ of Q. We show that an element β of $Q^{(1)}$ is prolonged to an element of $Q^{(2)}$ if and only if β satisfies a system of equations called the *second Gauss-Codazzi equation.* The second Gauss-Codazzi equation is obtained by the second order derivative of the Gauss equation and the integrability conditions of derivatives of $\boldsymbol{f}$ of order lesser than or equal to 4. Theoretically, the second Gauss-Codazzi equation may induce a reduction of $Q^{(1)}$, but in general, it cannot be represented explicitly. In some cases, the reduction induced from the second Gauss-Codazzi equation may influence Q. In case $Q^{(1)}$ has a fibered manifold structure over Q, it is known by the general theory that the second Gauss-Codazzi equation can be reduced to the problem related to the Spencer cohomology groups associated with symbol of

Q. We further show that if 1) (M, g) is locally Riemannian symmetric, and 2) the first prolongation of the symbol of Q vanishes, then the second Gauss-Codazzi equation is reduced to a non-linear algebraic equation imposed on the second fundamental forms which is independent from the Gauss equation (see Proposition 7).

To show the usefulness of the first and second Gauss-Codazzi equations we study the problem of isometric immersions of the two dimensional complex projective space $P^2(\boldsymbol{C})$.

As is known, the n-dimensional complex projective space $P^n(\boldsymbol{C})$ endowed with the *Fubini-Study metric* is globally isometrically imbedded into the euclidean space $\boldsymbol{R}^m$ with $m=n^2+2n$ (see [15]). Therefore $P^2(\boldsymbol{C})$ is isometrically imbedded into $\boldsymbol{R}^8$. Then it is a natural question to ask whether $P^2(\boldsymbol{C})$ can be isometrically immersed into a lower dimensional euclidean space or not.

We first note the fact that the Gauss equation associated with isometric immersions of $P^2(\boldsymbol{C})$ into $\boldsymbol{R}^6$ does not admit any solution. This fact follows frow Weinstein's theorem (see [24]) and a property of the Riemannian curvature of $P^2(\boldsymbol{C})$. (Another proof can be seen in Agaoka [2]). Consequently, it can be concluded that $P^2(\boldsymbol{C})$ cannot be isometrically immersed into $\boldsymbol{R}^6$ even locally. On the other hand, Agaoka [2] gave solutions of the Gauss equation associated with isometric immersions of $P^2(\boldsymbol{C})$ into $\boldsymbol{R}^7$. But it has not been known whether $P^2(\boldsymbol{C})$ can be isometrically immersed into $\boldsymbol{R}^7$ or not.

In this paper we first solve the Gauss equation associated with isometric immersions into $\boldsymbol{R}^7$ under an additional condition and obtain a class of solutions containing Agaoka's solutions. We can show that our solutions form an open subset in the set of all solutions of the Gauss equation. Then we prove that there are no solutions of the second Gauss Codazzi equation associated with these solutions. This implies that there are no isometric immersions of $P^2(\boldsymbol{C})$ into $\boldsymbol{R}^7$ whose second fundamental forms coincide with one of our solutions. This fact forms a contrast to the result concerning isometric immersions of 4-dimensional Riemannian manifolds in [6]. Our result, however, does not imply that $P^2(\boldsymbol{C})$ cannot be isometrically immersed into $\boldsymbol{R}^7$, because our solutions do not exhaust all of the solutions of the Gauss equation. It is still an open question whether $P^2(\boldsymbol{C})$ can be isometrically immersed into $\boldsymbol{R}^7$ or not.

Finally we refer to the relation between our Gauss-Codazzi equations and the classical *Codazzi equation* (for the definitions see [8]). Our first and second Gauss-Codazzi equations are just the purely algebraic formulations of the classical Codazzi equations (details are not quoted in this paper). Our formulation enables us to judge whether these equations have solutions or not and further suggests the possibility to obtain higher order obstructions by differentiations. But they are out of the scope of this work.

Throughout this paper we assume the differentiability of class C^∞.

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§A. Differential equations of isometric immersions.

In this section we consider differential equations of isometric immersions (cf. [14]). Notations used here are the same that are used in [14].

Let (M, g) be an n-dimensional Riemannian manifold. We denote by ∇ the *covariant differentiation* associated with the *Levi-Civita connection* of (M, g) and by R the *curvature tensor field* of type $(1, 3)$ with respect to ∇.

Let $\boldsymbol{f}$ be a differentiable mapping of M into the m-dimensional euclidean space $\boldsymbol{R}^m$. For each positive integer k we mean by $\overbrace{\nabla\cdots\nabla}^{k}\boldsymbol{f}$ the k-th k-th covariant derivative of $\boldsymbol{f}$. Then we have the following successive integrability conditions:

$$\nabla_x\nabla_y=\nabla_y\nabla_x\boldsymbol{f};$$

$$\nabla_z\nabla_x\nabla_y\boldsymbol{f}=\nabla_z\nabla_y\nabla_x\boldsymbol{f}=\nabla_x\nabla_z\nabla_y\boldsymbol{f}-\nabla_{R(z,x)y}\boldsymbol{f};$$

$$\begin{aligned}\nabla_u\nabla_z\nabla_x\nabla_y\boldsymbol{f}&=\nabla_u\nabla_z\nabla_y\nabla_x\boldsymbol{f}\\ &=\nabla_u\nabla_x\nabla_z\nabla_y\boldsymbol{f}-\nabla_u\nabla_{R(z,x)y}\boldsymbol{f}-\nabla_{\nabla_u R(z,x)y}\boldsymbol{f}\\ &=\nabla_z\nabla_u\nabla_x\nabla_y\boldsymbol{f}-\nabla_{R(u,z)x}\nabla_y\boldsymbol{f}-\nabla_x\nabla_{R(u,z)y}\boldsymbol{f};\end{aligned}$$

....

In the above equalities and in the following discussions, $x, y, z, \cdots$ mean elements in the tangent space T_p at an arbitrary point p of M.

Let $\boldsymbol{f}$ be an isometric immersion of (M, g) into $\boldsymbol{R}^m$. Then we have

$$(1)\qquad \langle\nabla_x\boldsymbol{f}, \nabla_y\boldsymbol{f}\rangle=g(x, y),$$

where $\langle\ ,\ \rangle$ stands for the standard inner product of $\boldsymbol{R}^m$. Differentiating (1) covariantly and using the integrability condition for $\nabla\nabla\boldsymbol{f}$, we have

$$(2)\qquad \langle\nabla_z\nabla_x\boldsymbol{f}, \nabla_y\boldsymbol{f}\rangle=0.$$

The equality (2) implies that the second covariant derivative $(\nabla\nabla\boldsymbol{f})_p$ takes its value in the normal vector space N_p, the orthogonal complement of $f_* T_p$ in $\boldsymbol{R}^m$ with respect to $\langle\ ,\ \rangle$. $(\nabla\nabla f)_p$ is called the *second fundamental form* of f at p. Differentiating (2) covariantly, we obtain

(3) $$\langle\nabla_u\nabla_z\nabla_x f, \nabla_y f\rangle+\langle\nabla_z\nabla_x f, \nabla_u\nabla_y f\rangle=0.$$

From (3) and the integrability conditions for $\nabla\nabla\nabla f$, we have the Gauss equation:

(4) $$\langle\nabla_u\nabla_x f, \nabla_z\nabla_y f\rangle-\langle\nabla_z\nabla_x f, \nabla_u\nabla_y f\rangle=C(u, z, x, y),$$

where C denotes the *curvature tensor field* of type (0, 4) given by

$$C(u, z, x, y)=-g(R(u, z)x, y).$$

Before proceeding to the further differentiations of the above equalities, we introduce an operator Ω in order to simplify the notations below.

Let $K(T_p)$ be the space of *curvature like tensors* of T_p, i. e., the space of all $H\in\Lambda^2 T_p^*\otimes\Lambda^2 T_p^*$ satisfying the *first Bianchi identity*

$$\underset{x,y,z}{\mathfrak{S}} H(x, y, z, w)=0,$$

where $\underset{x,y,z}{\mathfrak{S}}$ means the cyclic sum with respect to x, y, z.

Let $\xi, \eta\in S^2 T_p^*\otimes\boldsymbol{R}^m$. We define $\Omega(\xi, \eta)\in\Lambda^2 T_p^*\otimes\Lambda^2 T_p^*$ by

$$\Omega(\xi, \eta)(u, z, x, y)=\frac{1}{2}\{\langle\xi(u, x), \eta(z, y)\rangle+\langle\xi(z, y), \eta(u, x)\rangle$$
$$-\langle\xi(u, y), \eta(z, x)\rangle-\langle\xi(z, x), \eta(u, y)\rangle\}.$$

Then it is easily seen that: i) $\Omega(\xi, \eta)=\Omega(\eta, \xi)$; ii) $\Omega(\xi, \eta)\in K(T_p)$.

Utilizing the operator Ω, we can write the Gauss equation in the following simple form:

(4′) $$\Omega(\nabla\nabla f, \nabla\nabla f)=C.$$

The successive covariant differentiations of (3) and (4′) yield the following:

(5) $$\Omega(\nabla_v\nabla\nabla f, \nabla\nabla f)=\frac{1}{2}\nabla_v C.$$

(6) $$\langle\nabla_w\nabla_u\nabla_z\nabla_x f, \nabla_y f\rangle+\langle\nabla_u\nabla_z\nabla_x f, \nabla_w\nabla_y f\rangle$$
$$+\langle\nabla_w\nabla_z\nabla_x f, \nabla_u\nabla_y f\rangle+\langle\nabla_z\nabla_x f, \nabla_w\nabla_u\nabla_y f\rangle=0.$$

(7) $$\Omega(\nabla_w\nabla_v\nabla\nabla f, \nabla\nabla f)+\Omega(\nabla_v\nabla\nabla f, \nabla_w\nabla\nabla f)=\frac{1}{2}\nabla_w\nabla_v C.$$

....

Hereafter, we study the equalities (1)-(7).

Put $\boldsymbol{T}=T_p$, $\boldsymbol{T}^*=T_p^*$. Let $\omega\in\boldsymbol{T}^*\otimes\boldsymbol{R}^m(=\mathrm{Hom}(\boldsymbol{T},\boldsymbol{R}^m))$. We denote by $\omega(\boldsymbol{T})$ the image of $\boldsymbol{T}$ by ω and $\boldsymbol{N}_\omega$ the orthogonal complement of $\omega(\boldsymbol{T})$ in $\boldsymbol{R}^m$ with respect to $\langle\ ,\ \rangle$. Then we have the following natural decompositions:

$$\otimes^k\boldsymbol{T}^*\otimes\boldsymbol{R}^m=\otimes^k\boldsymbol{T}^*\otimes\omega(\boldsymbol{T})+\otimes^k\boldsymbol{T}^*\otimes\boldsymbol{N}_\omega\ (k=0,1,2,\cdots)\otimes^k\boldsymbol{T}^*\otimes\omega(\boldsymbol{T}).$$

For each $\xi\in\otimes^k\boldsymbol{T}^*\otimes\boldsymbol{R}^m$, we denote by ξ^0 (resp. ξ^1), the $\otimes^k\boldsymbol{T}^*\otimes\omega(\boldsymbol{T})$-component (resp. $\otimes^k\boldsymbol{T}^*\otimes\boldsymbol{N}_\omega$-component) of ξ with respect to the above decomposition.

Now let us put $\omega=\nabla\boldsymbol{f}$, $\alpha=\nabla\nabla\boldsymbol{f}$, $\beta=\nabla\nabla\nabla\boldsymbol{f}$, $\gamma=\nabla\nabla\nabla\nabla\boldsymbol{f}$. Then by equations (1)-(7) and the integrability conditions, we have:

PROPOSITION 1. *Let* $\boldsymbol{f}$ *be an isometric immersion of* (M,g) *into* $\boldsymbol{R}^m$. *Then* ω, α, $\beta(=\beta^0+\beta^1)$, $\gamma(=\gamma^0+\gamma^1)$ *satisfy the following* I-IV:

I. $\omega\in\boldsymbol{T}^*\otimes\omega(\boldsymbol{T})$;

(A 1) $\langle\omega(x),\omega(y)\rangle=g(x,y)$.

II. $\alpha\in S^2\boldsymbol{T}^*\otimes\boldsymbol{N}_\omega$;

(A 2) $\Omega(\alpha,\alpha)=C$.

III. 1°) $\beta^0\in\boldsymbol{T}^*\otimes S^2\boldsymbol{T}^*\otimes\omega(\boldsymbol{T})$;

(A 3) $\beta^0(u,z,x)=\beta^0(z,u,x)-\omega(R(u,z)x)$;

(A 4) $\langle\beta^0(u,z,x),\omega(y)\rangle+\langle\alpha(z,x),\alpha(u,y)\rangle=0$.

2°) $\beta^1\in S^3\boldsymbol{T}^*\otimes\boldsymbol{N}_\omega$;

(A 5) $\Omega(v\lrcorner\beta^1,\alpha)=\frac{1}{2}\nabla vC$.

IV. 1°) $\gamma^0\in S^2\boldsymbol{T}^*\otimes S^2\boldsymbol{T}^*\otimes\omega(\boldsymbol{T})$;

(A 6) $\gamma^0(w,u,z,x)=\gamma^0(w,z,u,x)-\omega(\nabla_wR(u,z)x)$;

(A 7) $\langle\gamma^0(w,u,z,x),\omega(y)\rangle+\langle\beta^1(u,z,x),\alpha(w,y)\rangle$

$+\langle\beta^1(w,z,x),\alpha(u,y)\rangle+\langle\alpha(z,x),\beta^1(w,u,y)\rangle=0$.

2°) $\gamma^1\in\otimes^2\boldsymbol{T}^*\otimes S^2\boldsymbol{T}^*\otimes\boldsymbol{N}_\omega$;

(A 8) $\gamma^1(u,z,y,x)=\gamma^1(u,y,z,x)-\alpha(u,R(z,y)x)$;

(A 9) $\gamma^1(u,z,y,x)=\gamma^1(z,u,y,x)-\alpha(R(u,z)y,x)$

$-\alpha(y,R(u,z)x)$;

(A10) $\Omega(w\otimes v)\lrcorner\gamma^1, \alpha)+\Omega(v\lrcorner\beta^0, w\lrcorner\beta^0)$
$$+\Omega(v\lrcorner\beta^1, w\lrcorner\beta^1)=\frac{1}{2}\nabla_w\nabla_v C.$$

§B. Gauss-Codazzi equations.

As we have seen in §**A**, isometric immersions $\boldsymbol{f}$ which are primarily defined as solutions of the first order differential equation (1) in §**A** satisfy many equations involving higher order derivatives. We now observe the equations (A1)-(A10) from the viewpoint of the theory of differential equations.

Let $J^k(\boldsymbol{R}^m)$ $(k=0,1,2,\cdots)$ be the bundle over M composed of all k-jets $j_p^k(\boldsymbol{f})$ of local differentiable mappings $\boldsymbol{f}$ of M into $\boldsymbol{R}^m$. For each k, the equations (A1)-(A10) satisfied by all k-jets of isometric immersions determine a subvariety in $J^k(\boldsymbol{R}^m)$. In order to give an explicit expression for this subvariety, we regard $J^k(\boldsymbol{R}^m)$ as a subbundle of the sum $T^k(\boldsymbol{R}^m)=\boldsymbol{R}^m+\boldsymbol{T}^*\otimes\boldsymbol{R}^m+\cdots+\otimes^k\boldsymbol{T}^*\otimes\boldsymbol{R}^m$ of vector bundles $\otimes^i\boldsymbol{T}^*\otimes\boldsymbol{R}^m(i=0,1,\cdots,k)$ by the assignment

$$j_p^k(\boldsymbol{f})\longrightarrow(p\,;\boldsymbol{f}(p),(\nabla\boldsymbol{f})_p,\cdots,(\overbrace{\nabla\cdots\nabla}^{k}\boldsymbol{f})_p).$$

Then the subbundle $J^k(\boldsymbol{R}^m)$ are characterized by the integrability conditions, i. e., an element $(p\,;\omega_0,\omega_1,\cdots,\omega_k)\in T^k(\boldsymbol{R}^m)$ $(\omega_i\in\otimes^i T_p^*\otimes\boldsymbol{R}^m,\ i=0,1,\cdots,k)$ is included in $J^k(\boldsymbol{R}^m)$ if and only if the integrability conditions in § **A** are all satisfied, where $\overbrace{\nabla\cdots\nabla}^{i}\boldsymbol{f}$ are replaced by ω_i $(i=0,1,\cdots,k)$. Then it is clear that $J^0(\boldsymbol{R}^m)=T^0(\boldsymbol{R}^m)$ and $J^1(\boldsymbol{R}^m)=T^1(\boldsymbol{R}^m)$.

Now let us denote by P the subset of $T^1(\boldsymbol{R}^m)$ composed of all $(p\,;\omega_0,\omega_1)\in T^1(\boldsymbol{R}^m)$ satisfyng (A1). P is the original differential equation of isometric immersions of (M,g) into $\boldsymbol{R}^m$. It is easy to see that under the assumption $m\geqq n$, P forms a submanifold of $J^1(\boldsymbol{R}^m)$. In order to simplify the notations below, for each element $(p\,;\omega_0,\omega_1)\in P$, we put $\omega=\omega_1$, $\boldsymbol{T}=T_p$, $\boldsymbol{T}^*=T_p^*$ and $\boldsymbol{N}=N_\omega$, Moreover, to represent an element of P, we formally write $\omega\in P$ instead of writing $(p\,;\omega_0,\omega)\in P$.

By $P^{(1)}$ (resp. $P^{(2)}$) we denote the first (resp. second) standard prolongation of P. The first prolongation $P^{(1)}$ is given by the subset of $T^2(\boldsymbol{R}^m)$ composed of all $(\omega,\alpha)\in T^2(\boldsymbol{R}^m)$ satisfying $\omega\in P$ and $\alpha\in S^2\boldsymbol{T}^*\otimes\boldsymbol{N}$. Let $\pi_k: T^k(\boldsymbol{R}^m)\longrightarrow T^{k-1}(\boldsymbol{R}^m)$ $(k=1,2,\cdots)$ be the canonical projection. Then it is easily seen that $P^{(1)}$ is mapped onto P by π_2. In other words, any element $\omega\in P$ can be prolonged to an element $(\omega,\alpha)\in P^{(1)}$. We note that $\pi_2: P^{(1)}\longrightarrow P$ has a vector bundle structure over P.

For simplicity, we symbolically write α in order to represent an element $(\omega, \alpha)\in P^{(1)}$. The second prolongation $P^{(2)}$ is given by the subset of $T^{3}(\boldsymbol{R}^{m})$ composed of all $(\alpha, \beta)\in T^{3}(\boldsymbol{R}^{m})$ satisfying $\alpha\in P^{(1)}$ and 1°) of III in Proposition 1 and $\beta^{1}\in S^{3}\boldsymbol{T}^{*}\otimes\boldsymbol{N}$. By (A3) and (A4) we know that an element α can be prolonged to an element $(\alpha, \beta)\in P^{(2)}$ if and only if α satisfies the Gauss equation (A2).

Let Q be the subset of $P^{(1)}$ composed of all α satisfying (A2). By $Q^{(1)}$ (resp. $Q^{(2)}$) we denote the first (resp. second) standard prolongation of Q. Then the first prolongation $Q^{(1)}$ is given by the subset of $T^{3}(\boldsymbol{R}^{m})$ composed of all $(\alpha, \beta)\in P^{(2)}$ satisfying $\alpha\in Q$ and III in Proposition 1. And the second prolongation $Q^{(2)}$ is given by the subset of $T^{4}(\boldsymbol{R}^{m})$ composed of all $(\alpha, \beta, \gamma)\in T^{4}(\boldsymbol{R}^{m})$ satisfying $(\alpha, \beta)\in Q^{(1)}$ and IV in Proposition 1.

Now let us consider the condition under that an element $\alpha\in Q$ (resp. $(\alpha, \beta)\in Q^{(1)}$) can be prolonged to an element of $Q^{(1)}$ (resp. $Q^{(2)}$). Let $\alpha\in P^{(1)}$. We define a linear mapping $\Omega_{\alpha}: S^{2}\boldsymbol{T}^{*}\otimes\boldsymbol{N}\longrightarrow K(\boldsymbol{T})$ by $\Omega_{\alpha}(\xi)=\Omega(\xi, \alpha)$ for $\xi\in S^{2}\boldsymbol{T}^{*}\otimes\boldsymbol{N}$. For each positive integer k, we extend Ω_{α} to a linear mapping $\Omega_{\alpha}^{k}: \otimes^{k}\boldsymbol{T}^{k}\otimes S^{2}\boldsymbol{T}^{*}\otimes\boldsymbol{N}\longrightarrow\otimes^{k}\boldsymbol{T}^{*}\otimes K(\boldsymbol{T})$ by setting $\Omega_{\alpha}^{k}=\boldsymbol{1}_{k}\otimes\Omega_{\alpha}$, where $\boldsymbol{1}_{k}$ denotes the identity mapping of $\otimes^{k}\boldsymbol{T}^{*}$ onto itself. Let $K^{(1)}(\boldsymbol{T})$ be the linear subspace of $\boldsymbol{T}^{*}\otimes K(\boldsymbol{T})$ consisting of all elements $H\in\boldsymbol{T}^{*}\otimes K(\boldsymbol{T})$ satisfying the *second Bianchi identity*, i.e.,

$$\underset{x,y,z}{\mathfrak{S}} H(x, y, z, u, v)=0.$$

Then it is known that: i) $(\nabla C)_{p}\in K^{(1)}(\boldsymbol{T})$; ii) $\Omega_{\alpha}^{1}(S^{3}\boldsymbol{T}^{*}\otimes\boldsymbol{N})\subset K^{(1)}(\boldsymbol{T})$. We have

$$\dim K(\boldsymbol{T})=\frac{1}{12}n^{2}(n^{2}-1)\ ;\ \dim K^{(1)}(\boldsymbol{T})=\frac{1}{24}n^{2}(n^{2}-1)(n+2)$$

(see [14]).

PROPOSITION 2. *An element $\alpha\in Q$ is prolonged to an element of $Q^{(1)}$ if and only if there exists an element $\beta^{1}\in S^{3}\boldsymbol{T}^{*}\otimes\boldsymbol{N}$ satisfying*

$$\text{(B1)} \qquad \Omega_{\alpha}^{1}(\beta^{1})=\frac{1}{2}\nabla C.$$

In particular if (M, g) is locally symmetric, i.e., $\nabla C=0$, then the canonical projection $\pi_{3}: Q^{(1)}\longrightarrow Q$ is necessarily surjective.

PROOF. Let $\alpha\in Q$. We take an element $\beta^{0}\in\boldsymbol{T}^{*}\otimes S^{2}\boldsymbol{T}^{*}\otimes\omega(\boldsymbol{T})$ satisfying (A4). We note that such β^{0} always exists and is determined uniquely by (A4). By virtue of the Gauss equation (A2), we can show that β^{0}

satisfies (A3). Now let us assume that there exists $\beta^1 \in S^3 \boldsymbol{T}^* \otimes \boldsymbol{N}$ satisfyig (B1). Then putting $\beta=\beta^0+\beta^1$, we have $(\alpha, \beta) \in Q^{(1)}$. This proves that any element $\alpha \in Q$ can be prolonged to an element of $Q^{(1)}$. The converse is obvious from III in Proposition 1. Q. E. D.

We call (B1) the *first Gauss-Codazzi equation*. The following example shows that even if the Gauss equation admits solutions at each point of M, the first Gauss-Codazzi equation does not necessarily admit solutions. Thus the first Gauss-Codazzi equation plays a role of an obstruction to the existence of isometric immersions.

EXAMPLE 1. Let x_1, x_2, x_3 be the canonical field coordinates of $\boldsymbol{R}^3$ and let $g=\sum g_{ij} dx_i dx_j$ be a covariant symmetric 2-tensor field on $\boldsymbol{R}^3$ whose coefficients $g_{ij} (g_{ij}=g_{ji})$ are polynomials of x_1, x_2, x_3 written in the form

$$g_{ij}=\delta_{ij}+\frac{1}{2}(\sum_k x_k^2)\delta_{ij}+\frac{1}{6}\sum_{k,l,m} E^0_{klmij}\, x_k x_l x_m,$$

where E^0_{klmij} are constants satisfying $E^0_{klmij}=E^0_{klmji}=E^0_{lkmji}=E^0_{kmlij}$ for $i, j, k, l, m=1, 2, 3$. As is easily seen, g is positive definite on a small open neighborhood M of the origin $o \in \boldsymbol{R}^3$. At the origin o, the Riemannian curvature $C=(C_{ijkl})$ and its first covariant derivative $\nabla C=(\nabla_m C_{ijkl})$ of the Riemannian manifold (M, g) are given as follows:

$$C_{ijkl}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\,;$$

$$\nabla_m C_{ijkl}=\frac{1}{2}(E^0_{mikjl}+E^0_{mjlik}-E^0_{miljk}-E^0_{mjkil}).$$

Now we consider isometric immersions of (M, g) into the euclidean space $\boldsymbol{R}^4$. Let $\omega=(o\,; \omega_0, \omega_1) \in P$ and let $\boldsymbol{n}$ be a unit vector of $\boldsymbol{N}=\boldsymbol{N}_\omega$. We put $\alpha_0=(\sum_i (dx_i)^2)\boldsymbol{n}$. By a direct calculation we can show that the solution of the Gauss equation (A2) are limited to $\pm\alpha_0$, i.e., $Q \cap \pi_2^{-1}(\omega)=\{(\omega, \pm\alpha_0)\}$. We note that since $\dim S^2\boldsymbol{T}^* \otimes \boldsymbol{N}=\dim K(\boldsymbol{T})=6$ and the type number of α_0 is equal to 3, the Gauss equation admits a unique solution (up to sign) at each point p of M sufficiently close to the origin o (for the definition of the type number and related facts, see [16]).

We now consider the differential equation $Q^{(1)}$ at o. Since $\dim S^3\boldsymbol{T}^* \otimes \boldsymbol{N}=10$ and $\dim K^{(1)}(\boldsymbol{T})=15$, the linear mapping Ω^1_α: $S^3\boldsymbol{T}^* \otimes \boldsymbol{N} \longrightarrow K^{(1)}(\boldsymbol{T})$ $(\alpha=\pm\alpha_0)$ is not surjective. Let us take an element $L=(L_{mijkl}) \in K^{(1)}(\boldsymbol{T})$ such that $L \notin \Omega^1_\alpha(S^3\boldsymbol{T}^* \otimes \boldsymbol{N})$. Then there exists an element $E=(E_{klmij}) \in S^3\boldsymbol{T}^* \otimes S^2\boldsymbol{T}^*$ satisfying

$$L_{mijkl}=\frac{1}{2}(E_{mikjl}+E_{mjlik}-E_{miljk}-E_{mjkil})$$

for $i, j, k, l, m=1, 2, 3$ (see Proposition 1 in Appendix in [14]). If the constants E^0_{klmij} in the definition of g_{ij}'s are so chosen to be equal to the coefficients E_{klmij} of the above E, we have $\nabla C=L\notin\Omega_\alpha(S^3\boldsymbol{T}^*\otimes\boldsymbol{N})$. In other words, the first Gauss-Codazzi equation (B1) does not admit any solution for any $\alpha\in Q\cap\pi_2^{-1}(\omega)$. Accordingly (M, g) cannot be isometrically immersed into $\boldsymbol{R}^4$.

In order to obtain a higher order obstruction, we prepare some notations and lemmas.

Let $\alpha\in P^{(1)}$. We define $S_\alpha\in S^2\boldsymbol{T}^*\otimes\boldsymbol{T}^*\otimes\boldsymbol{T}$ by

$$g(S_\alpha(z, x)u, y)=-\langle\alpha(z, x), \alpha(u, y)\rangle.$$

Further we define $\chi_\alpha\in\otimes^2\boldsymbol{T}^*\otimes S^2\boldsymbol{T}^*\otimes\boldsymbol{N}$ and $\Phi(\alpha)\in\Lambda^2\boldsymbol{T}^*\otimes S^2\boldsymbol{T}^*\otimes\boldsymbol{N}$ by setting

$$\chi_\alpha(w, v, u, x)=\alpha(w, S_\alpha(u, x)v)$$

$$\Phi(\alpha)\,(w, v, u, x)=-\alpha(R(w, v)u, x)-\alpha(u, R(w, v)x)$$
$$-\chi_\alpha(w, v, u, x)+\chi_\alpha(v, w, u, x).$$

LEMMA 3. (1) *Let* $\alpha\in Q$. *Then*:

$$R(u, z)x=S_\alpha(u, x)z-S_\alpha(z, x)u.$$

(2) *Let* $(\alpha, \beta)\in Q^{(1)}$. *Then*:
$$\beta^0(u, z, x)=\omega(S_\alpha(z, x)u).$$

PROOF. The assertion (1) is clear from (A2). Let $(\alpha, \beta)\in Q^{(1)}$. Because of (A4) and (A1), we have

$$\langle\beta^0(u, z, x), \omega(y)\rangle=-\langle\alpha(z, x), \alpha(u, y)\rangle$$
$$=g(S_\alpha(z, x)u, y)=\langle\omega(S_\alpha(z, x)u), \omega(y)\rangle.$$

Since $\beta^0(u, z, x)\in\omega(\boldsymbol{T})$, it follows that $\beta^0(u, z, x)=\omega(S_\alpha(z, x)u)$. This proves the assertion (2). Q. E. D.

Let s and k be any integers with $0\leqq s\leqq k-1$. We define a linear mapping $\delta_s:\otimes^k\boldsymbol{T}^*\longrightarrow\otimes^k\boldsymbol{T}^*$ by

$$(\delta_s X)\,(x_1,\cdots, x_k)=\sum_{i=1}^{s+1}(-1)^i X(x_1,\cdots, \hat{x}_i,\cdots, x_{s+1}, x_i, x_{s+2},\cdots, x_k)$$

for $X\in\otimes^k\boldsymbol{T}^*$. Putting $\tilde{\delta}_s=\delta_s\otimes 1_N$, we extend δ_s to a linear mapping $\tilde{\delta}_s$: $\otimes^k\boldsymbol{T}^*\otimes\boldsymbol{N}\longrightarrow\otimes^k\boldsymbol{T}^*\otimes\boldsymbol{N}$.

LEMMA 4. *Let* $(\alpha, \beta)\in Q^{(1)}$. *Then*:

(1) $\Omega((w\otimes v)\lrcorner\chi_\alpha, \alpha)+\Omega(v\lrcorner\beta^0, w\lrcorner\beta^0)=0.$

(2) $\tilde{\delta}_2\Phi(\alpha)=0$; $\Omega_\alpha^2(\Phi(\alpha))=\frac{1}{2}\delta_1(\nabla\nabla C).$

PROOF. By Lemma 3 and by the definitions of χ_α and S_α, we have

$$\begin{aligned}&\langle\chi_\alpha(w, v, u, x), \alpha(z, y)\rangle=\langle\alpha(w, S_\alpha(u, x)v), \alpha(z, y)\rangle\\&=-g(S_\alpha(u, x)v, S_\alpha(z, y)w)=-\langle\omega(S_\alpha(u, x)v), \omega(S_\alpha(z, y)w)\rangle\\&=-\langle\beta^0(v, u, x), \beta^0(w, z, y)\rangle=-\langle(v\lrcorner\beta^0)(u, x), (w\lrcorner\beta^0)(z, y)\rangle.\end{aligned}$$

Then by the definition of Ω we have the desired equality (1).

We now show the assertion (2). The first equality can be easily proved by a simple calculation. Next we prove the second equality. By the Ricci formula we have:

$$\begin{aligned}&-\delta_1(\nabla\nabla C)(w, v, u, z, x, y)\\&=-\nabla_w\nabla_v C(u, z, x, y)+\nabla_v\nabla_w C(u, z, x, y)\\&=C(R(w, v)u, z, x, y)+C(u, R(w, v)z, x, y)\\&\quad+C(u, z, R(w, v)x, y)+C(u, z, x, R(w, v)y)\\&=\langle\alpha(R(w, v)u, x), \alpha(z, y)\rangle-\langle\alpha(R(w, v)u, y), \alpha(z, x)\rangle\\&\quad+\langle\alpha(u, x), \alpha(R(w, v)z, y)\rangle-\langle\alpha(u, y), \alpha(R(w, v)z, x)\rangle\\&\quad+\langle\alpha(u, R(w, v)x), \alpha(z, y)\rangle-\langle\alpha(u, y), \alpha(z, R(w, v)x)\rangle\\&\quad+\langle\alpha(u, x), \alpha(z, R(w, v)y)\rangle-\langle\alpha(u, R(w, v)y), \alpha(z, x)\rangle.\end{aligned}$$

In view of the above and the equality (1), we obtain

$$\begin{aligned}\Omega((w\otimes v)\lrcorner\Phi(\alpha), \alpha)&=\frac{1}{2}(w\otimes v)\lrcorner\delta_1(\nabla\nabla C)\\&\quad-\Omega(v\lrcorner\beta^0, w\lrcorner\beta^0)+\Omega(w\lrcorner\beta^0, v\lrcorner\beta^0)\\&=\frac{1}{2}(w\otimes v)\lrcorner\delta_1(\nabla\nabla C).\end{aligned}$$

Hence we have $\Omega_\alpha^2(\Phi(\alpha))=\frac{1}{2}\delta_1(\nabla\nabla C)$. This completes the proof of the assertion (2). Q. E. D.

LEMMA 5. *Let* $(\alpha, \beta, \gamma)\in Q^{(2)}$. *Put* $\kappa=\gamma^1-\chi_\alpha$. *Then* $\kappa\in \boldsymbol{T}^*\otimes S^3\boldsymbol{T}^*\otimes\boldsymbol{N}$ *and*

(B2) $\Omega((w\otimes v)\lrcorner\kappa, \alpha)+\Omega(v\lrcorner\beta^1, w\lrcorner\beta^1)=\frac{1}{2}\nabla_w\nabla_v C$;

(B3) $\tilde{\delta}_1 \varkappa = \Phi(\alpha)$.

PROOF. Since both γ^1 and χ_α belong to $\otimes^2 \boldsymbol{T}^* \otimes S^2 \boldsymbol{T}^* \otimes \boldsymbol{N}$, we have $\varkappa \in \otimes^2 \boldsymbol{T}^* \otimes S^2 \boldsymbol{T}^* \otimes \boldsymbol{N}$. Moreover by (A8) we have

$$\begin{aligned}
&\varkappa(w, v, u, x) - \varkappa(w, u, v, x) \\
&= \gamma^1(w, v, u, x) - \gamma^1(w, u, v, x) \\
&\quad - \chi_\alpha(w, v, u, x) + \chi_\alpha(w, u, v, x) \\
&= -\alpha(w, R(v, u)x) - \alpha(w, S_\alpha(u, x)v - S_\alpha(v, x)u) \\
&= -\alpha(w, R(v, u)x) - \alpha(w, R(u, v)x) \\
&= 0.
\end{aligned}$$

This implies that $\varkappa \in \boldsymbol{T}^* \otimes S^3 \boldsymbol{T}^* \otimes \boldsymbol{N}$. The equality (B2) is immediately obtained by (A10) and (1) of Lemma 4. Finally we prove (B3). By (A9) we have

$$\begin{aligned}
\tilde{\delta}_1 \varkappa(w, v, u, x) &= \gamma^1(w, v, u, x) - \gamma^1(v, w, u, x) \\
&\quad - \chi_\alpha(w, v, u, x) + \chi(v, w, u, x) \\
&= -\alpha(R(w, v)u, x) - \alpha(u, R(w, v)x) \\
&\quad - \chi_\alpha(w, v, u, x) + \chi_\alpha(v, w, u, x) \\
&= \Phi(\alpha)\,(w, v, u, x).
\end{aligned}$$

Hence we have $\tilde{\delta}_1 \varkappa = \Phi(\alpha)$. Q. E. D.

We are now in a position to prove

THEOREM 6. *An element* $(\alpha, \beta) \in Q^{(1)}$ *is prolonged to an element of* $Q^{(2)}$ *if and only if there exists* $\varkappa \in \boldsymbol{T}^* \otimes S^3 \boldsymbol{T}^* \otimes \boldsymbol{N}$ *satisfying* (B2) *and* (B3).

PROOF. Let $(\alpha, \beta) \in Q^{(1)}$. We assume that there exists $\varkappa \in \boldsymbol{T}^* \otimes S^3 \boldsymbol{T}^* \otimes \boldsymbol{N}$ satisfying (B2) and (B3). Put $\gamma^1 = \varkappa + \chi_\alpha$. Then it is easily checked that $\gamma^1 \in \otimes^2 \boldsymbol{T}^* \otimes S^2 \boldsymbol{T}^* \otimes \boldsymbol{N}$ satisfies (A8), (A9) and (A10). Let γ^0 be the element of $S^2 \boldsymbol{T}^* \otimes S^2 \boldsymbol{T}^* \otimes \omega(\boldsymbol{T})$ determined uniquely by (A7). Then by (B1) we know that γ^0 satisfies (A6). Thus putting $\gamma = \gamma^0 + \gamma^1$, we have $(\alpha, \beta, \gamma) \in Q^{(2)}$. The converse is obvious. Q. E. D.

We call the system composed of (B2) and (B3) the *second Gauss-Codazzi equation*.

Let $\alpha \in P^{(1)}$. We denote by $\mathfrak{g}_\alpha$ the kernel of the linear mapping Ω_α, i.

e., $\mathfrak{g}_\alpha=\{\xi\in S^2\boldsymbol{T}^*\otimes\boldsymbol{N}|\ \Omega(\xi,\alpha)=0\}$. If $\alpha\in Q$, $\mathfrak{g}_\alpha$ is called the *symbol* of the differential equation Q at α. For each positive integer k, we define the k-th prolongation $\mathfrak{g}_\alpha^{(k)}$ of $\mathfrak{g}_\alpha$ by $\mathfrak{g}_\alpha^{(k)}=S^k\boldsymbol{T}^*\otimes\mathfrak{g}_\alpha\cap S^{k+2}\boldsymbol{T}^*\otimes\boldsymbol{N}$. Then it is easy to see that

$$\mathfrak{g}_\alpha^{(k)}=\mathrm{Ker}\ \Omega_\alpha^k\cap S^{k+2}\boldsymbol{T}^*\otimes\boldsymbol{N}.$$

We note that a solution β^1 of the first Gauss-Codazzi equation, if exists, is uniquely determined up to modulo $\mathfrak{g}_\alpha^{(1)}$.

We now consider a special case.

PROPOSITION 7. *Let (M,g) be a locally Riemannian symmetric space and let $\alpha\in Q$ satisfy $\mathfrak{g}_\alpha^{(1)}=0$. Then α is uniquely prolonged to an element $(\alpha,\beta)\in Q^{(1)}$. And (α,β) is prolonged to an element of $Q^{(2)}$ if and only if α satisfies the equality $\Phi(\alpha)=0$.*

PROOF. Putting $\nabla C=0$ into (B1), we have $\beta^1\in\mathfrak{g}_\alpha^{(1)}$. Since $\mathfrak{g}_\alpha^{(1)}=0$, it follows that $\beta^1=0$. This implies the uniqueness of $(\alpha,\beta)\in Q^{(1)}$. Now suppose that $(\alpha,\beta)\in P^{(1)}$ is prolonged to an element of $Q^{(2)}$. Then there exists an element $\varkappa\in\boldsymbol{T}^*\otimes S^3\boldsymbol{T}^*\otimes\boldsymbol{N}$ satisfying (B2) and (B3) (Theorem 6). From (B2) we have $\varkappa\in\otimes^2\boldsymbol{T}^*\otimes\mathfrak{g}_\alpha\cap\boldsymbol{T}^*\otimes S^3\boldsymbol{T}^*\otimes\boldsymbol{N}=\boldsymbol{T}^*\otimes\mathfrak{g}_\alpha^{(1)}=0$. Hence we have $\varkappa=0$. Consequently it holds that $\Phi(\alpha)=0$. Conversely if $\Phi(\alpha)=0$, then $\varkappa=0$ satisfies both (B2) and (B3) and hence (α,β) can be prolonged to an element of $Q^{(2)}$ (Theorem 6). Q. E. D.

As is seen in the above proposition, the quantity $\Phi(\alpha)$ plays an important role. It seems an interesting problem to determine isometric immersions satisfying $\Phi(\alpha)=0$. In the following, we show examples of isometric immersions satisfying $\Phi(\alpha)=0$. Particularly, Example 3 indicates that the equation $\Phi(\alpha)=0$ is independent from the Gauss equation (A2) in general.

EXAMPLE 2. Let $M=G/K$ be a symmetric $\boldsymbol{R}$-space. Then there exists a standard isometric imbedding of M into a euclidean space induced by a group representation of G associated with a graded Lie algebra of the first kind. The second fundamental form α of this isometric imbedding satisfies the equality $\Phi(\alpha)=0$ at every point of M (see Proposition 5. 5 in [14]).

EXAMPLE 3. Let (M,g) be a connected 2-dimensional Riemannian manifold of constant curvature $K(\neq 0)$. As is known, there are infinitely many local isometric immersions of (M,g) into $\boldsymbol{R}^3$ not equivalent to each other under the action of euclidean motions of $\boldsymbol{R}^3$. Now let us assume that an isometric immersion $\boldsymbol{f}: M\longrightarrow\boldsymbol{R}^3$ satisfies the equation $\Phi(\alpha)=0$ at each point of M. Let $p \in M$ and let $\boldsymbol{n}$ be a unit normal vector at $\boldsymbol{f}(p)$. Then the second fundamental form α_p can be written in the form $\alpha_p = A \otimes \boldsymbol{n}$, where $A \in S^2 \boldsymbol{T}^*$. We choose an orthonormal basis $\{e_1, e_2\}$ of $\boldsymbol{T}$ such that $A(e_i, e_j) = \lambda_i \delta_{ij}$. Then we have

$$K = \lambda_1 \lambda_2 \,; \; R(e_i, e_j) e_i = -K e_j \; (i \neq j) \,;$$

$$S_\alpha(e_i, e_i) e_i = -\lambda_i^2 e_i \; (i = 1, 2),$$

$$S_\alpha(e_i, e_i) e_j = -\lambda_1 \lambda_2 e_j \,; \; S_\alpha(e_i, e_j) e_k = 0 \; (i \neq j).$$

By a simple calculation we obtain

$$\Phi(\alpha)(e_1, e_2, e_i, e_i) = 0 \;\; (i = 1, 2) \,; \; \Phi(\alpha)(e_1, e_2, e_1, e_2) = K(\lambda_2 - \lambda_1) \boldsymbol{n}.$$

Consequently we have $\lambda_1 = \lambda_2$ and hence $K = \lambda_1^2 > 0$. Therefore (M, g) is a space of constant positive curvature and $\boldsymbol{f}(M)$ is totally umbilic in $\boldsymbol{R}^3$. Hence $\boldsymbol{f}(M)$ is contained in a sphere in $\boldsymbol{R}^3$.

REMARK. The assertion of Theorem 6 also holds even if the equation (B3) is replaced by the following weaker condition

(B3′) $\quad \tilde{\delta}_1 \varkappa = \Phi(\alpha) \qquad \mathrm{mod}\ \tilde{\delta}_1(\boldsymbol{T}^* \otimes \mathfrak{g}_\alpha^{(1)}).$

In fact if there is an element $\varkappa_0 \in \boldsymbol{T}^* \otimes \mathfrak{g}_\alpha^{(1)}$ such that $\tilde{\delta}_1 \varkappa_0 = \tilde{\delta}_1 \varkappa - \Phi(\alpha)$, we put $\varkappa' = \varkappa - \varkappa_0$. Then it is easy to see that $\varkappa'$ satisfies both the equations (B2) and (B3).

Generally for an element $\varkappa \in \boldsymbol{T}^* \otimes S^3 \boldsymbol{T}^* \otimes \boldsymbol{N}$ satisfying (B2), it holds that $\tilde{\delta}_1 \varkappa - \Phi(\alpha) \in \Lambda^2 \boldsymbol{T}^* \otimes \mathfrak{g}_\alpha$. This fact follows from $\Omega_\alpha^2(\Phi(\alpha)) = \frac{1}{2} \tilde{\delta}_1(\nabla \nabla C)$ ((2) of Lemma 4) and from the equality $\Omega_\alpha^2(\tilde{\delta}_1 \varkappa) = \frac{1}{2} \delta_1(\nabla \nabla C)$ that can be obtained by (B2). Further by the definition of $\tilde{\delta}_2$ and (2) of Lemma 4, we have $\tilde{\delta}_2(\tilde{\delta}_1 \varkappa - \Phi(\alpha)) = 0$. Now consider the following complex:

$$\boldsymbol{T}^* \otimes \mathfrak{g}_\alpha^{(1)} \xrightarrow{\tilde{\delta}_1} \Lambda^2 \boldsymbol{T}^* \otimes \mathfrak{g}_\alpha \xrightarrow{\tilde{\delta}_2} \Lambda^3 \boldsymbol{T}^* \otimes \boldsymbol{T}^* \otimes \boldsymbol{N}.$$

We denote by $\epsilon(\alpha, \beta)$ the cohomology class at $\Lambda^2 \boldsymbol{T}^* \otimes \mathfrak{g}_\alpha$ determined by $\tilde{\delta}_1 \varkappa - \Phi(\alpha)$. The equation (B3′) requires that $\epsilon(\alpha, \beta) = 0$. Thus in case the cohomology group of the above complex vanishes, the equation (B3′) is automatically satisfied.

The cohomology class $\epsilon(\alpha, \beta)$ that does not depend on the choice of $\varkappa$, is called the *curvature* of the differential equation $Q^{(1)}$ (see [10]).

§C. Isometric immersions of $P^2(\boldsymbol{C})$.

As an application of the discussions in §**B**, we study the problem of isometric immersions of 2-dimensional *complex projective space* $P^2(\boldsymbol{C})$ into the euclidean space $\boldsymbol{R}^7$. Here we assume that $P^2(\boldsymbol{C})$ is endowed with the *Fubini-Study metric* with constant holomorphic sectional curvature 2.

Let $\boldsymbol{T}^c$(resp. $\boldsymbol{N}^c$) be the complexification of $\boldsymbol{T}$(resp. $\boldsymbol{N}$). We extend the complex structure J to a complex linear endomorphism of $\boldsymbol{T}^c$ and extend the inner product g (resp. $\langle\ ,\ \rangle$) to a non-degenerate complex symmetric bilinear form of T^c(resp. $\boldsymbol{N}^c$) in a natural manner. For $z\in \boldsymbol{T}^c$(resp. $w\in\boldsymbol{N}^c$) we denote by $\bar{z}$ (resp. $\bar{w}$) the complex conjugate of z (resp. w) with respect ot $\boldsymbol{T}$(resp. $\boldsymbol{N}$). We choose and fix a basis $\{e_1, e_2, e_{\bar 1}, e_{\bar 2}\}$ of $\boldsymbol{T}^c$ such that:

$$Je_i=\sqrt{-1}\,e_i,\ e_{\bar i}=\bar{e}_i(i=1,2)\,;$$

$$g(e_i, e_{\bar i})=1(i=1,\ 2)\,;\ g(e_1, e_{\bar 2})=g(e_2, e_{\bar 1})=0.$$

In the following discussions we promise that the indices $i, j,\cdots$ run over the range $1, 2$ and the indices λ, μ, $\nu\cdots$ run over the range $1, 2, \bar 1, \bar 2$ and promise that $\bar{\bar 1}=1, \bar{\bar 2}=2$.

Now let us solve the Gauss equation associated with isometric immersions of $(P^2(\boldsymbol{C}), g)$ into $\boldsymbol{R}^7$. For this purpose we extend the Riemannian curvature tensor C to an element of $\Lambda^2\boldsymbol{T}^{*c}\otimes_c\Lambda^2\boldsymbol{T}^{*c}$ and the unknown $\alpha\in S^2\boldsymbol{T}^*\otimes\boldsymbol{N}$ to an element of $S^2\boldsymbol{T}^{*c}\otimes_c\boldsymbol{N}^c$ in a natural way. We put $C_{\lambda\mu\nu\sigma}= C(e_\lambda, e_\mu, e_\nu, e_\sigma)$ and $\alpha_{\lambda\mu}=\alpha(e_\lambda, e_\mu)$. Then we have $\overline{C_{\lambda\mu\nu\sigma}}=C_{\bar\lambda\bar\mu\bar\nu\bar\sigma}$; $\overline{\alpha_{\lambda\mu}}=\alpha_{\bar\lambda\bar\mu}$ and

$$C_{i\bar j k\bar l}=-C_{\bar j i k\bar l}=-C_{i\bar j\bar l k}=-(\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk})\,;$$

$$C_{\lambda\mu\nu\sigma}=0 \qquad \text{otherwise,}$$

(see [16]). The Gauss equation $\Omega(\alpha, \alpha)=C$ is then given by the following system of equations:

(C1212) $\quad C_{1212}=\langle\alpha_{11}, \alpha_{22}\rangle-\langle\alpha_{12},\alpha_{12}\rangle=0.$

(C121$\bar 1$) $\quad C_{121\bar 1}=\langle\alpha_{11}, \alpha_{2\bar 1}\rangle-\langle\alpha_{1\bar 1},\alpha_{21}\rangle=0.$

(C121$\bar 2$) $\quad C_{121\bar 2}=\langle\alpha_{11}, \alpha_{2\bar 2}\rangle-\langle\alpha_{1\bar 2},\alpha_{21}\rangle=0.$

(C122$\bar 1$) $\quad C_{122\bar 1}=\langle\alpha_{12}, \alpha_{2\bar 1}\rangle-\langle\alpha_{1\bar 1},\alpha_{22}\rangle=0.$

(C122$\bar 2$) $\quad C_{122\bar 2}=\langle\alpha_{12}, \alpha_{2\bar 2}\rangle-\langle\alpha_{1\bar 2}, \alpha_{22}\rangle=0.$

(C1$\bar 1$1$\bar 1$) $\quad C_{1\bar 11\bar 1}=\langle\alpha_{11}, \alpha_{\bar 1\bar 1}\rangle-\langle\alpha_{1\bar 1}, \alpha_{\bar 11}\rangle=-2.$

(C2$\bar{2}$2$\bar{2}$) $C_{2\bar{2}2\bar{2}}=\langle\alpha_{22},\alpha_{\bar{2}\bar{2}}\rangle-\langle\alpha_{2\bar{2}},\alpha_{\bar{2}2}\rangle=-2.$

(C1$\bar{1}$2$\bar{2}$) $C_{1\bar{1}2\bar{2}}=\langle\alpha_{12},\alpha_{\bar{1}\bar{2}}\rangle-\langle\alpha_{1\bar{2}},\alpha_{\bar{1}2}\rangle=-1.$

(C1$\bar{2}$2$\bar{1}$) $C_{1\bar{2}2\bar{1}}=\langle\alpha_{12},\alpha_{\bar{2}\bar{1}}\rangle-\langle\alpha_{1\bar{1}},\alpha_{\bar{2}2}\rangle=-1.$

(C1$\bar{1}$1$\bar{2}$) $C_{1\bar{1}1\bar{2}}=\langle\alpha_{11},\alpha_{\bar{1}\bar{2}}\rangle-\langle\alpha_{1\bar{2}},\alpha_{\bar{1}1}\rangle=0.$

(C1$\bar{2}$2$\bar{2}$) $C_{1\bar{2}2\bar{2}}=\langle\alpha_{12},\alpha_{\bar{2}\bar{2}}\rangle-\langle\alpha_{1\bar{2}},\alpha_{\bar{2}2}\rangle=0.$

(C1$\bar{2}$1$\bar{2}$) $C_{1\bar{2}1\bar{2}}=\langle\alpha_{11},\alpha_{\bar{2}\bar{2}}\rangle-\langle\alpha_{1\bar{2}},\alpha_{1\bar{2}}\rangle=0.$

It seems difficult to obtain all the solutions of the system of equations (C1212)-(C1$\bar{2}$1$\bar{2}$). In the following we solve the above system under the additional condition $\alpha_{11}=0$.

PROPOSITION 8. *For each solution $\alpha\in S^2\boldsymbol{T}^*\otimes\boldsymbol{N}$ of the Gauss equation satisfying $\alpha_{11}=0$, there exists an orthonoraml basis $\{\xi_1,\xi_2,\xi_3\}$ of $\boldsymbol{N}$ and complex numbers a, b, p with $|a|^2\geqq 3$, $|b|^2=|a|^2+1$ and $|p|^2=(1/2)|a|^2|b|^2(|a|^2-3)$ such that*:

$$\alpha_{11}=0,\ \alpha_{12}=a\eta,\ \alpha_{22}=c\xi+k\eta+m\zeta\,;$$

$$\alpha_{1\bar{1}}=\sqrt{2}\,\xi,\ \alpha_{1\bar{2}}=b\eta,\ \alpha_{2\bar{2}}=d\xi+p\eta+\bar{p}\zeta\,;$$

where $\xi=\xi_1$, $\eta=(1/\sqrt{2})(\xi_2-\sqrt{-1}\,\xi_3)$, $\zeta=(1/\sqrt{2})(\xi_2+\sqrt{-1}\,\xi_3)$, $c=(1/\sqrt{2})a\bar{b}$, $d=(1/\sqrt{2})|b|^2$, $k=(\bar{b}/\bar{a})p$, $m=(a/b)\bar{p}$.

PROOF. We first observe (C1$\bar{1}$1$\bar{1}$). Since $\alpha_{11}=0$, we have $\langle\alpha_{1\bar{1}},\alpha_{1\bar{1}}\rangle=2$. Put $\xi_1=(1/\sqrt{2})\alpha_{1\bar{1}}$. Then we have $\langle\xi_1,\xi_1\rangle=1$, $\bar{\xi}_1=\xi_1$ and hence $\xi_1\in\boldsymbol{N}$. We select two vectors ξ_2,ξ_3 of $\boldsymbol{N}$ so that $\{\xi_1,\xi_2,\xi_3\}$ forms an orthonormal basis of $\boldsymbol{N}$. By (C121$\bar{1}$) and (C1$\bar{1}$1$\bar{2}$) we know that α_{12} and $\alpha_{1\bar{2}}$ can be represented by linear combinations of ξ_2,ξ_3. Now we write $\alpha_{12}=a_1\xi_2+a_2\xi_3$, $\alpha_{1\bar{2}}=b_1\xi_2+b_2\xi_3$. Then by (C1212), (C1$\bar{2}1\bar{2}$) and (C121$\bar{2}$) we have $a_1^2+a_2^2=0$, $b_1^2+b_2^2=0$ and $a_1b_1+a_2b_2=0$. By exchanging ξ_3 and $-\xi_3$ if necessary, we can assume that $b_2=-\sqrt{-1}\,b_1$. Since $\alpha_{1\bar{2}}\neq 0$ (see (C1$\bar{1}$2$\bar{2}$)), we have $b_1\neq 0$ and hence $a_2=-\sqrt{-1}\,a_1$. Thus setting $\eta=(1/\sqrt{2})(\xi_2-\sqrt{-1}\,\xi_3)$, we have $\alpha_{12}=a\eta$, $\alpha_{1\bar{2}}=b\eta$ $(a=\sqrt{2}\,a_1,\ b=\sqrt{2}\,b_1)$. Put $\zeta=\bar{\eta}$. Then it holds that $\langle\eta,\eta\rangle=\langle\zeta,\zeta\rangle=0$, $\langle\eta,\zeta\rangle=1$ and that $\{\xi,\eta,\zeta\}$ $(\xi=\xi_1)$ forms a basis of $\boldsymbol{T}^c$.

Now let us set $\alpha_{22}=c\xi+k\eta+m\zeta$ and $\alpha_{2\bar{2}}=d\xi+p\eta+\bar{p}\zeta$, where $c,k,m,p\in\boldsymbol{C}$ and $d\in\boldsymbol{R}$. Then we have:

(C122$\bar{1}$) $a\bar{b}-\sqrt{2}\,c=0.$

(C122$\bar{2}$) $a\bar{p}-bm=0.$

(C2$\bar{2}$2$\bar{2}$) $|c|^2+|k|^2+|m|^2-(d^2+2|p|^2)=-2.$

(C1$\bar{1}$2$\bar{2}$) $|a|^2-|b|^2=-1.$

(C1$\bar{2}$2$\bar{1}$) $|a|^2-\sqrt{2}\,d=-1.$

(C1$\bar{2}$2$\bar{2}$) $a\bar{k}-b\bar{p}=0.$

First we assume that $a=0$. Then by the above equations except for (C2$\bar{2}$2$\bar{2}$), we obtain $|b|^2=1, c=0, d=1/\sqrt{2}, m=0$ and $p=0$. Putting these into (C2$\bar{2}$2$\bar{2}$), we have $|k|^2=-(3/2)$. This is a contradiction. Consequently we have $a\neq 0$ and hence $|b|^2=|a|^2+1, c=(1/\sqrt{2})a\bar{b}, d=(1/\sqrt{2})|b|^2, m=(a/b)\bar{p}, k=(\bar{b}/\bar{a})p$ and $|p|^2=(1/2)|a|^2|b|^2(|a|^2-3)$. By the last equality, we obtain the inequality $|a|^2\geqq 3$. Q. E. D.

In what follows we fix an orthonormal basis $\{\xi_1, \xi_2, \xi_3\}$ of $\boldsymbol{N}$. Let us denote by $\alpha(a, b, p)$ the solution given in Proposition 8, where a, b, p are complex numbers satisfying the relations stated there. We note that the solutions of the form $\alpha=\alpha(a, b, 0)$ have been obtained in Agaoka [2]. $\alpha(a, b, p)$ is a special solution in the sense that it satisfies the additional condition $\alpha_{11}=0$, but it turns out that $\alpha(a, b, p)$ is generic in the set of solutions $\Omega(\alpha, \alpha)=C$ (see Proposition 9 and Remark after it).

Let $\alpha=\alpha(a, b, p)$. We define a basis $\{E_1, E_{\bar{1}}, E_2, E_{\bar{2}}\}$ of $\boldsymbol{T}^c$ by setting $E_1=e_1, E_{\bar{1}}=e_{\bar{1}}, E_2=be_2-ae_{\bar{2}}$ and $E_{\bar{2}}=-\bar{a}e_2+\bar{b}e_{\bar{2}}$. Clearly we have $\overline{E_\lambda}=E_{\bar{\lambda}}$ and $\overline{\alpha(E_\lambda, E_\mu)}=\alpha(E_{\bar{\lambda}}, E_{\bar{\mu}})$. Moreover we have:

$$\alpha(E_1, E_1)=0\,;\ \alpha(E_1, E_2)=0\,;\ \alpha(E_2, E_2)=\sqrt{2}\,A\,\xi+P\eta\,;$$

$$\alpha(E_1, E_{\bar{1}})=\sqrt{2}\,\xi\,;\ \alpha(E_1, E_{\bar{2}})=\eta\,;\ \alpha(E_2, E_{\bar{2}})=\sqrt{2}B\xi,$$

where we set $A=-ab/2,\ B=|b|^2/2,\ P=p/\overline{ab}$.

PROPOSITION 9. *Let $\alpha=\alpha(a, b, p)$. Then:*

(1) $\dim \mathfrak{g}_\alpha=10.$

(2) $$\dim \mathfrak{g}_\alpha^{(1)}=\begin{cases} 0 & \text{if } p\neq 0. \\ 4 & \text{if } p=0. \end{cases}$$

In case $p=0$, the complexification $\mathfrak{g}_\alpha^{(1)c}$ of $\mathfrak{g}_\alpha^{(1)}$ is generated by the following four elements: $(E_{\bar{1}}^*)^3\otimes\zeta, (E_{\bar{1}}^*)^3\otimes\eta, (E_2^*)^3\otimes\eta, (E_{\bar{2}}^*)^3\otimes\zeta$, where $\{E_{\bar{1}}^*, E_{\bar{1}}^*, E_2^*, E_{\bar{2}}^*\}$ is the dual basis of $\{E_1, E_{\bar{1}}, E_2, E_{\bar{2}}\}$ and $(E_{\bar{1}}^*)^3=E_{\bar{1}}^*\otimes E_{\bar{1}}^*\otimes E_{\bar{1}}^*$, etc.

The results corresponding to the case $p=0$ were first obtained by Agaoka. In the next section we give a proof of the above proposition in a unified manner.

REMARK. Let W_1 be the set of all $\alpha \in P^{(1)}$ satisfying $\dim \mathfrak{g}_\alpha = 10$. Then we have: 1) W_1 is an open dense subset of $P^{(1)}$ such that $P^{(1)} \setminus W_1$ is an algebraic subvariety of $P^{(1)}$; 2) $Q \cap W_1$ forms a submanifold of $P^{(1)}$. In fact since $\mathfrak{g}_\alpha$ is the kernel of the map $\Omega_\alpha : S^2 \boldsymbol{T}^* \otimes \boldsymbol{N} \longrightarrow K(\boldsymbol{T})$, it follows that $\dim \mathfrak{g}_\alpha \geqq \dim S^2 \boldsymbol{T}^* \otimes \boldsymbol{N} - \dim K(\boldsymbol{T}) = 30 - 20 = 10$ (see [14]). Moreover we know that the function $P^{(1)} \ni \alpha \longrightarrow \dim \mathfrak{g}_\alpha \in \boldsymbol{Z}$ is upper semi-continuous. Hence W_1 is an open subset (not empty because $\alpha(a, b, p) \in W_1$) of $P^{(1)}$ and $P^{(1)} \setminus W_1$ is an algebraic subvariety of $P^{(1)}$. The assertion 2) follows by the fact that the rank of the map $P^{(1)} \ni \alpha \longrightarrow \Omega(\alpha, \alpha) \in K(\boldsymbol{T})$ is constant around each $\alpha \in W_1$. Consequently, our solutions $\alpha(a, b, p)$ are generic points of Q. Let W_2 be the set of all $\alpha \in P^{(1)}$ satisfying $\dim \mathfrak{g}_\alpha = 10$ and $\mathfrak{g}_\alpha{}^{(1)} = 0$. Then W_2 has the above properties 1) and 2), where W_1 is replaced by W_2. (Recall that $\mathfrak{g}_\alpha{}^{(1)}$ is the kernel of the map Ω^1_α: $S^3 \boldsymbol{T}^* \otimes \boldsymbol{N} \longrightarrow K^{(1)}(\boldsymbol{T})$.) In this sense Agaoka's solutions $\alpha(a, b, 0)$ are singular points, i. e., $\alpha(a, b, 0) \in W_1 \setminus W_2$.

We now prove the main theorem of this section.

THEOREM 10. *Let $\alpha = \alpha(a, b, p)$. Then any element $(\alpha, \beta) \in Q^{(1)}$ cannot be prolonged to an element of $Q^{(2)}$. Accordingly there is no isometric immersion of $P^2(\boldsymbol{C})$ into $\boldsymbol{R}^7$ whose second fundamental form is equal to α.*

We first prove

LEMMA 11. *Let $\alpha = \alpha(a, b, p)$. Then $\Phi(\alpha)(E_1, E_{\bar{1}}, E_1, E_{\bar{2}}) \neq 0$.*

PROOF. By simple calculations we have:

1) $S_\alpha(E_1, E_1) E_{\bar{1}} = 0$, $S_\alpha(E_1, E_{\bar{1}}) E_1 = -2E_1$;

2) $S_\alpha(E_1, E_{\bar{2}}) E_{\bar{1}} = -2\bar{a}\bar{b} E_2 - (|a|^2 + |b|^2) E_{\bar{2}}$,

$S_\alpha(E_{\bar{1}}, E_{\bar{2}}) E_1 = 0$;

3) $S_\alpha(E_1, E_{\bar{2}}) E_1 = 0$.

Thus we have

$$R(E_1, E_{\bar{1}}) E_1 = S_\alpha(E_1, E_1) E_{\bar{1}} - S_\alpha(E_{\bar{1}}, E_1) E_1 = 2E_1;$$

$$R(E_1, E_{\bar{1}}) E_{\bar{2}} = S_\alpha(E_1, E_{\bar{2}}) E_{\bar{1}} - S_\alpha(E_{\bar{1}}, E_{\bar{2}}) E_1$$
$$= -2\bar{a}\bar{b}\, \mathrm{E}_2 - (|a|^2 + |b|^2) E_{\bar{2}}.$$

Therefore

$$\begin{aligned}\Phi(\alpha)(E_1, E_{\bar 1}, E_1, E_{\bar 2}) &= -\alpha(R(E_1, E_{\bar 1})E_1, E_{\bar 2}) - \alpha(E_1, R(E_1, E_{\bar 1})E_{\bar 2}) \\ &\quad + \alpha(S_\alpha(E_1, E_{\bar 2})E_1, E_{\bar 1}) - \alpha(E_1, S_\alpha(E_1, E_{\bar 2})E_{\bar 1} \\ &= -\alpha(2E_1, E_{\bar 2}) - \alpha(E_1, -2\bar a \bar b E_2 \\ &\quad -(|a|^2+|b|^2)E_{\bar 2}) \\ &\quad + \alpha(0, E_{\bar 1}) - \alpha(E_1, -2\bar a \bar b E_{\bar 2} - (|a|^2+|b|^2)E_{\bar 2}) \\ &= 4|a|^2\eta \neq 0. \end{aligned}$$

Q. E. D.

PROOF OF THEOREM 10. First consider the case $p\neq 0$. Then our assertion immediately follows from Proposition 7, because we have $\mathfrak{g}_\alpha^{(1)}=0$ (Proposition 9) and $\Phi(\alpha)\neq 0$ (Lemma 11).

We next consider the case $p=0$. Suppose that there exist $\beta^1\in\mathfrak{g}_\alpha^{(1)}$ and $\varkappa\in \boldsymbol{T}^*\otimes S^3\boldsymbol{T}^*\otimes \boldsymbol{N}$ satisfying (B2) and (B3). We note that by using complex numbers X and Y, β^1 can be written in the form

$$\beta^1 = X(E_1^*)^3\otimes\zeta + \bar X(E_{\bar 1}^*)^3\otimes\eta + Y(E_2^*)^3\otimes\eta + \bar Y(E_{\bar 2}^*)^3\otimes\zeta,$$

(Proposition 9). Then clearly we have $\Omega(E_1 \lrcorner \beta^1, E_1 \lrcorner \beta^1)=0$. Hence by (B2) we have $\Omega((E_1\otimes E_1)\lrcorner \varkappa, \alpha)=0$. Consequently we have $\Omega((E_1\otimes E_1)\lrcorner \varkappa, \alpha)(E_{\bar 1}, E_{\bar 2}, E_{\bar 1}, E_{\bar 2})=\frac{\sqrt{2}}{2}\bar A\langle\varkappa(E_1, E_1, E_{\bar 1}, E_{\bar 1}), \xi\rangle=0$. On the other hand since $\Omega((E_1\otimes E_{\bar 1})\lrcorner \varkappa, \alpha)(E_1, E_{\bar 1}, E_1, E_{\bar 1})=-\sqrt{2}\langle\varkappa(E_1, E_{\bar 1}, E_1, E_{\bar 1}), \xi\rangle$, it follows that

$$\Omega((E_1\otimes E_{\bar 1})\lrcorner \varkappa, \alpha)(E_1, E_{\bar 1}, E_1, E_{\bar 1})=0.$$

Therefore by (B2) we have

$$\begin{aligned} 0 &= 2\Omega(E_1\lrcorner\beta^1, E_1\lrcorner\beta^1)(E_1, E_{\bar 1}, E_1, E_{\bar 1}) \\ &= \langle\beta^1(E_1, E_1, E_1), \beta^1(E_{\bar 1}, E_{\bar 1}, E_{\bar 1})\rangle + \langle\beta^1(E_1, E_{\bar 1}, E_{\bar 1}), \beta^1(E_{\bar 1}, E_1, E_1)\rangle \\ &\quad -\langle\beta^1(E_1, E_1, E_{\bar 1}), \beta^1(E_{\bar 1}, E_{\bar 1}, E_1)\rangle - \langle\beta^1(E_1, E_{\bar 1}, E_1), \beta_1(E_{\bar 1}, E_1, E_{\bar 1})\rangle \\ &= |X|^2. \end{aligned}$$

Hence we have $X=0$. Consequently $E_1\lrcorner\beta^1=E_{\bar 1}\lrcorner\beta^1=0$. Thus by (B2) we have $\Omega((E_1\otimes E_\lambda)\lrcorner\varkappa, \alpha)=\Omega((E_{\bar 1}\otimes E_\lambda)\lrcorner\varkappa, \alpha)=0$ $(\lambda=1, \bar 1, 2, \bar 2)$. This implies that $E_1\lrcorner\varkappa$ and $E_{\bar 1}\lrcorner\varkappa\in\mathfrak{g}_\alpha^{(1)c}$. In view of the basis of $\mathfrak{g}_\alpha^{(1)c}$, we know that $\varkappa(E_1, E_{\bar 1}, E_1, E_{\bar 2})=\varkappa(E_{\bar 1}, E_1, E_1, E_{\bar 2})=0$. Therefore by (B3) we have $\Phi(\alpha)(E_1, E_{\bar 1}, E_1, E_{\bar 2})=0$. This contradicts Lemma 11. Therefore our assertion holds even for the case $p=0$. Q. E. D.

REMARK. Let W be the subset of $P^{(1)}$ consisting of all $\alpha\in P^{(1)}$ such that $\dim \mathfrak{g}_\alpha=10$, $\dim \mathfrak{g}_\alpha^{(1)}=0$ and $\Phi(\alpha)\neq 0$. Then it is obvious that W forms an open dense subset of $P^{(1)}$. By Proposition 7 we know that for any $\alpha\in Q\cap W$ there is no isometric immersion $\boldsymbol{f}$ of $P^2(\boldsymbol{C})$ into $\boldsymbol{R}^7$ whose second fundamental form at $p\in P^2(\boldsymbol{C})$ is equal to α.

§D. Proof of Proposition 9.

Let $\alpha=\alpha(a, b, p)\in Q$. We define a linear map $\Theta: \boldsymbol{N}\longrightarrow S^2\boldsymbol{T}^*$ by setting $\Theta(v)(x, y)=\langle v, \alpha(x, y)\rangle$ for $v\in\boldsymbol{N}$, $x, y\in\boldsymbol{T}$. Since the vectors $\{\alpha(x, y)|x, y\in\boldsymbol{T}\}$ generate $\boldsymbol{N}$, it follows that Θ is injective. Put $\mathfrak{n}=\Theta(\boldsymbol{N})$. We set $\mathfrak{h}=\mathfrak{n}+\Lambda^2\boldsymbol{T}^*(\subset\otimes^2\boldsymbol{T}^*)$ and denote by π the canonical projection of $\mathfrak{h}$ onto $\mathfrak{n}$. For each non-negative integer k, we mean by $\mathfrak{h}^{(k)}$ the k-th prolongation of $\mathfrak{h}$, i.e., $\mathfrak{h}^{(k)}=S^{k+1}\boldsymbol{T}^*\otimes\boldsymbol{T}^*\cap S^k\boldsymbol{T}^*\otimes\mathfrak{h}$ and by $\mathfrak{q}^k$ the image of $\mathfrak{h}^{(k)}$ via the map $\pi^k=\mathbf{1}_k\otimes\pi$. Then for each $k\geqq 2$, we have

$$\mathfrak{h}^{(k)}\cong\mathfrak{q}^k; \quad \mathfrak{q}^k=\Theta^k(\mathfrak{g}_\alpha^{(k-2)}),$$

where we put $\Theta^k=\mathbf{1}_k\otimes\Theta$ (see [14]). Since Θ^k is injective, we have $\dim\mathfrak{h}^{(k)}=\dim\mathfrak{g}_\alpha{}^{(k-2)}$. Therefore, to determine $\dim\mathfrak{g}_\alpha$ and $\dim\mathfrak{g}_\alpha^{(1)}$, it suffices to determine $\dim\mathfrak{h}^{(2)}$ and $\dim\mathfrak{h}^{(3)}$.

Let $\mathrm{Ann}(\mathfrak{h}^{(k)})(k=0, 1, 2,\cdots)$ be the *annihilator* of $\mathfrak{h}^{(k)}$ in $S^{k+1}\boldsymbol{T}\otimes\boldsymbol{T}$, the dual space of $S^{k+1}\boldsymbol{T}^*\otimes\boldsymbol{T}^*$. We define a linear map $\Psi^k: S^k\boldsymbol{T}\otimes\mathrm{Ann}(\mathfrak{h})\longrightarrow S^{k+1}\boldsymbol{T}\otimes\boldsymbol{T}$ by setting $\Psi^k(w\otimes\tau)=w\odot\tau$ for $w\in S^k\boldsymbol{T}$ and $\tau\in\mathrm{Ann}(\mathfrak{h})$. Here the product $\odot$ is defined to satisfy the following:

$$(s_1\cdots s_k)\odot(t_1\otimes t_2)=(s_1\cdots s_k\cdot t_1)\otimes t_2,$$

where $s_1,\cdots, s_k$ and $t_1, t_2\in\boldsymbol{T}$. As is well known, it holds that $\mathrm{Ann}(\mathfrak{h}^{(k)})=S^k\boldsymbol{T}\odot\mathrm{Ann}(\mathfrak{h})$, i.e., $\mathrm{Ann}(\mathfrak{h}^{(k)})=\mathrm{Im}\,\Psi^k(k=0, 1, 2,\cdots)$ (see [17]). Hence we have

$$\dim\mathfrak{h}^{(k)}=\dim S^{k+1}\boldsymbol{T}\otimes\boldsymbol{T}-\dim S^k\boldsymbol{T}\otimes\mathrm{Ann}(\mathfrak{h})+\dim\mathrm{Ker}\,\Psi^k.$$

We now determine $\dim\mathrm{Ker}\,\Psi^k(k=2, 3)$.

Utilizing the basis $E_1, E_{\bar 1}, E_2, E_{\bar 2}$ of $\boldsymbol{T}^c$, let us regard the complexification $S^k\boldsymbol{T}^c$ of $S^k\boldsymbol{T}$ as the complex vector space of homogeneous polynomials of degree k of the variables $E_1, E_{\bar 1}, E_2, E_{\bar 2}$. We define elements F_1, $F_{\bar 1},\cdots, F_4$ of $S^2\boldsymbol{T}^c$ by:

$$F_1=E_1^2, F_{\bar 1}=E_{\bar 1}{}^2;\ F_2=E_1\cdot E_2, F_{\bar 2}=E_{\bar 1}\cdot E_{\bar 2};$$

$$F_3=E_2^2-E_1\cdot(AE_{\bar 1}+PE_{\bar 2}), F_{\bar 3}=E_{\bar 2}{}^2-E_{\bar 1}\cdot(\bar{A}E_1+\bar{P}E_2);$$

$$F_4=E_2\cdot E_{\bar 2}-BE_1\cdot E_{\bar 1}.$$

Then it is clearly seen that $F_1, F_{\bar 1},\cdots, F_4$ are linearly independent and $\bar{F}_i=F_{\bar i}(i=1, 2, 3)$ and $\bar{F}_4=F_4$.

LEMMA 12. *The elements* $F_1, F_{\bar{1}}, \cdots, F_4$ *form a basis of* $\mathrm{Ann}(\mathfrak{h}^c)$.

PROOF. Since $\mathfrak{h}=\mathfrak{n}+\Lambda^2 \boldsymbol{T}^*$, $\mathrm{Ann}(\mathfrak{h})$ coincides with the annihilator $\mathrm{Ann}(\mathfrak{n})$ of $\mathfrak{n}$ in $S^2\boldsymbol{T}$. Viewing the expression of α with respect to the basis $\{E_1, E_{\bar{1}}, E_2, E_{\bar{2}}\}$ of $\boldsymbol{T}^c$, we know that the elements $F_1, F_{\bar{1}} \cdots, F_4$ belong to $\mathrm{Ann}(\mathfrak{n})^c$. Hence $F_1, F_{\bar{1}}, \cdots, F_4 \in \mathrm{Ann}(\mathfrak{h})^c$. On the other hand, since $\dim \mathfrak{n}=3$, we have $\dim \mathrm{Ann}(\mathfrak{h})=\dim \mathrm{Ann}(\mathfrak{n})=\dim S^2\boldsymbol{T}-\dim \mathfrak{n}=10-3=7$. Thus our assertion follows immediately.

LEMMA 13. (1) $\dim \mathrm{Ker}\, \Psi^2=0$.

(2) $\dim \mathrm{Ker}\, \Psi^3=\begin{cases} 0 & \text{if } p\neq 0. \\ 4 & \text{if } p=0. \end{cases}$

PROOF. We first prove the assertion (2). Let Γ be an element of $S^3\boldsymbol{T}\otimes \mathrm{Ann}(\mathfrak{h})$. In view of Lemma 12, we can write

$$\Gamma=\sum_{i=1}^{3}(f_i\otimes F_i+\bar{f}_i\otimes \bar{F}_i)+f_4\otimes F_4,$$

where $f_1, f_2, f_3, f_4(\bar{f}_4=f_4)\in S^3\boldsymbol{T}^c$. Write $\Psi^3(\Gamma)$ in the form $\Psi^3(\Gamma)=\sum_{i=1}^{2}(g_i\otimes E_i+\bar{g}_i\otimes \bar{E}_i)(g_1, g_2\in S^4\boldsymbol{T}^c)$.

Then we have

$$\begin{aligned} g_1&=2f_1E_1+f_2E_2-f_3(AE_{\bar{1}}+PE_{\bar{2}}) \\ &\qquad -\bar{A}\bar{f}_3E_{\bar{1}}-B\, f_4E_{\bar{1}}, \\ g_2&=f_2E_1+2f_3E_2-\bar{P}\bar{f}_3E_{\bar{1}}+f_4E_{\bar{2}}. \end{aligned}$$

Let $\Gamma\in \mathrm{Ker}\, \Psi^3$. Then we have $g_1=g_2=0$. Putting this into the above equalities, we have

$$\begin{aligned} 2f_1E_1^2&=\{f_3(AE_{\bar{1}}+PE_{\bar{2}})+(\bar{A}\bar{f}_3+Bf_4)\ E_{\bar{1}}\}\ E_1 \\ &\qquad +(2f_3E_2-\bar{P}\bar{f}_3E_{\bar{1}}+f_4E_{\bar{2}})E_2\,; \\ f_2E_1&=-2f_3E_2+\bar{P}\bar{f}_3E_{\bar{1}}-f_4E_{\bar{2}}. \end{aligned}$$

Now let us set $\lambda=f_3$, $\mu=f_4$ and

$$\begin{aligned} \mathrm{I}&=2\lambda E_2-\bar{P}\bar{\lambda}E_{\bar{1}}+\mu E_{\bar{2}}, \\ \mathrm{II}&=(A\lambda+\bar{A}\bar{\lambda}+B\mu)E_{\bar{1}}+P\lambda E_{\bar{2}}+(\mathrm{I}/E_1)E_2. \end{aligned}$$

Then we have $E_1|\mathrm{I}$, $E_1|\mathrm{II}$, $f_1=\frac{1}{2}(\mathrm{II}/E_1)$ and $f_2=-(\mathrm{I}/E_1)$. Conversely, it is easily checked that if $E_1|\mathrm{I}$, $E_1|\mathrm{II}$, $f_1=\frac{1}{2}(\mathrm{II}/E_1)$ and $f_2=-(\mathrm{I}/E_1)$, then $g_1=g_2=0$, i. e., $\Gamma\in\mathrm{Ker}\ \Psi^3$.

We now assume $\Gamma\in\mathrm{Ker}\ \Psi^3$. We determine λ and μ by the conditions $E_1|\mathrm{I}$ and $E_1|\mathrm{II}$.

Let us regard I and II as polynomials of $E_1, E_{\bar{1}}$ and denote by $\mathrm{I}_{i,j}$ (resp. $\mathrm{II}_{i,j}$) the coefficient of $E_1^i E_{\bar{1}}^j$ $(i, j=0, 1, 2,\cdots)$ in I (resp. II). Then $\mathrm{I}_{i,j}$ and $\mathrm{II}_{i,j}$ are homogeneous polynomials of E_2 and $E_{\bar{2}}$ of degree $4-(i+j)$. Then the conditions $E_1|\mathrm{I}$ and $E_1|\mathrm{II}$ are stated in the form $\mathrm{I}_{0,j}=\mathrm{II}_{0,j}=0$ $(j=0, 1, 2,\cdots)$. Write

$$\lambda=\lambda_1E_1E_{\bar{1}}+\lambda_2E_1+\lambda_3E_{\bar{1}}+\lambda_4,$$

$$\mu=\mu_1E_1E_{\bar{1}}+\mu_2E_1+\mu_3E_{\bar{1}}+\mu_4\ ;$$

$$\lambda_1=rE_1+sE_{\bar{1}}+\varphi_1,$$

$$\mu_1=tE_1+\bar{t}E_{\bar{1}}+\psi_1\ ;$$

$$\lambda_2=l_2E_1^2+\varphi_2E_1+\sigma_2,$$

$$\mu_2=m_2E_1^2+\psi_2E_1+\tau_2\ ;$$

$$\lambda_3={}_{l3}E_{\bar{1}}^{\,2}+\varphi_3E_{\bar{1}}+\sigma_3,$$

$$\mu_3=\bar{m}_2E_{\bar{1}}^{\,2}+\bar{\psi}_2E_{\bar{1}}+\bar{\tau}_2\ ;$$

where r, s, t, l_2, l_3, m_2 are complex numbers and $\varphi_1, \varphi_2, \varphi_3, \psi_1$ $(\bar{\psi}_1=\psi_1)$, $\psi_2\in\boldsymbol{T}^C$; $\sigma_2, \sigma_3, \tau_2\in S^2\boldsymbol{T}^c$; λ_4, μ_4 $(\bar{\mu}_4=\mu_4)\in S^3\boldsymbol{T}^c$ are homogeneous polynomials of the variables E_2 and $E_{\bar{2}}$. Putting the above into I and II, we have:

(D 1) $\mathrm{I}_{0,0}=2\lambda_4E_2+\mu_4E_{\bar{2}}=0.$

(D 2) $\mathrm{I}_{0,1}=2\sigma_3E_2-\bar{P}\bar{\lambda}_4+\bar{\tau}_2E_{\bar{2}}=0.$

(D 3) $\mathrm{I}_{0,2}=2\dot{\varphi}_3E_2-\bar{P}\bar{\sigma}_2+\bar{\psi}_2E_{\bar{2}}=0.$

(D 4) $\mathrm{I}_{0,3}=2l_3E_2-\bar{P}\bar{\varphi}_2+\bar{m}_2E_{\bar{2}}=0.$

(D 5) $\mathrm{I}_{0,4}=-\bar{P}\bar{l}_2=0.$

(D 6) $\mathrm{II}_{0,0}=P\lambda_4E_{\bar{2}}+(2\sigma_2E_2+\tau_2E_{\bar{2}})\ E_2=0.$

(D 7) $\mathrm{II}_{0,1}=(A\lambda_4+\bar{A}\bar{\lambda}_4+B\ \mu_4)+P\sigma_3E_{\bar{2}}$
$$+(2\varphi_1E_2-\bar{P}\bar{\sigma}_3+\psi_1E_{\bar{2}})E_2=0.$$

(D 8) $\mathrm{II}_{0,2}=(A\sigma_3+\bar{A}\bar{\sigma}_2+B\bar{\tau}_2)+P\ \varphi_3E_{\bar{2}}$
$$+(2sE_2-\bar{P}\bar{\varphi}_1+\bar{t}E_{\bar{2}})E_2=0.$$

(D 9) $\mathrm{II}_{0,3}=(A\varphi_3+\bar{A}\bar{\varphi}_2+B\bar{\psi}_2)+Pl_3E_{\bar{2}}-\bar{P}\bar{r}E_2=0.$

(D10) $\mathrm{II}_{0,4}=Al_3+\bar{A}\,\bar{l}_2+B\bar{m}_2=0.$

We consider the equations (D1)-(D10). First we assume that $p\neq 0$. Then by (D5) and (D6) we have $l_2=0$ and $E_2|\lambda_4$. Since $E_2|\lambda_4$, it follows from (D1) that $E_{\bar{2}}^2|\mu_4$. On the other hand since $\bar{\mu}_4=\mu_4$, we have $E_2^2E_{\bar{2}}{}^2|\mu_4$. This implies that $\mu_4=0$, because $\mu_4\in S^3\boldsymbol{T}^c$. Hence by (D1) we have $\lambda_4=0$. Putting $\lambda_4=\mu_4=0$ into (D6) and (D7), we have $E_2|\tau_2$ and $E_2|\sigma_3$. Then from (D2), it follows that $E_{\bar{2}}{}^2|\sigma_3$ and hence $E_2E_{\bar{2}}{}^2|\sigma_3$. This implies that $\sigma_3=0$, because $\sigma_3\in S^2\boldsymbol{T}^c$. Therefore by (D2) and (D6) we have $\sigma_2=\tau_2=0$. Similarly by (D8) and (D3) we obtain $\varphi_3=\psi_2=0$. Putting the above results into (D4) and (D7)-(D10), we have:

(D 4′) $2l_3E_2-\bar{P}\bar{\varphi}_2+\bar{m}_2E_{\bar{2}}=0.$

(D 7′) $2\varphi_1E_2+\psi_1E_{\bar{2}}=0.$

(D 8′) $2sE_2-\bar{P}\bar{\varphi}_1+\bar{t}E_{\bar{2}}=0.$

(D 9′) $\bar{A}\bar{\varphi}_2+Pl_3E_{\bar{2}}-\bar{P}\bar{r}E_2=0.$

(D10′) $Al_3+B\bar{m}_2=0.$

By (D7′) we have $E_2|\psi_1$. On the other hand since $\bar{\psi}=\psi_1$, it follows that $E_2E_{\bar{2}}|\psi_1$. This implies $\psi_1=0$, because $\psi_1\in\boldsymbol{T}^c$. Consequently we have $\varphi_1=0$. Hence by (D8′) we obtain that $s=t=0$. Eliminating φ_2 from (D4′) and (D9′), we have $|P|^2l_3+\bar{A}\bar{m}_2=0$. This together with (D10′) means that $l_3=m_2=0$, because $|P|^2B-|A|^2=-(3/4)|b|^2\neq 0$. Therefore by (D4′) and (D9′) we have $\varphi_2=0$ and $r=0$. Thus we have $\lambda=\mu=0$ and hence $\mathrm{I}=\mathrm{II}=0$. Therefore we have $f_1=f_2=f_3=f_4=0$, i.e., $\Gamma=0$. This shows that Ker $\Psi^3=0$ in case $p\neq 0$.

Next we consider the case $p=0$. By a similar argument as in the case $p\neq 0$, we know that there are two complex numbers x and y such that

$$\lambda=\{yE_{\bar{1}}-(\bar{x}/A)(\bar{A}E_2-2BE_{\bar{2}})\}\,E_1{}^2+xE_{\bar{2}}E_{\bar{1}}^2\,;$$

$$\mu=-2\ \{xE_2E_{\bar{1}}{}^2+\bar{x}E_{\bar{2}}E_{\bar{1}}^2\}.$$

As above $f_3=\lambda$ and $f_4=\mu$ are parameterized by two arbitrary complex numbers x and y. On the other hand since $f_1=\dfrac{1}{2}(\mathrm{II}/E_1)$ and $f_2=-(\mathrm{I}/E_1)$, f_1 and f_2 are uniquely determined by f_3 and f_4. Consequently, we know that Ker Ψ^3 is isomorphic to $\boldsymbol{C}^2$ as a real vector space. Hence we have dim Ker $\Psi^3=4$ in case $p=0$.

Finally we show the assertion (1). Let us suppose that there are four elements h_1, h_2, h_3, h_4 ($\bar{h}_4=h_4$) of $S^2\boldsymbol{T}^c$ such that $\Xi=\sum_{i=1}^{3}(h_i\otimes F_i+\bar{h}_i\otimes\bar{F}_i)$ $+h_4\otimes F_4\in$ Ker Ψ^2. Then we have $(E_1+E_{\bar{1}})\Xi=\sum_{i=1}^{3}\{(E_1+E_{\bar{1}})h_i\otimes F_i+(E_1+E_{\bar{1}})\bar{h}_i\otimes\bar{F}_i\}+(E_1+E_{\bar{1}})h_4\otimes F_4\in$Ker Ψ^3. However as we have seen in the proof of the assertion (1), Ker Ψ^3 does not contain any non-trivial element of the above form. This proves $\Xi=0$ and hence Ker $\Psi^2=0$. Q. E. D.

PROOF OF PROPOSITION 9. First we note $\dim S^2\boldsymbol{T}^*=10$; $\dim S^3\boldsymbol{T}^*=20$; $\dim S^4\boldsymbol{T}^*=35$. Since Ker $\Psi^2=0$ ((1) of Lemma 13) and $\dim \mathrm{Ann}(\mathfrak{h})=7$, we have

$$\begin{aligned}\dim \mathfrak{h}^{(2)}&=\dim S^3\boldsymbol{T}\otimes\boldsymbol{T}-\dim S^2\boldsymbol{T}\otimes\mathrm{Ann}(\mathfrak{h})\\&=80-70=10;\\ \dim \mathfrak{h}^{(3)}&=\dim S^4\boldsymbol{T}\otimes\boldsymbol{T}-\dim S^3\boldsymbol{T}\otimes\mathrm{Ann}(\mathfrak{h})+\dim \mathrm{Ker}\ \Psi^3\\&=140-140+\dim\mathrm{Ker}\ \Psi^3=\dim\mathrm{Ker}\ \Psi^3.\end{aligned}$$

Since $\mathfrak{g}_\alpha\cong\mathfrak{h}^{(2)}$, the assertion (1) follows from the first equality. By (2) of Lemma 13 and by the second equality, we have $\dim\mathfrak{g}_\alpha^{(1)}=0$ in case $p\neq 0$ and $\dim\mathfrak{g}_\alpha^{(1)}=4$ in case $p=0$. This proves the assertion (2).

Finally we show that if $p=0$, then $\mathfrak{g}_\alpha^{(1)c}$ contains elements of the form $(E_1^*)^3\otimes\zeta$, $(E_{\bar{1}}^*)^3\otimes\eta$, $(E_2^*)^3\otimes\eta$ and $(E_{\bar{2}}^*)^3\otimes\zeta$. Viewing the expression of α with respect to the basis $\{E_1^*, E_{\bar{1}}^*, E_2^*, E_{\bar{2}}^*\}$, we have $\Theta(\eta)=E_{\bar{1}}^*\circledcirc E_2^*$, $\Theta(\zeta)=E_1^*\circledcirc E_{\bar{2}}^*$, where $\circledcirc$ is the symmetric product. Hence $E_{\bar{1}}^*\otimes E_2^*$, $E_2^*\otimes E_{\bar{1}}^*$, $E_1^*\otimes E_{\bar{2}}^*$ and $E_{\bar{2}}^*\otimes E_1^*\in\mathfrak{h}^c$. Therefore it can be easily checked that $\mathfrak{h}^{(3)c}$ contains elements of the form $(E_{\bar{1}}^*)^4\otimes E_2^*$, $(E_2^*)^4\otimes E_{\bar{1}}^*$, $(E_1^*)^4\otimes E_{\bar{2}}^*$ and $(E_{\bar{2}}^*)^4\otimes E_1^*$. Consequently $(E_{\bar{1}}^*)^3\otimes(E_{\bar{1}}^*\circledcirc E_2^*)$, $(E_2^*)^3\otimes(E_{\bar{1}}^*\circledcirc E_2^*)$, $(E_1^*)^3\otimes(E_1^*\circledcirc E_{\bar{2}}^*)$ and $(E_{\bar{2}}^*)^3\otimes(E_1^*\circledcirc E_{\bar{2}}^*)\in\mathfrak{q}^{3c}$. Considering the inverse of the isomorphism $\Theta^3:\mathfrak{g}_\alpha^{(1)c}\longrightarrow\mathfrak{q}^{3c}$, we know that $\mathfrak{g}_\alpha^{(1)c}$ contains elements of the form $(E_{\bar{1}}^*)^3\otimes\eta$, $(E_2^*)^3\otimes\eta$, $(E_1^*)^3\otimes\zeta$ and $(E_{\bar{2}}^*)^3\otimes\zeta$. This completes the proof of the assertion (2). Q. E. D.

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