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The moment problem on divisible abelian semigroups

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1. Introduction

The moment problem is concerned with the integral representation of positive definite functions on semigroups. A recent detailed study of the moment problem is found in [2]. The purpose of this paper is to prove that every positive definite function on a divisible countable semigroup admits a unique integral representation.

Let S be an abelian semigroup with zero element 0. A *semicharacter* on S is a function $\rho: S\rightarrow\boldsymbol{R}$ such that $\rho(0)=1$, $\rho(s+t)=\rho(s)\rho(t)$ for all s, $t\in S$. The set S^{*} of all semicharacters on S is called the *dual semigroup* of S. We equip S^{*} with the topology of pointwise convergence. A function $\varphi: S\rightarrow\boldsymbol{R}$ is called *positive definite* if

$$\sum_{i,j=1}^{n}c_{i}c_{j}\varphi(s_{i}+s_{j})\geqq 0$$

for all $n\in\boldsymbol{N}$, $\{s_{1},\cdots,s_{n}\}\subset S$ and $\{c_{1},\cdots,c_{n}\}\subset\boldsymbol{R}$. A function $\psi: S\rightarrow\boldsymbol{R}$ is called *negative definite* if

$$\sum_{i,j=1}^{n}c_{i}c_{j}\psi(s_{i}+s_{j})\leqq 0$$

for all $n\in\boldsymbol{N}$, $\{s_{1},\cdots,s_{n}\}\subset S$ and $\{c_{1},\cdots,c_{n}\}\subset\boldsymbol{R}$ with $\sum_{i=1}^{n}c_{i}=0$. Let $M_{+}(S^{*})$ denote the set of all nonnegative Radon measures on S^{*}, and let $E_{+}(S^{*})$ denote the set of $\mu\in M_{+}(S^{*})$ such that

$$\int_{S^{*}}|\rho(s)|d\mu(\rho)<\infty \quad \text{for all } s\in S.$$

A function $f: S\rightarrow\boldsymbol{R}$ is called a *moment function* if there exists a measure $\mu\in E_{+}(S^{*})$ such that

$$f(s)=\int_{S^{*}}\rho(s)d\mu(\rho) \quad \text{for } s\in S.$$

Every moment function is positive definite. It is known (see [4]) that every bounded positive definite function is a moment function whose representing measure is unique. But a positive definite function is not necessarily a moment function, and also a representing measure for a moment function is not necessarily unique. For instance, according to the classical Hamburger moment problem, every positive definite function on the additive semigroup of nonnegative integers $\boldsymbol{N}_0$ is a moment function, but there exists a positive definite function whose representing measure is not unique.

An abelian semigroup S is called *perfect* if every positive definite function is a moment function whose representing measure is unique. For instance, the additive semigroup of nonnegative rational numbers $\boldsymbol{Q}_+$ is perfect (see [2], Proposition 6.5.6). Prefect semigroups form a rather restrictive class, while they have some very nice properties:

(1) The direct sum of a countable family of perfect semigroups is perfect (see [2], Note VI).

(2) Any homomorphic image of a perfect semigroup is perfect (see [2], Theorem 6.5.5).

An abelian semigroup S is called *2-divisible* if every $s\in S$ can be written $s=t+t$ for some $t\in S$.

Berg [1] proved the following results.

THEOREM A. *The abelian semigroup* $(\boldsymbol{D}, +)$ *of dyadic numbers* (*i.e.* $\boldsymbol{D}=\{k2^{-n}|k, n\in\boldsymbol{N}_0\}$) *is perfect.*

THEOREM B. *If a countable abelian semigroup* S *is 2-divisible, then* S *is perfect.*

We say that an abelian semigroup S is *divisible* if every $s\in S$ can be written $s=nt$ for some $n\geqq 2$ and some $t\in S$. In § 2 of this paper, we shall generalize the above Berg's results to the wider class of divisible abelian semigroups. In § 3, we shall characterize the completely negative difinite functions on a divisible abelian semigroup by the notion of Schur monotonicity.

2. Main results

For each sequence $\vec{m}=\{m_n\}_{n\geqq 1}$ of integers $m_n\geqq 2$, we define the abelian semigroup

$$T(\vec{m})=\left\{\frac{k}{m_1\cdots m_n}|k\in\boldsymbol{N}_0,\ n\geqq 1\right\}.$$

As particular cases, we have $T(\vec{m})=\boldsymbol{Q}_+$ if $m_n=n+1$ for $n\geqq 1$, and $T(\vec{m})=\boldsymbol{D}$ if $m_n=2$ for $n\geqq 1$. We shall prove that $(T(\vec{m}), +)$ is perfect for each $\vec{m}$.

First, we consider the case when m_n is odd for every $n \geqq 1$. For $x \in \boldsymbol{R}$ the function $\rho_x : T(\vec{m}) \longrightarrow \boldsymbol{R}$ defined by $\rho_x(r) = e^{rx}$ is a semicharacter and so is $\rho_{-\infty} := \boldsymbol{1}_{\{0\}}$, the indicator function of $\{0\}$. Since each m_n is odd, the function $\chi\left(\frac{k}{m_1 \cdots m_n}\right) := (-1)^k$ is well defined and multiplicative on $T(\vec{m})$. Then $\chi \rho_x$ is also a semicharacter for $x \in \boldsymbol{R}$. Note that $\chi = \boldsymbol{1}_{2T(\vec{m})} - \boldsymbol{1}_{T(\vec{m}) \setminus 2T(\vec{m})}$. Conversely let $\rho \in T(\vec{m})^*$. Then $\rho(1) \in \boldsymbol{R}$ and $x = \log|\rho(1)| \in \underline{\boldsymbol{R}} (:= [-\infty, \infty))$. It is easy to see that, for $r = \frac{k}{m_1 \cdots m_n} \in T(\vec{m})$, $\rho(r) = \rho(1)^r$ if $\rho(1) \geqq 0$ and $\rho(r) = (-1)^k (-\rho(1))^r$ if $\rho(1) < 0$. Hence $\rho = \rho_x$ if $\rho(1) \geqq 0$ and $\rho = \chi \rho_x$ if $\rho(1) < 0$. Moreover the mapping $\rho \longmapsto \rho(1)$ is a topological semigroup isomorphism of $T(\vec{m})^*$ onto $(\boldsymbol{R}, \cdot)$. Thus we may identify $T(\vec{m})^*$ with $\boldsymbol{R}$ and also with the disjoint union $\underline{\boldsymbol{R}} \cup \boldsymbol{R}$.

THEOREM 2.1. *Let $\vec{m} = \{m_n\}_{n \geqq 1}$ be a sequence of odd numbers greater than 2. Then the semigroup $(T(\vec{m}), +)$ is perfect. Every positive definite function φ on $T(\vec{m})$ has a unique representation*

$$\varphi(r) = a \boldsymbol{1}_{\{0\}}(r) + \int_{\underline{\boldsymbol{R}}} e^{rx} d\mu(x) + \int_{\boldsymbol{R}} \chi(r) e^{rx} d\nu(x)$$

for all $r \in T(\vec{m})$, where $a \geqq 0$ and $\mu, \nu \in M_+(\boldsymbol{R})$ satisfy

$$\int_{\boldsymbol{R}} e^{rx} d\mu(x) < \infty, \quad \int_{\boldsymbol{R}} e^{rx} d\nu(x) < \infty \quad \textit{for } r \in T(\vec{m}).$$

PROOF: Let $\ell_n = m_1 \cdots m_n$ for $n \geqq 1$. Let φ be a positive definite function on $T(\vec{m})$. For each $n \geqq 1$, $\left\{\varphi\left(\frac{k}{\ell_n}\right)\right\}_{k \geqq 0}$ is a Hamburger moment sequence because $k \longmapsto \varphi\left(\frac{k}{\ell_n}\right)$ is positive definite on $(\boldsymbol{N}_0, +)$. Therefore it follows (see [2], Theorem 6.2.2) that there exists a $\mu_n \in M_+(\boldsymbol{R})$ such that

$$\int_{\boldsymbol{R}} |x|^k d\mu_n(x) < \infty \quad \text{for } k \geqq 0,$$
$$\varphi\left(\frac{k}{\ell_n}\right) = \int_{\boldsymbol{R}} x^k d\mu_n(x) \quad \text{for } k \geqq 0.$$

Define the mappings $f_n : \underline{\boldsymbol{R}} \longrightarrow [0, \infty)$ and $g_n : \underline{\boldsymbol{R}} \longrightarrow (-\infty, 0]$ by

$$f_n(x) = \exp\left(\frac{x}{\ell_n}\right), \quad g_n(x) = -\exp\left(\frac{x}{\ell_n}\right) \quad \text{for } x \in \underline{\boldsymbol{R}}.$$

Then f_n and g_n are homeomorphisms, so there exist $\tau_n, \sigma_n \in M_+(\underline{\boldsymbol{R}})$ such that $\tau_n \circ f_n^{-1} = \mu_n|[0, \infty)$ and $\sigma_n \circ g_n^{-1} = \mu_n|(-\infty, 0)$. Hence we have

$$\varphi\left(\frac{k}{\ell_n}\right)=\int_{\underline{R}}\exp\left(\frac{k}{\ell_n}x\right)d\tau_n(x)+\int_{\underline{R}}x\left(\frac{k}{\ell_n}\right)\exp\left(\frac{k}{\ell_n}x\right)d\sigma_n(x).$$

Since $\tau_n(\underline{\boldsymbol{R}})+\sigma_n(\underline{\boldsymbol{R}})=\varphi(0)<\infty$, $\{\tau_n\}_{n\geqq 1}$ and $\{\sigma_n\}_{n\geqq 1}$ are relatively compact in the vague topology on $M_+(\underline{\boldsymbol{R}})$ (see [2], Proposition 2.4.6). Since the vague topology on $M_+(\underline{\boldsymbol{R}})$ is metrizable (*see* [2], Proposition 2.4.10), there is an increasing sequence $n_1<n_2<\cdots$ such that τ_{n_i} and σ_{n_i} converge vaguely to $\tau,\sigma\in M_+(\underline{\boldsymbol{R}})$, respectively, with total masses uniformly bounded by $\varphi(0)$.

Let $r=\frac{k}{\ell_n}\in T(\vec{m})$ be fixed. For $i\geqq 1$ such that $n_i\geqq n$, we have

$$\begin{aligned}\varphi(r)&=\varphi(km_{n+1}\cdots m_{n_i}/\ \ell_{n_i})\\&=\int_{\underline{R}}e^{rx}d\tau_{n_i}(x)+\int_{\underline{R}}x(r)e^{rx}d\sigma_{n_i}(x).\end{aligned}$$

Using the fact that, for each nonnegative continuous function f, the integral $\int fd\mu$ is lower semicontinuous in μ with respect to the vague topology (see [2], p. 50), we have

$$\int_{\underline{R}}e^{2rx}d\tau(x)\leqq\liminf_{i\to\infty}\int_{\underline{R}}e^{2rx}d\tau_{n_i}(x)\leqq\varphi(2r),$$
$$\int_{\underline{R}}e^{2rx}d\sigma(x)\leqq\liminf_{i\to\infty}\int_{\underline{R}}e^{2rx}d\sigma_{n_i}(x)\leqq\varphi(2r).$$

Since $e^{rx}\leqq(1+e^{2rx})/2$, e^{rx} is integrable with respect to τ and σ. Let $h(x)=1+e^{2(r+1)x}$ for $x\in\underline{\boldsymbol{R}}$. Since the sequence $\{h(x)\tau_{n_i}\}_{i\geqq 1}$ converges vaguely to $h(x)\tau$, with total masses bounded uniformly by $\varphi(0)+\varphi(2(r+1))$ and since $e^{rx}/h(x)\in C_0(\underline{\boldsymbol{R}})$, it follows (see [2], Proposition 2.4.4) that

$$\begin{aligned}\lim_{i\to\infty}\int_{\underline{R}}e^{rx}d\tau_{n_i}(x)&=\lim_{i\to\infty}\int_{\underline{R}}\frac{e^{rx}}{h(x)}h(x)d\tau_{n_i}(x)\\&=\int_{\underline{R}}\frac{e^{rx}}{h(x)}h(x)d\tau(x)\\&=\int_{\underline{R}}e^{rx}d\tau(x).\end{aligned}$$

Similarly

$$\lim_{i\to\infty}\int_{\underline{R}}e^{rx}d\sigma_{n_i}(x)=\int_{\underline{R}}e^{rx}d\sigma(x).$$

Since $r\in T(\vec{m})$ is arbitrary, we have

$$\varphi(r)=\int_{\underline{R}}e^{rx}d\tau(x)+\int_{\underline{R}}x(r)e^{rx}d\sigma(x)\quad\text{for all } r\in T(\vec{m}).$$

Defining $a=\tau(\{-\infty\})+\sigma(\{-\infty\})$, $\mu=\tau|(-\infty,\infty)$ and $\nu=\sigma|(-\infty,\infty)$ we have

$$\varphi(r)=a\mathbf{1}_{\{0\}}(r)+\int_{\mathbf{R}}e^{rx}d\mu(x)+\int_{\mathbf{R}}\chi(r)e^{rx}d\nu(x),$$

which shows that φ is a moment function on $(T(\vec{m}),+)$.

Next we prove the uniqueness of the triple (a,μ,ν). Since

$$\lim_{r\to 0}\varphi(2r)=\mu(\mathbf{R})+\nu(\mathbf{R})=\varphi(0)-a,$$

a is uniquely determined by φ. Suppose that $\mu',\nu'\in M_+(\mathbf{R})$ satisfy

$$\varphi(r)=a\mathbf{1}_{\{0\}}(r)+\int_{\mathbf{R}}e^{rx}d\mu'(x)+\int_{\mathbf{R}}\chi(r)e^{rx}d\nu'(x)$$

for all $r\in T(\vec{m})$. Then, for $r\in 2T(\vec{m})$, we have

$$\int_{\mathbf{R}}e^{rx}d(\mu+\nu)(x)=\int_{\mathbf{R}}e^{rx}d(\mu'+\nu')(x).$$

Define the function Φ on the closed right half-plane $\mathbf{C}_+=\{z\in\mathbf{C}|\mathrm{Re}\,z\geqq 0\}$ by

$$\Phi(z)=\int_{\mathbf{R}}e^{zx}d(\mu+\nu-\mu'-\nu')(x),$$

which is well defined and continuous on $\mathbf{C}_+$ and holomorphic in its interior. Since $2T(\vec{m})$ is dense in $[0,\infty)$, $\Phi(z)=0$ for $\mathrm{Re}\,z>0$ by uniqueness theorem, so that $\Phi(e^{iy})=0$ for $y\in\mathbf{R}$ by continuity. By the injectivity of Fourier-Stieltjes transform (see [5], p. 17), we have $\mu+\nu-\mu'-\nu'=0$. Since

$$\int_{\mathbf{R}}e^{rx}d(\mu-\nu)(x)=\int_{\mathbf{R}}e^{rx}d(\mu'-\nu')(x)$$

for $r\in T(\vec{m})\backslash 2T(\vec{m})$, by the similar argument we have $\mu-\nu-\mu'+\nu'=0$. Therefore $\mu=\mu'$ and $\nu=\nu'$. Thus the triple (a,μ,ν) is unique. □

Secondly, we consider the case when $\{m_n|n\in\mathbf{N}\}\cap 2\mathbf{N}$ is nonvoid and finite. Assume that m_p is even and m_n is odd for all $n>p$, and let $\ell=m_1\cdots m_p$. In this case, the function $\chi=\mathbf{1}_{2T(\vec{m})}-\mathbf{1}_{T(\vec{m})\backslash 2T(\vec{m})}$ is given by $\chi\left(\dfrac{k}{m_1\cdots m_n}\right)=(-1)^k$ where $n\geqq p$, so that χ is multiplicative. Then the functions $\rho_x(r)=e^{rx}(x\in\mathbf{R})$, $\rho_{-\infty}=\mathbf{1}_{\{0\}}$ and $\chi\rho_x(x\in\mathbf{R})$ are semicharacters. Conversely let $\rho\in T(\vec{m})^*$. Then it is easy to see that, for $r=\dfrac{k}{m_1\cdots m_n}$ where $n\geqq p$, $\rho(r)=\rho(1)^r$ if $\rho\left(\dfrac{1}{\ell}\right)\geqq 0$ and $\rho(r)=(-1)^k(-\rho(1))^r$ if $\rho\left(\dfrac{1}{\ell}\right)$

<0. Hence $\rho=\rho_x$ if $\rho\left(\frac{1}{\ell}\right)\geqq 0$ and $\rho=\chi\rho_x$ if $\rho\left(\frac{1}{\ell}\right)<0$, where $x=\log \rho(1)\in\underline{\boldsymbol{R}}$. The mapping $\rho\longmapsto\rho\left(\frac{1}{\ell}\right)$ is a topological semigroup isomorphism of $T(\vec{m})^*$ onto $(\boldsymbol{R},\cdot)$. Thus we may identify $T(\vec{m})^*$ with $\boldsymbol{R}$ and also with the disjoint union $\underline{\boldsymbol{R}}\cup\boldsymbol{R}$. Just as before, we can prove that Theorem 2.1 remains valid for this case.

Finally, we consider the case when $\{m_n|n\in\boldsymbol{N}\}\cap 2\boldsymbol{N}$ is infinite. Note in this case that $T(\vec{m})$ is 2-divisible, so that it is perfect by Theorem B. Moreover, the set of semicharacters are $\{\rho_x\}_{x\in\underline{\boldsymbol{R}}}$ and we identify $T(\vec{m})^*$ with $\underline{\boldsymbol{R}}$. Hence, for every positive definite function φ on $T(\vec{m})$, we have a unique representation

$$\varphi(r)=a\boldsymbol{1}_{\{0\}}(r)+\int_{\boldsymbol{R}}e^{rx}d\mu(x)$$

for all $r\in T(\vec{m})$, where $a\geqq 0$, and $\mu\in M_+(\boldsymbol{R})$ satisfies

$$\int_{\boldsymbol{R}}e^{rx}d\mu(x)<\infty \quad \text{for } r\in T(\vec{m}).$$

Consequently, we have the next theorem.

THEOREM 2.2. *Let $\vec{m}=\{m_n\}_{n\geqq 1}$ be a sequence of integers $m_n\geqq 2$. Then the semigroup $(T(\vec{m}),+)$ is perfect.*

Using this theorem and the properties (1) and (2) stated in § 1, we have the following.

THEOREM 2.3. *Every countable divisible abelian semigroup S is perfect.*

PROOF: Suppose $S=\{0, s_1, s_2, \cdots\}$. Since S is divisible, for every s_j there exist a sequence $\vec{m}^{(j)}=\{m_n^{(j)}\}_{n\geqq 1}$ of integers $m_n^{(j)}\geqq 2$ and a sequence $\{t_n^{(j)}\}_{n\geqq 1}$ of elements in S such that

$$s_j=m_1^{(j)}t_1^{(j)},\ t_n^{(j)}=m_{n+1}^{(j)}t_{n+1}^{(j)} \quad \text{for } n\geqq 1.$$

Note that for $r=k/m_1^{(j)}\cdots m_n^{(j)}\in T(\vec{m}^{(j)})$ the element $rs_j:=kt_n^{(j)}$ is well defined. We define the mapping $\pi:\bigoplus_{j=1}^{\infty}T(\vec{m}^{(j)})\longrightarrow S$ by

$$\pi(r_1, r_2, \cdots)=\sum_{j=1}^{\infty}r_js_j.$$

Then π is a surjective homomorphism. Every $T(\vec{m}^{(j)})$ is perfect by Theorem 2.2, so that $\bigoplus_{j=1}^{\infty}T(\vec{m}^{(j)})$ is perfect by (1) in § 1. Hence $S=$

$\pi(\bigoplus_{j=1}^{\infty} T(\vec{m}^{(j)}))$ is prefect by (2) in § 1. This completes the proof. $\square$

We further give the next theorem concerning the integral representation of negative definite functions on $(T(\vec{m}), +)$. The proof can be done by modifying that in [2, proposition 6. 5. 13], for the integral representation of negative definite functions on $(\boldsymbol{Q}_+, +)$.

THEOREM 2. 4. *Let $\vec{m}=\{m_n\}_{n\geqq 1}$ be a sequence of integers $m_n\geqq 2$. Let ψ be a negative definite function on $T(\vec{m})$.*

(i) *If $\{m_n | n\in \boldsymbol{N}\}\cap 2\boldsymbol{N}$ is finite, then ψ has a form*

$$\begin{aligned}\psi(r)=a+br-cr^2+d\boldsymbol{1}_{\{0\}}(r)\\ +\int_{\boldsymbol{R}\setminus\{0\}}(1-e^{rx}-r(1-e^x))d\mu(x)\\ +\int_{\boldsymbol{R}}(1-\chi(r)e^{rx})d\nu(x),\end{aligned}$$

where $a, b\in \boldsymbol{R}$, $c, d\geqq 0$, $\mu\in M_+(\boldsymbol{R}\setminus\{0\})$ and $\nu\in M_+(\boldsymbol{R})$ satisfy

$$\int_{0<|x|\leqq 1} x^2 d\mu(x)<\infty,$$

$$\int_{|x|>1} e^{rx}d\mu(x)<\infty, \quad \int_{\boldsymbol{R}} e^{rx}d\nu(x)<\infty \quad \text{for } r\in T(\vec{m}).$$

The sextuple (a, b, c, d, μ, ν) is uniquely determined by ψ.

(ii) *If $\{m_n | n\in \boldsymbol{N}\}\cap 2\boldsymbol{N}$ is infinite, then ψ has a form*

$$\begin{aligned}\psi(r)=a+br-cr^2+d\boldsymbol{1}_{\{0\}}(r)\\ +\int_{\boldsymbol{R}\setminus\{0\}}(1-e^{rx}-r(1-e^x))d\mu(x),\end{aligned}$$

where $a, b\in \boldsymbol{R}$, $c, d\geqq 0$, $\mu\in M_+(\boldsymbol{R}\setminus\{0\})$ satisfies

$$\int_{0<|x|\leqq 1} x^2 d\mu(x)<\infty,$$

$$\int_{|x|>1} e^{rx}d\mu(x)<\infty \quad \text{for } r\in T(\vec{m}).$$

The quintuple (a, b, c, d, μ) is uniquely determined by ψ.

3. Application to Schur monotonicity

In this section, applying Theorem 2. 2, we characterize the completely negative definite functions on a divisible abelian semigroup in terms of Schur monotonicity.

Let A be a convex subset of some real linear space E. For two vectors $x=(x_1, \cdots, x_n)$ and $y=(y_1, \cdots, y_n)$ in E^n whose components x_i and

y_i belong to A, we say x is *majorized* by y and write $x<y$ if there exists an $n\times n$ doubly stochastic matrix $P=(p_{ij})$ such that

$$x_i=\sum_{j=1}^{n}p_{ij}y_j \quad \text{for } i=1,\cdots,n.$$

Let S be an abelian semigroup. A function $\psi: S\longrightarrow \boldsymbol{R}$ is called *completely negative definite* if $\psi(\cdot+a)$ is negative definite for all $a\in S$. For each $n\in\boldsymbol{N}$, a function $\psi: S\longrightarrow\boldsymbol{R}$ is called *Schur increasing of order n* if, for every $\nu=(\nu_1,\cdots,\nu_n)$ and $\mu=(\mu_1,\cdots,\mu_n)$ in $\mathrm{Mol}_+^1(S)^n$ such that $\nu<\mu$, the inequality

$$\int\psi d(\nu_1*\cdots*\nu_n)\leqq\int\psi d(\mu_1*\cdots*\mu_n)$$

holds, where $\mathrm{Mol}_+^1(S)$ denotes the set of all Radon probability measures with finite support.

Note (see [2, Chapter 7]) that a function $\psi: S\longrightarrow\boldsymbol{R}$ is Schur increasing of order 2 if and only if ψ is negative definite, and that if ψ is Schur increasing of order $n\geqq 3$, then ψ is completely negative difinite. Conversely, Berg [1] proved the following.

THEOREM C. *Let S be a 2-divisible abelian semigroup. Then every negative definite function on S is Schur increasing of all orders.*

The next theorem extends Theorem C to the case of a divisible abelian semigroup. Here we note that a negative definite function on a divisible abelian semigroup is not necessarily completely negative definite (for example, $\psi(k3^{-n})=-(-1)^k$ on $\{k3^{-n}|k\in\boldsymbol{N}_0,\ n\geqq 1\}$).

THEOREM 3.1. *Let S be a divisible abelian semigroup. Then every completely negative definite function on S is Schur increasing of all orders.*

PROOF: Let ψ be a completely negative definite function on S. Let $\mu=(\mu_1,\cdots,\mu_n)$ and $\nu=(\nu_1,\cdots,\nu_n)$ in $\mathrm{Mol}_+^1(S)^n$ be given such that $\nu<\mu$. There exists a finite set $F\subset S$ on which all μ_i (and hence all ν_i) are concentrated. Suppose $F=\{s_1,\cdots,s_d\}$. Since S is divisible, for every s_j there exists a sequence $\vec{m}^{(j)}=\{m_n^{(j)}\}_{n\geqq 1}$ of integers $m_n^{(j)}\geqq 2$ and a sequence $\{t_n^{(j)}\}_{n\geqq 1}$ in S such that

$$s_j=m_1^{(j)}t_1^{(j)},\quad t_n^{(j)}=m_{n+1}^{(j)}t_{n+1}^{(j)}\quad \text{for } n\geqq 1.$$

Let S_0 be a subsemigroup of S generated by $\{t_n^{(j)}|n\geqq 1,\ j=1,2,\cdots,d\}$. Then $S_0\supset F$. It is seen as in the proof of Theorem 2.3 that S_0 becomes a homomorphic image of $\bigoplus_{j=1}^{\infty}T(\vec{m}^{(j)})$. Hence S_0 is perfect by Theorem 2.2

and (1), (2) in § 1. Since every completely negative definite function on a perfect abelian semigroup is Schur increasing of all orders (see [2], Theorem 7.3.9), $\psi'=\psi|S_0$ is Schur increasing of all orders as a function on S_0, and hence

$$\int\psi' d(\nu_1*\cdots*\nu_n)\leqq\int\psi' d(\mu_1*\cdots*\mu_n).$$

Since $\mu_1*\cdots*\mu_n$ and $\nu_1*\cdots*\nu_n$ are concentrated on S_0, we have

$$\int\psi d(\nu_1*\cdots*\nu_n)\leqq\int\psi d(\mu_1*\cdots*\mu_n),$$

which shows that ψ is Schur increasing of order n. □

REMARK. After completing the paper, the author has known that Bisgaard and Ressel [3] recently proved a more general result than Theorem 2.3 by a different method.

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