Hokkaido Mathematical Journal Vol. 20(1991) p. 593-600

Kakutani's example on product spectral measures

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(Received August 20, 1990)

Spectral operators of scalar-type, briefly scalar operators, were introduced by N. Dunford. They are natural analogues in Banach spaces of normal operators in Hilbert space and are precisely those operators which have an integral representation of the form $\int f dP$ for some spectral measure P and some P-integrable function f. The question of whether the sum and product of commuting scalar operators are again scalar operators was affirmatively answered in the Hilbert space setting by J. Wermer [10]. The answer is negative for Banach spaces; the first example was due to S. Kakutani [4]. A further example, in a "nicer" Banach space, was provided by C. A. McCarthy [7]. This example is usually considered as a modification of Katutani's example (which it is in some sense) and is usually quoted to show that "the same things can go wrong" in a nice separable, reflexive Banach space.

The fact is that these two examples actually illustrate somewhat different (though related) phenomena and are not simply two versions of the same point. The example of Kakutani is based on the interpretation that a spectral measure P is a uniformly bounded, multiplicative, projection-valued set function which is finitely additive and whose domain is an algebra of sets. The example of McCarthy is based on countably additive spectral measures whose domains are σ-algebras of sets. The point is that the spectral measures exhibited by Kakutani cannot be extended to spectral measures on the generated σ-algebras of sets. In particular (unlike McCarthy's example), the ranges of his spectral measures do not form a σ-complete Boolean algebra of projections in the sense of W. Bade [1] nor can they be imbedded in such a Boolean algebra of projections. These differences appear to get confused in the literature and, consequently, Kakutani's example is often misquoted; see [2; p. 192], [3; p. 2099], [6; p. 253], [7; p. 295], [8; p. 359] and [9; p. 657], for example. The purpose of this note is to draw explicit attention to this difference with the hope of clarifying it somewhat.

We begin by recalling Kakutani's construction. Let $S=S'$ be the

Cantor set in $[0, 1]$. Let $C(S)$ and $C(S')$ be the Banach spaces of all continuous $\boldsymbol{C}$-valued functions on S and S', respectively, equipped with the sup-norm. Then $C(S)$ and $C(S')$ are closed subalgebras of $C(S\times S')$. Let $\mathrm{sim}(S, S')$ be the dense subalgebra of $C(S\times S')$ consisting of all functions φ of the form

$$(1)\quad \varphi(s, s')=\sum_{i=1}^{n} f_j(s)g_j(s'),\quad s\in S,\ s'\in S',$$

with n a positive integer and $f_j\in C(S)$, $g_j\in C(S')$ for $1\leq i\leq n$. The norm of $\varphi\in\mathrm{sim}(S, S')$ is defined by

$$(2)\quad |||\varphi|||=\inf\sum_{j=1}^{n}\|f_j\|_\infty\|g_j\|_\infty$$

where the infimum is taken over all possible representations of φ in the form (1). It follows that

$$(3)\quad \|\varphi\|_\infty\leq|||\varphi|||,\ \varphi\in\mathrm{sim}(S, S').$$

Let $C(S)\hat{\otimes}_\pi C(S')$ be the completion of $\mathrm{sim}(S, S')$ with respect to the norm $|||\cdot|||$. Then $C(S)\hat{\otimes}_\pi C(S')$ may be considered as a linear subspace of $C(S\times S')$.

Let $\mathscr{B}$ and $\mathscr{B}'$ be the algebras of subsets of S and S', respectively, consisting of all subsets which are simultaneously open and closed. For each $A\in\mathscr{B}$, let χ_A be the characteristic function of A. Define

$$P(A)\varphi : (s, s') \mapsto \chi_A(s)\varphi(s, s'),\quad s\in S,\ s'\in S,$$

and

$$P'(A)\varphi : (s, s') \mapsto \chi_A(s')\varphi(s, s'),\quad s\in S,\ s'\in S',$$

for every $\varphi\in C(S)\hat{\otimes}_\pi C(S')$. Then P is a uniformly bounded spectral measure on $\mathscr{B}$ whose values are continuous projections in $C(S)\hat{\otimes}_\pi C(S')$. This means that $P(A\cap B)=P(A)P(B)$ for every A, $B\in\mathscr{B}$, that P is finitely additive with $P(S)=I$ (the identity operator in $C(S)\hat{\otimes}_\pi C(S')$) and that $\sup\{\|P(A)\|;\ A\in\mathscr{B}\}<\infty$. The analogous properties are true of P' on $\mathscr{B}'$. In addition, P and P' commute, that is, $P(A)P'(A')=P'(A')P(A)$ for every $A\in\mathscr{B}$ and $A'\in\mathscr{B}'$. It was shown by Kakutani that the product spectral measure $P\otimes P'$ is not uniformly bounded, that is,

$$\sup\{\|\sum_{j=1}^{n}P(A_j)P'(A'_j)\|\}=\sup\{\|P\otimes P'(\bigcup_{j=1}^{n}A_j\times A'_j)\|\}=\infty$$

where the supremum in taken over all finite collections of pairwise disjoint " rectangles "

$$A_j\times A'_j,\ 1\leq j\leq n,\ \text{in } S\times S' \text{ with } A_j\in\mathscr{B} \text{ and } A'_j\in\mathscr{B}'.$$

Every $s \in S$ has a unique representation of the form

$$(4) \quad s=2\sum_{n=1}^{\infty}\varepsilon_n(s)3^{-n}$$

where $\varepsilon_n(s)\in\{0, 1\}$, for every $n=1, 2,\cdots$. Let $f\in C(S)$ be the function defined by

$$(5) \quad f(s)=3\sum_{n=1}^{\infty}\varepsilon_n(s)4^{-n}, \quad s\in S.$$

Then bounded linear operators T and T' can be defined in $C(S)\hat{\otimes}_\pi C(S')$ by

$$T\varphi : (s, s') \mapsto f(s)\varphi(s, s'), \quad s\in S,\ s'\in S',$$

and

$$T'\varphi : (s, s') \mapsto f(s')\varphi(s, s'), \quad s\in S,\ s'\in S',$$

for every $\varphi\in C(S)\hat{\otimes}_\pi C(S')$. The operators T and T' are considered "scalar" operators because $T=\int_S f(s)dP(s)$ and $T'=\int_{S'} f(s')dP'(s')$ in a certain sense.

The interpretation of these "integrals" is an important point. If

$$\psi=\sum_{j=1}^{n}\alpha_j\chi_{A_j}, \quad A_j\in\mathscr{B},\ 1\leq j\leq n,$$

is any $\mathscr{B}$-simple function, then we can define the operator $\int_S \psi dP$ (unambiguously) by $\int_S \psi dP=\sum_{j=1}^{n}\alpha_j P(A_j)$; it satisfies the estimate

$$\left\|\int_S \psi dP\right\|\leq 4\|\psi\|_\infty \sup\{\|P(A)\| ;\ A\in\mathscr{B}\}=4\|\psi\|_\infty.$$

Of course, $\int_S \psi dP$ is a bounded linear operator in $C(S)\hat{\otimes}_\pi C(S')$. We note that the only P-null set (i.e. a set $A\in\mathscr{B}$ such that $P(A)=0$) is the empty set. It follows, by a continuous extension argument, that a bounded linear operator $\int_S \varphi dP$ can be defined in $C(S)\hat{\otimes}_\pi C(S')$ for every function $\varphi : S\rightarrow \boldsymbol{C}$ which can be approximated in the sup-norm $\|\cdot\|_\infty$ by $\mathscr{B}$-simple functions. Of course, we then have the estimate

$$\left\|\int_S \varphi dP\right\|\leq 4\|\varphi\|_\infty.$$

Since the compact metric space S is totally disconnected the sets in $\mathscr{B}$

form a base for the topology in S and it follows that the collection of all such functions φ is precisely $C(S)$. It is in this sense that the integral $T=\int_S fdP$ is to be interpreted as a "scalar" operator. The same is true of $T'=\int_{S'} f(s')dP'(s')$. It is these two commuting "scalar" operators T and T' which Kakutani used to show that the operators TT' and $T+T'$ are not scalar.

According to the monographs [2] and [3] a scalar operator means something more restrictive than that stated above. Namely, it is again a bounded operator T in a Banach space X which has an integral representation of the form $T=\int hdQ$ for some spectral measure Q and some bounded measurable function h. The difference is that a spectral measure means a set function $Q:\Sigma\rightarrow L(X)$, whose domain Σ is *σ-algebra* of subsets of some set Ω, which satisfies $Q(\Omega)=I$ and $Q(A\cap B)=Q(A)Q(B)$, for every $A, B\in\Sigma$, and which is *σ-additive* with respect to the strong operator topology in $L(X)$; here $L(X)$ denotes the space of all bounded linear operators of X into itself. In this case T can also be expressed in the form $T=\int_C \lambda dQ_T(\lambda)$ where Q_T is a (unique) spectral measure defined on the σ-algebra of all Borel subsets of $\boldsymbol{C}$ and is supported by the spectrum $\sigma(T)$, of T. Usually Q_T is called the resolution of the identity for T. For the purpose of this note let us call scalar operators in this more restricted sense, σ-scalar.

After these remarks we wish to indicate some special features of Kakutani's example which seem not to have been observed in the literature.

PROPOSITION 1. *The domain $\mathscr{B}$ of P (resp. $\mathscr{B}'$ of P') is an algebra of subsets of S (resp. S'), but is not a σ-algebra.*

PROOF. That $\mathscr{B}$ is an algebra of sets is clear. For each $n=1,2,\cdots$, equip $X_n=\{0, 1\}$ with the discrete topology and let the infinite product space $X=\Pi_{n=1}^{\infty}X_n$ have its usual product topology. Elements $x\in X$ will be denoted by $(x_n)_{n=1}^{\infty}$. The map $\Lambda: X\rightarrow S$ defined by

$$\Lambda(x)=2\sum_{n=1}^{\infty}x_n 3^{-n},\quad x=(x_n)_{n=1}^{\infty}\in X,$$

is a homeomorphism of X onto S. Each set $U_k=\Pi_{n=1}^{\infty}Y_n$, $k=1, 2,\cdots$, where $Y_k=\{1\}$ and $Y_n=X_n$, for $n\neq k$, is both open and closed. Since $\bigcap_{k=1}^{\infty}U_k$ is the singleton set consisting of the element $x\in X$ given by $x_n=1$,

$n=1, 2,\cdots$, it is not an open set in X. Accordingly, the intersection of the sets $\Lambda(U_k)\in\mathscr{B}$, $k=1, 2,\cdots$, is not open in S and so cannot belong to $\mathscr{B}$. □

PROPOSITION 2. *Let $Y=C(S)\hat{\otimes}_\pi C(S')$. The finitely additive spectral measures $P:\mathscr{B}\to L(Y)$ and $P':\mathscr{B}'\to L(Y)$ are actually σ-additive.*

PROOF. Let $\{A_n\}_{n=1}^\infty$ be a sequence of pairwise disjoint elements from $\mathscr{B}$ whose union belongs to $\mathscr{B}$. It is to be shown that $P(\bigcup_{n=1}^\infty A_n)=\sum_{n=1}^\infty P(A_n)$ where the series converges in the strong operator topology of $L(Y)$. Since $\bigcup_{n=1}^\infty A_n$ belongs to $\mathscr{B}$ it is a closed subset of S and hence, is compact. Accordingly, the open cover $\{A_n\}_{n=1}^\infty$ has a finite subcover, say $A_{n_1},\ldots,A_{n_k}$. By disjointness it follows that $A_r=\phi$ if $r\notin\{n_1,\ldots,n_k\}$. Then

$$P(\bigcup_{n=1}^\infty A_n)=P(\bigcup_{j=1}^k A_{n_j})=\sum_{j=1}^k P(A_{n_j})=\sum_{n=1}^\infty P(A_n). \qquad \square$$

In view of Propositions 1 and 2 and the fact that the range $P(\mathscr{B})=\{P(A)\,;A\in\mathscr{B}\}$, of P, is uniformly bounded in $L(Y)$, it may be anticipated that P has an extension to a spectral measure on the σ-algebra, $\mathscr{B}_\sigma$, of sets generated by $\mathscr{B}$. On the other hand, the space $Y=C(S)\hat{\otimes}_\pi C(S')$ is not weakly sequentially complete and so it is not clear that such an extension from $\mathscr{B}$ to $\mathscr{B}_\sigma$ is possible.

PROPOSITION 3. *The σ-additive set function $P:\mathscr{B}\to L(Y)$ cannot be extended to any σ-additive measure $\tilde{P}:\mathscr{B}_\sigma\to L(Y)$. An analogous statement is true for $P':\mathscr{B}'\to L(Y)$.*

PROOF. Let $\{U_k\}_{k=1}^\infty$ be the sequence of sets in the proof of Proposition 1. Then $V_n=\Lambda(\bigcup_{k=1}^n U_k)$, $n=1, 2,\cdots$, in an increasing sequence of sets in $\mathscr{B}$ such that $\{P(V_n)1\}_{n=1}^\infty$ is not convergent in Y, where 1 denotes the constant function 1. Accordingly, P has no σ-additive extension to $\mathscr{B}_\sigma$. □

Let Z be a Banach space. A Boolean algebra of projections $\mathscr{M}\subseteq L(Z)$ is σ-complete (complete) in the sense of W. Bade [1] if it is σ-complete (complete) as an abstract Boolean algebra (where the partial order $\leq$ is range inclusion) and, whenever $\{A_\alpha\}_\alpha$ is a sequence (family) of elements from $\mathscr{M}$ it follows that

$$(\wedge_\alpha A_\alpha)(Z)=\bigcap_\alpha A_\alpha(Z) \text{ and } (\vee_\alpha A_\alpha)(Z)=\overline{sp}(\bigcup_\alpha A_\alpha(Z)),$$

the closed subspace of Z spanned by $\bigcup_\alpha A_\alpha(Z)$.

In [7 ; p. 295] it is claimed that the Boolean algebras of projections constructed by Kakutani (i. e. the ranges of P and P' in $L(Y)$) are Bade complete. This is actually false as seen by the following.

PROPOSITION 4. *The Boolean algebra of projections $P(\mathscr{B})\subseteq L(Y)$ is not Bade σ-complete nor can it be imbedded in a σ-complete Boolean algebra of projections in $L(Y)$. The same is true of $P'(\mathscr{B}')$.*

PROOF. Suppose that $\mathscr{M}\subseteq L(Y)$ is a σ-complete Boolean algebra of projections containing $P(\mathscr{B})$. Let $\{E_n\}_{n=1}^{\infty}\subseteq\mathscr{B}$ be an increasing sequence of sets, in which case $P(E_n)\leq P(E_{n+1})$, for every $n=1,2,\ldots$. Then $\lim_{n\to\infty}P(E_n)=\vee_{n=1}^{\infty}P(E_n)$ in the strong operator topology [1; Lemma 2.3]. In view of Proposition 2 it follows from the Theorem of Extension (v) for vector measures in [5] that there is a σ-additive measure $\tilde{P}:\mathscr{B}_{\sigma}\to L(Y)$ agreeing with P on $\mathscr{B}$. This contradicts Proposition 3. □

In view of Proposition 3 and earlier remarks the "integrals" $T=\int_S fdP$ and $T'=\int_{S'} fdP'$ are not with respect to spectral meausures on σ-algebras of sets. Now, the function $\Phi: X\to[0,1]$ defined by

$$\Phi(x)=3\sum_{n=1}^{\infty}x_n4^{-n},\quad x=(x_n)_{n=1}^{\infty}\in X,$$

is injective and continuous with range the compact set $f(S)$. It follows that Φ^{-1} is also continuous and hence Φ is a homeomorphism of X onto $f(S)$. So, $f=\Phi o\Lambda^{-1}$ (see (5)) is a homeomorphism of S onto its range $f(S)$, where $\Lambda: X\to S$ is defined in the proof of Proposition 1. Accordingly, $f(S)$ is a totally disconnected, compact Hausdorff space. Let $\mathscr{B}_T$ denote the algebra of all simultaneously open and closed subsets of $\sigma(T)=f(S)$. Then f induces a Boolean algebra isomorphism between $\mathscr{B}$ and $\mathscr{B}_T$. It follows from Proposition 2 that the set function

$$(6)\quad Q: E\to P(f^{-1}(E)),\quad E\in\mathscr{B}_T,$$

is a σ-additive spectral measure on the algebra of sets $\mathscr{B}_T$ and that

$$T=\int_S f(s)\,dP(s)=\int_{\sigma(T)}\lambda dQ(\lambda).$$

Now, the sets $\mathscr{B}_T$ form a base for the topology in $\sigma(T)=f(S)$. If T were a σ-scalar operator, with resolution of the identity Q_T defined on the σ-algebra, $\mathscr{B}(\sigma(T))$, of all Borel subsets of $\sigma(T)$, then it ought to follow that Q_T agrees with Q on $\mathscr{B}_T$. From the definition (6) and the identification of $\mathscr{B}_T$ with $\mathscr{B}$ this would imply that $P:\mathscr{B}\to L(Y)$ has a σ-additive extension to a measure on $\mathscr{B}_{\sigma}$, thereby contradicting Proposition 3. This suggests the following result (which we prove formally).

PROPOSITION 5. *The "scalar" operators T and T' are not σ-scalar operators in Y.*

PROOF. If $\delta\in\mathscr{B}_T$, then $Q(\delta)$-given by (6) - is the spectral projection for T (defined via the usual Cauchy integral formula) corresponding to the open-closed subset $\delta\subseteq\sigma(T)$ and satisfies $Q(\delta)T=TQ(\delta)$. In addition, $Q(\mathscr{B}_T)$ is uniformly bounded in $L(Y)$. Let Y^* be the dual space of Y. It follows [2; Theorem 5.36] that there is a spectral measure $R:\mathscr{B}(\sigma(T))\rightarrow L(Y^*)$ of class Y (see [2; p. 119] for the definition) such that the adjoint operator $T^*=\int_{\sigma(T)}\lambda dR(\lambda)$ is a prescalar operator of class Y and

$$Q(\delta)^*=R(\delta),\quad \delta\in\mathscr{B}_T.$$

Suppose that $T=\int_{\sigma(T)}\lambda dQ_T(\lambda)$ was a σ-scalar operator in Y with resolution of the identity $Q_T:\mathscr{B}(\sigma(T))\rightarrow L(Y)$. Then the adjoint operator $T^*\in L(Y^*)$ satisfies $T^*=\int_{\sigma(T)}\lambda dQ_T^*(\lambda)$ where $Q_T^*:\mathscr{B}(\sigma(T))\rightarrow L(Y^*)$ is the spectral measure of class Y given by $Q_T^*(E)=Q_T(E)^*$, $E\in\mathscr{B}(\sigma(T))$. Since T^* has a unique resolution of the identity of class Y [2; Theorem 5.13] it follows that R and Q_T^* coincide on $\mathscr{B}(\sigma(T))$. In particular,

$$Q(\delta)^*=Q_T^*(\delta)=Q_T(\delta)^*,\quad \delta\in\mathscr{B}_T,$$

and hence, Q and Q_T agree on $\mathscr{B}_T$. Accordingly, Q has a σ-additive extension from $\mathscr{B}_T$ to $(\mathscr{B}_T)_\sigma\subseteq\mathscr{B}(\sigma(T))$ and it follows that $P:\mathscr{B}\rightarrow L(Y)$ has a σ-additive extension to $\mathscr{B}_\sigma$ (which is isomorphic to the σ-algebra $(\mathscr{B}_T)_\sigma$). This contradicts Proposition 3 thereby showing that T cannot be σ-scalar. □

In conclusion, we point out that McCarthy produced in [7] two commuting σ-scalar operators T and T' whose sum and product are not σ-scalar. This can be seen, for example, from the fact that the underlying Banach space constructed there is separable and reflexive and hence, in particular, is weakly sequentially complete. Furthermore, the Boolean algebras of projections exhibited there, say $\mathscr{M}$ and $\mathscr{M}'$, are uniformly bounded and contain T and T', respectively, in the weak operator closed algebras that they generate. It follows that T and T' are necessarily σ-scalar [3; XVII Theorem 3.19]. In contrast to Kakutani's example, the Boolean algebras $\mathscr{M}$ and $\mathscr{M}'$ can be imbedded in σ-complete Boolean algebras of projections [1; Lemma 2.9] which, by separability, are even complete [3; XVII Lemma 3.21].

REMARK. Let $\mathscr{B}\times\mathscr{B}'=\{E\times E'\,;\,E\in\mathscr{B},\,E'\in\mathscr{B}'\}$ and let $\mathscr{B}\otimes\mathscr{B}'$ denote the algebra of sets in $S\times S'$ generated by $\mathscr{B}\times\mathscr{B}'$. Then the set function $P\otimes P':\mathscr{B}\times\mathscr{B}'\to L(Y)$ defined by

$$(P\otimes P')(E\times E')=P(E)P'(E')=P'(E')P(E),$$

for every $E\times E'\in\mathscr{B}\times\mathscr{B}'$, has a unique extension to a finitely additive spectral measure, again denoted by $P\otimes P'$, on the algebra $\mathscr{B}\otimes\mathscr{B}'$; it is called the product spectral measure. Since the range of $P\otimes P'$ on $\mathscr{B}\otimes\mathscr{B}'$ is not a uniformly bounded subset of $L(Y)$-see [4] -the following observation is of some interest. Its proof follows the lines of that of Proposition 2, after noting that elements of $\mathscr{B}\otimes\mathscr{B}'$ are finite disjoint unions of elements from $\mathscr{B}\times\mathscr{B}'$ and hence, are subsets of the compact space $S\times S'$ which are simultaneously open and closed.

PROPOSITION 6. *The product spectral measure* $P\otimes P':\mathscr{B}\otimes\mathscr{B}'\to L(Y)$ *is* σ*-additive on the algebra* $\mathscr{B}\otimes\mathscr{B}'$.

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