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On groups G of p-length 2 whose nilpotency indices of J(KG) are a(p-1)+1

Dedicated to Professor Tosiro TSUZUKU on his 60th birthday

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1. Introduction

Let G be a finite p-solvable group with a Sylow p-subgroup P of order p^{a}, K a field of characteristic p, KG the group algebra of G over K, and $t(G)$ the nilpotency index of the radical $J(KG)$ of KG.

D. A. R. Wallace [9] proved that $a(p-1)+1\leqq t(G)\leqq p^{a}$. Y. Tsushima [8] proved that the second equality $t(G)=p^{a}$ holds if and only if P is cyclic. Here we shall study the structure of G with $t(G)=a(p-1)+1$. If G has p-length 1, then $t(G)=t(P)$ by Clarke [1]. From this, we can easily see that $t(G)=a(p-1)+1$ if and only if P is elementary abelian. Therefore we shall be interested in the structure of G of p-length 2 with $t(G)=a(p-1)+1$. As such examples, we know the followings.

We set $q=p^{r}$ and $l=(q^{p}-1)/(q-1)$. Then $q-1$ and l are relatively prime. Let $F=GF(q^{p})$ be a finite field of q^{p} elements, λ a generator of the multiplicative group F^{*} of F, and $\nu=\lambda^{q-1}$. Let V be the additive group of F. If we define $v^{x}=\nu v$, where νv means a multiplication in the field F, then $x\in\mathrm{Aut}(V)$. Let U be the Galois group of F over $GF(q)$, and $H=\langle x\rangle$. Then $HU\subseteqq\mathrm{Aut}(V)$. So we can consider the semidirect product of V by HU. We set $M_{p,r}=VHU$. Then HU is a Frobenius group and $|H|=l$, $|U|=p$, and VU is a Sylow p-subgroup of $M_{p,r}$ of order p^{pr+1}. In [5], Motose proved $t(M_{p,r})=(pr+1)(p-1)+1$.

Let $G=M_{p,r}$, then $G=O_{p,p',p}(G)$ and $G/O_{p}(G)$ is a Frobenius group. So can we consider conversely that if G satisfies such conditions and $t(G)=a(p-1)+1$, then is G isomorphic to $M_{p,r}$? Concerning this problem, we have the following result.

THEOREM. *Let V be a normal p-subgroup of G with $G=VN$ and $V\cap N=1$ for some Frobenius group N with complement U and kernel H, where U and H are p-group and abelian p'-group, respectively. Then the* *follwing conditions are equivalent.*

(i) $t(G)=a(p-1)+1$, *where* p^a *is the order of a Sylow* p*-subgroup of* G.

(ii) (A) U *and* V *are elementary abelian.*

(B) $C_G(x)$ *contains a Sylow* p*-subgroup of* G *for every element* x *of* V.

(iii) *Set* $\overline{G}=G/O_{p'}(G)$. *Then* $\overline{G}=[\overline{V}, \overline{H}]\overline{N}\times C_{\overline{V}}(\overline{H})$ *and* $[\overline{V}, \overline{H}]=\overline{V}_1\times\cdots\times\overline{V}_m$, *where* V_i *is a minimal normal subgroup of* VH, $1\leqq i\leqq m$. *Furthermore* $[\overline{V}, \overline{H}]\overline{H}=(\overline{V}_1\overline{H}_1)\times\cdots\times(\overline{V}_m\overline{H}_m)$, *where* H_i *is a subgroup of* H *and* $\overline{V}_i\overline{H}_i\overline{U}\simeq M_{p,r_i}$ *for some* r_i.

2. Preliminaries

In this section we shall prove some lemmas which will be used to prove the theorem.

From Theorem 3. 1 of Wallace [9], we have immediately the following result.

LEMMA 1. *Let* G *be a* p*-solvable group of order* $p^a m$, $(p, m)=1$, *and have the following normal series*

$$1=P_0\leqq N_0\leqq P_1\leqq N_1\leqq\cdots\cdots\leqq P_n=G$$

such that P_{i+1}/N_i, N_i/P_i *are* p *and* p'*-groups, respectively,* $0\leqq i\leqq n-1$. *If* $t(G)=a(p-1)+1$, *then* P_{i+1}/N_i *is elementary abelian,* $0\leqq i\leqq n-1$.

LEMMA 2. ([7, Proposition 19. 8]). *Let* G *act faithfully on vector space* V *of order* q^n *over* $GF(q)$. *Suppose* A *is a normal cyclic subgroup of* G *which acts irreducibly on* V. *Then we can identify* V *with the additive group of* $GF(q^n)$ *in such a way that* $G\leqq T(q^n)$ *where* $T(q^n)$ *is the set of semilinear transformations of the form* $x\rightarrow ax^\sigma$ *with* $a\in GF(q^n)$, $a\neq 0$ *and* σ *a field automorphism. Furthermore* $C_G(A)$ *is contained in the subgroup of* $T(q^n)$ *consisting of linear transformations (that is,* $\sigma=1$).

LEMMA 3. ([3, Theorem 4. 6, p409]). *Let* G *satisfy the assumption of the theorem. Then the following conditions are equivalent.*

(1) $t(G)=t(U)+t(V)-1$.

(2) $\sum_{h\in H} z^{hu}-\sum_{h\in H} z^h\in J(KV)^{i+1}$ *for all* $i\geqq 0$, *all* $z\in J(KV)^i$ *and all* $u\in U$.

LEMMA. 4 *Let* G *satisfy the assumption and* (i) *of the theorem. Then the following conditions hold.*

(1) *$|U|=p$ and V is elementary abelian.*
(2) *If W is a H-invariant subgroup of V, then W is N-invariant.*
(3) $\sum_{h\in H} z^h+(\sum_{h\in H} z^h)^s+\cdots+(\sum_{h\in H} z^h)^{s^{p-1}}\in J(KV)^{i+(p-1)}$ *for all $i\geqq 0$ and all $z\in J(KV)^i$.*

PROOF. (1) Since $N=HU$ is a Frobenius group, (1) follows immediately by Lemma 1.

(2) Since H is a p'-group, V is completely reducible as H-module. So we may assume that W is an irreducible H-module. Then we have (2) by the proof of [6, Lemma 11(7)].

(3) HU acts on KV by conjugation. So KHU acts on KV by linear extention. Then

$$z^{(\sum_{h\in H} h)(s-1)}=(\sum_{h\in H} z^h)^s-\sum_{h\in H} z^h\in J(KV)^{i+1}$$

by Lemma 3. *Similarly*

$$\{z^{(\sum_{h\in H} h)(s-1)}\}^{(\sum_{h\in H} h)(s-1)}=|H|z^{(\sum_{h\in H} h)(s-1)^2}\in J(KV)^{i+2}$$

by Lemma 3. Hence $z^{(\sum_{h\in H} h)(s-1)^2}\in J(KV)^{i+2}$. If we repeat this method, we have $z^{(\sum_{h\in H} h)(s-1)^{p-1}}\in J(KV)^{i+(p-1)}$. On the other hand $(s-1)^{p-1}=1+s+\cdots+s^{p-1}$, *and so*

$$z^{(\sum_{h\in H} h)(1+s+\cdots+s^{p-1})}=\sum_{h\in H} z^h+(\sum_{h\in H} z^h)^s+\cdots+(\sum_{h\in H} z^h)^{s^{p-1}}\in J(KV)^{i+(p-1)}.$$

3. Proof of the Theorem

(ii)→(i) For $z\in V$, $z^{h_0}\in Z(VU)$ for some $h_0\in H$ by (ii)(B). Then $(\sum_{h\in H} z^h)^u=\sum_{h\in H} z^{hu}=\sum_{h\in H} z^{h_0h_0^{-1}hu}=\sum_{h\in H}(z^{h_0})^{hu}=\sum_{h\in H}(z^{h_0})^{u^{-1}hu}=\sum_{h\in H} z^h$.

Hence $t(G)=t(U)+t(V)-1$ by Lemma 3, and so $t(G)=a(p-1)+1$ by (ii)(A).

(iii)→(ii) It is clear that (ii)(A) holds. Let $v\in V$, then $v=v_1\cdots v_m$ for some $v_i\in V_i$, $1\leqq i\leqq m$. Since $V_iHU\simeq M_{p,r_i}$, there exists an element $h_i\in H_i$ such that $v_i^{h_i}\in C_{V_i}(U)\subseteqq Z(VU)$. Set $h=h_1\cdots h_m\in H$, then $v^h=v_1^{h_1}\cdots v_m^{h_m}\in Z(VU)$. Hence $C_G(v^h)\supseteqq VU$, and so $C_G(v)\supseteqq(VU)^{h^{-1}}\in Syl_p(G)$. This proves that (ii)(B) holds.

In the next proof, we denote by e the identity of G.

(i)→(iii) Let $\overline{G}=G/O_{p'}(G)$. Then $\overline{G}$ satisfies (i), and so we may assume that $O_{p'}(G)=\{e\}$. By Lemma 4(1), $|U|=p$ and V is elementary abelian. N acts on V by conjugation. So we can regard V as $N=HU$

module. Since H is a p'-group, V is a completely reducible H-module. Let $V=V_1\times\cdots\times V_m$, where V_i *is an irreducible* H-module, $1\leqq i\leqq m$. Then by Lemma 4(2), V_i is N-invariant.

If $C_V(H)\neq\{e\}$, then we may assume that $[V_1,H]=\{e\}$. Since $G\triangleright V_2\times\cdots\times V_mH$ and $G/V_2\times\cdots\times V_mH\simeq V_1U$, V_1U is elementary abelian by Lemma 1, in particular $[V_1, U]=\{e\}$. Hence $G=(V_2\times\cdots\times V_mHU)\times V_1$. Since $V_2\times\cdots\times V_mHU\simeq G/V_1$ satisfies (i), we may assume that $C_V(H)=\{e\}$.

Let $L_i=V_iHU$ and $\bar{L}_i=V_iHU/C_H(V_i)$, $1\leqq i\leqq m$. Then $\bar{L}_i=\bar{V}_i\bar{H}\bar{U}$ and $\bar{H}$ is cyclic since $\bar{H}$ is abelian and $\bar{H}$ acts irreducibly and faithfully on $\bar{V}_i$. Since $\bar{V}_i\simeq V_i$ and $\bar{U}\simeq U$, we can identify $\bar{V}_i$ and $\bar{U}$ with V_i and U, respectively. By Lemma 2, we can identify V_i with the additive group of $GF(p^{n_i})$ in such a way that $\bar{H}\bar{U}\subseteqq T(p^{n_i})$, where $p^{n_i}=|V_i|$. Furthermore $\bar{H}$ is contained in the subgroup of $T(p^{n_i})$ consisting of linear transformations.

Let $\mathrm{Gal}(GF(p^{n_i})/GF(p))=\langle\tau\rangle$ and $\sigma=\tau^{r_i}$, where $n_i=pr_i$. Furthermore $C_{V_i}(\sigma)=GF(q_i)$, where $q_i=p^{r_i}$. By considering the structure of $T(p^{n_i})$, a conjugate subgroup of $\bar{H}\bar{U}$ in $T(p^{n_i})$ contains σ. So let $U=\langle s\rangle$, then we may assume that $s=\sigma$. When we identify V_i with $GF(p^{n_i})$, let V_i^* be the multiplicative group of V_i. Set $\langle\lambda_i\rangle=C_{V_i*}(s)$ and $\langle\nu_i\rangle=[V_i^*, \langle s\rangle]$, then $V_i^*=\langle\lambda_i\rangle\times\langle\nu_i\rangle$. Let $x\in H$ such that $\langle\bar{x}\rangle=\bar{H}$. Then there exists $\xi_i\in V_i^*$ such that $v^x=\xi_iv$ for every element $v\in V_i$, where ξ_iv means a multiplication of ξ_i and v in $GF(p^{n_i})(=V_i)$. Since $\bar{H}\bar{U}$ is a Frobenius group, $\overline{x^s}=\bar{x}^j$ for some $\bar{x}^j\neq\bar{x}$. Let 1 be the identity of V_i^*, then $(1^x)^s=\xi_i^s$ and $(1^x)^s=1^{xs}=1^{s^{-1}xs}=1^{x^j}=\xi_i^j$. Hence $\xi_i^s=\xi_i^j\neq\xi_i$. This implies that s acts fixted-point-freely on $\langle\xi_i\rangle$. Hence $\langle\xi_i\rangle=[\langle\xi_i\rangle, \langle s\rangle]\subseteqq\langle\nu_i\rangle$, and so $\bar{L}_i\subseteqq M_{p,r_i}$.

Now we shall divide the remainder of the proof into several steps.

STEP 1. *The following conditions hold.*

(1) *If* $C_{V_i}(h)\neq\{e\}$ *for some* V_i *and* $h\in H$, *then* $[V_i, h]=\{e\}$.

(2) *If* $a^h\in C_{V_i*}(s)$ *for some* $a\in C_{V_i*}(s)$ *and* $h\in H$, *then* $[V_i, h]=\{e\}$.

PROOF. (1) Since H is abelian, $C_{V_i}(h)$ is H-invariant. Moreover, since V_i is an irreducible H-module, $C_{V_i}(h)=V_i$, and so $[V_i, h]=\{e\}$.

(2) Two elements of $Z(UV)$ are conjugate in G if and only if they are conjugate in $N_G(VU)$ (See [Gorenstein, Finite groups, p. 240 Th. 1.1]). Since $N=UH$ is a Frobenius group, $N_G(VU)=VU$. Hence $a^h=a$ and so $e\neq a\in C_{V_i*}(h)$. Then (2) follows from (1).

Let $A=\{(\mu_1, \cdots, \mu_m)|\mu_i\in<\nu_i>, 1\leqq i\leqq m\}$. Next we define the action of HU on A as follows:

$$(\mu_1, \cdots \mu_m)^g=(\mu_1{}^g, \cdots, \mu_m{}^g) \text{ for } g\in HU.$$

STEP 2. *The following conditions holds.*
(1) *H acts regularly on A.*
(2) *If H acts transitively on A, then* (ii) (B) *holds.*

PROOF. (1) If $(\mu_1, \cdots, \mu_m)^h=(\mu_1, \cdots, \mu_m)$, then $\mu_i\in C_{V_i}(h)$, $1\leqq i\leqq m$. By Step 1(1), $[V_i, h]=\{e\}$, and so $[V, h]=\{e\}$. Since $O_{p'}(G)=\{e\}$, $h=e$. Hence H acts regularly on A.

(2) Let $a\in V$, then $a=(\alpha_1\mu_1)\cdots(\alpha_m\mu_m)$, where $\alpha_i\in C_{V_i}(s)$ and $\mu_i\in\langle\nu_i\rangle$, and $\alpha_i\mu_i$ means a multiplication of α_i and μ_i in $GF(p^{n_i})$ $(=V_i)$, $1\leqq i\leqq m$. Since H acts transitively on A, $(\mu_1, \cdots, \mu_m)^h=(1, \cdots, 1)$ for some $h\in H$, where 1 is the identity of V_i^*. Since $\mu_i{}^h=1$, $v^h=\mu_i^{-1}v$ for every element $v \in V_i$. In particular $(\alpha_i\mu_i)^h=\mu_i^{-1}\alpha_i\mu_i=\alpha_i$, $1\leqq i\leqq m$. Hence $a^h=\{(\alpha_1\mu_1)\cdots(\alpha_m\mu_m)\}^h=(\alpha_1\mu_1)^h\cdots(\alpha_m\mu_m)^h=\alpha_1\cdots\alpha_m\in C_V(s)\subseteqq Z(VU)$.

Let $A_0, A_1, \cdots, A_r$ be the H-orbits of A, where $(1, \cdots, 1)\in A_0$. Since U normalizes H, $U=\langle s\rangle$ induces a permutation on the set of the H-orbits of A.

STEP 3. *A_0 is the only orbit which is fixed by s.*

PROOF. Since $(|s|, |H|)=1$, a s-invariant orbit contains an element which is fixed by s. Since $C_{\langle\nu_i\rangle}(s)=1$, $(1, \cdots, 1)$ is the only element of A which in fixed by s and the assertion follows.

STEP 4. *Let $(\eta_1, \cdots \eta_m)\in A$, then there exists $y\in Aut(V)$ which satisfies the following conditions.*

(1) *y fixes V_i, $1\leqq i\leqq m$.*
(2) *If we define the action of y as follows:*
$$(\mu_1, \cdots, \mu_m)^y=(\mu_1{}^y, \cdots, \mu_m{}^y) \text{ for } (\mu_1, \cdots, \mu_m)\in A,$$
then y acts on A.
(3) $(\eta_1, \cdots, \eta_m)^y=(1, \cdots, 1)$.
(4) *y commutes with every element of H in Aut (V).*

PROOF. If we define $y\in\text{Aut}(V)$ as follows:
$$(v_1\cdots v_m)^y=(\eta_1^{-1}v_1)\cdots(\eta_m^{-1}v_m), \text{ where } v_i\in V_i,\ 1\leqq i\leqq m,$$
then $y\in\text{Aut}(V)$ and y fixes V_i. Furthermore $(\eta_1, \cdots, \eta_m)^y=(\eta_1{}^y, \cdots, \eta_m{}^y)=(1, \cdots, 1)$. Next, since $\eta_i\in\langle\nu_i\rangle$, $1\leqq i\leqq m$, y acts on A. Let $h\in H$, then $(v_1\cdots v_m)^h=(\xi_1v_1)\cdots(\xi_mv_m)$, where $\xi_i\in\langle\nu_i\rangle$ and $v_i\in V_i$, $1\leqq i\leqq m$. Then

$$(v_1\cdots v_m)^{hy}=\{(\xi_1 v_1)\cdots(\xi_m v_m)\}^y$$
$$=(\eta_1^{-1}\xi_1 v_1)\cdots(\eta_m^{-1}\xi_m v_m)$$

and

$$(v_1\cdots v_m)^{yh}=\{(\eta_1^{-1}v_1)\cdots(\eta_m^{-1}v_m)\}^h$$
$$=(\xi_1\eta_1^{-1}v_1)\cdots(\xi_m\eta_m^{-1}v_m),$$

and so (4) follows.

Let $\hat{S}=\sum_{v\in S} v(\in KG)$ for any subset S of V and let $V_1^*\times\cdots\times V_m^*=\{v_1\cdots v_m | v_i\in V_i^*,\ 1\leqq i\leqq m\}$. Set $z=\hat{V}-V_1^*\times\cdots\times V_m^*$ and

$$z_n=z+\sum_{(\mu_1,\cdots,\mu_m)\in A_n}\ \sum_{\substack{a_i\in C_{V_i^*}(s)\\ 1\leqq i\leqq m}}(a_1\mu_1)\cdots(a_m\mu_m),\quad 0\leqq n\leqq r.$$

Since $\{(a_1\mu_1)\cdots(a_m\mu_m)\}^h=(a_1\mu_1{}^h)\cdots(a_m\mu_m{}^h)$,

$$z_n=z+\sum_{h\in H}\sum_{a_i}\{(a_1\eta_1)\cdots(a_m\eta_m)\}^h$$

for some fixed element $(\eta_1,\cdots,\eta_m)\in A_n$, and so $z_n{}^h=z_n$ for $^\forall h\in H$.

STEP 5. *Let t be the maximal positive integer such that $z_0\in J(KV)^t$, then for z_n, $1\leqq n\leqq r$, t is also the maximal positive integer such that $z_n\in J(KV)^t$. Furthermore, if $t=d(p-1)$, where $|V|=p^d$, then* (ii) (B) *holds.*

PROOF. Let $y\in\mathrm{Aut}(V)$ as in Step 4. Then

$$z_n^y=z^y+\{\sum_{h\in H}\sum_{a_i}(a_1\eta_1)\cdots(a_m\eta_m)\}^{hy}$$
$$=z+\{\sum_{h\in H}\sum_{a_i}(a_1\eta_1)\cdots(a_m\eta_m)\}^{hy}\qquad\text{(by Step 4(1))}$$
$$=z+\{\sum_{h\in H}\sum_{a_i}(a_1\eta_1)\cdots(a_m\eta_m)\}^{yh}\qquad\text{(by Step 4(4))}$$
$$=z+\{\sum_{h\in H}\sum_{a_i}(a_1 1)\cdots(a_m 1)\}^{h}\qquad\text{(by Step 4(3))}$$
$$=z_0.$$

Since $y\in\mathrm{Aut}(V)$, the first assertion follows.

We remark $J(KV)^{d(p-1)}=K\hat{V}$. Hence, if $z_0\in J(KV)^{d(p-1)}$, $A_0=A$, and so H acts transitively on A. By Step 2(2), the second assertion follows.

STEP 6. (ii) (B) *holds.*

PROOF. Suppose false, then $r\geqq 1$ by Step 2(2). Let $\{A_1,\cdots,A_p\}$ be a s-orbit. For example, if $A_1{}^s=A_2$, then

$$\{\sum_{(\mu_1,\cdots,\mu_m)\in A_1}\ \sum_{\substack{a_i\in C_{V_i^*}(s)\\ 1\leqq i\leqq m}}(a_1\mu_1)\cdots(a_m\mu_m)\}^s$$
$$=\sum_{(\mu_1{}^s,\cdots,\mu_m{}^s),\in A_2}\ \sum_{a_i}(a_1\mu_1{}^s)\cdots(a_m\mu_m{}^s).$$

Since $a_i \in C_{V_i*}(s)$, $1 \leqq i \leqq m$, $z_1{}^s = z_2$.

By Lemma 4(3),

$$\sum_{h\in H} z_1{}^h + (\sum_{h\in H} z_1{}^h)^s + \cdots + (\sum_{h\in H} z_1{}^h)^{s^{p-1}} = |H|(z_1 + z_1{}^s + \cdots + z_1{}^{s^{p-1}})$$
$$= |H|(z_1 + \cdots + z_p) \in J(KV)^{t+(p-1)}$$

The same thing holds for another orbits, and so $z_1 + \cdots + z_r \in J(KV)^{t+(p-1)} \subseteqq J(KV)^{t+1}$ by Step 3. Then

$$\begin{aligned} & z_0 + z_1 + \cdots + z_r \\ &= z + \sum_{(\mu_1,\cdots,\mu_m)\in A_0} \sum_{a_i} (a_1\mu_1)\cdots(a_m\mu_m) + \sum_{j=1}^{p}\{\sum_{(\mu_1,\cdots,\mu_m)\in A_j} \sum_{a_i}(a_1\mu_1)\cdots(a_m\mu_m)\} + \cdots \\ &= z + \sum_{j=0}^{r} \sum_{(\mu_1,\cdots,\mu_m)\in A_j} \sum_{a_i}(a_1\mu_1)\cdots(a_m\mu_m) \\ &= z + \sum_{(\mu_1,\cdots,\mu_m)\in A} \sum_{a_i}(a_1\mu_1)\cdots(a_m\mu_m) \\ &= z + V_1^* \times \cdots \times V_m^* \\ &= \hat{V} \in J(KV)^{d(p-1)} \subseteqq J(KV)^{t+1} \text{ by Step 5.} \end{aligned}$$

Hence $z_0 = \hat{V} - (z_1 + \cdots + z_r) \in J(KV)^{t+1}$, this contradicts the choice of t and the assertion follows.

STEP 7. *Let $H_i = C_H(V_1 \times \cdots \times V_{i-1} \times V_{i+1} \times \cdots \times V_m)$, then $H = C_H(V_i) \times H_i$, $1 \leqq i \leqq m$.*

PROOF. Let $h \in H$, and let $a_i \in C_{V_i}^*(s)$, $1 \leqq i \leqq m$. Since $a_1 \cdots a_i{}^h \cdots a_m \in V$, there exists an element $x \in H$ such that $(a_1 \cdots a_i{}^h \cdots a_m)^x \in C_V(s) = \prod_i C_{V_i}(s)$ by Step 6. Then $a_1{}^x \in C_{V_1*}(s), \cdots, a_i{}^{hx} \in C_{V_i*}(s), \cdots, a_m{}^x \in C_{V_m*}(s)$. By Step 1(2), $x \in H_i$ and $hx \in C_H(V_i)$, hence $h = hxx^{-1} \in C_H(V_i)H_i$. Furthermore $C_H(V_i) \cap H_i = C_H(V) \subseteqq O_{p'}(G) = \{e\}$ and the assertion follows.

STEP 8. (iii) *holds.*

PROOF. Since $H = C_H(V_2) \times H_2$ by Step 7, $C_H(V_1) = H_2 \times C_H(V_1 \times V_2)$. Hence $H = H_1 \times H_2 \times C_H(V_1 \times V_2)$. Similarly we have $C_H(V_1 \times V_2) = H_3 \times C_H(V_1 \times V_2 \times V_3)$, and hence $H = H_1 \times H_2 \times H_3 \times C_H(V_1 \times V_2 \times V_3)$. By the similar argument, $H = H_1 \times \cdots \times H_m \times C_H(V)$. Since $O_{p'}(G) = \{e\}$, $C_H(V) = \{e\}$, and so $H = H_1 \times \cdots \times H_m$ and $VH = (V_1H_1) \times \cdots \times (V_mH_m)$. Then, by the first paragraph of the proof, $M_{p,r_i} \supseteqq V_iHU/C_H(V_i) = V_iHU/H_1 \times \cdots \times H_{i-1} \times H_{i+1} \times \cdots \times H_m \simeq V_iH_iU$. If $M_{p,r_i} \supsetneqq V_iHU/C_H(V_i)$, $\langle \nu_i \rangle \supsetneqq \langle \xi_i \rangle$, then ν_i is not conjugate to any element of $C_{V_1}(s)$. This contradicts Step 6. Therefore $V_iH_iU \simeq M_{p,r_i}$ and the assertion follows.

This proves (i)→(iii) and completes the proof of the theorem.

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