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A note on the Lonergan-Hosack presentation

A. C. KIM, M. NIEMENMAA and G. ROSENBERGER
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Introduction.

The problem of finding different methods to prove groups finite or infinite was discussed by M. Newman in a lecture which he gave at the GROUPS-KOREA 1988 conference in Pusan. As an example he considered the Lonergan-Hosack presentation

$$G(m)=\langle x, z: z^{3}x^{m-4}z^{3}x^{-1}=z^{5}x^{m-3}z^{2}x^{m-3}=1\rangle.$$

For the two cases $m=1$ and $m=5$ we have fundamental group presentations of closed, orientable 3-dimensional manifolds. For a while it was an open problem whether these presentations are those of finite or infinite groups. The first solution came from M. Slattery [**4**] who managed to show that $G(1)$ and $G(5)$ are infinite by using the computer algebra package CAYLEY. In his conference talk M. Newman welcomed all contributions to this area and the purpose of this note is to show that the use of some supporting theory provides us with more information about the structure of $G(m)$.

Results.

Consider the presentation $G(m)$ given in the introduction.

THEOREM 1. *Let $m\geq 5$ or $m=3$ or $m<0$. Then*

(1) *$G(m)$ has a subgroup of finite index mapping onto a free group of rank 2 and $G(m)$ has a free subgroup of rank 2.*

(2) *$G(m)$ has a generating pair $\{u, v\}$ such that the subgroup generated by elements u^{k} and v^{k} is free of rank 2 for a sufficiently large integer k.*

(3) *$G(m)$ is SQ-universal.*

PROOF: If $m=5$, then the result follows directly from [**3**]. If $m=3$, then $G(3)\cong Z_{2}\star Z_{7}$ and the result is well known (see [**2**]). Then assume

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that either $m \geq 6$ or $m \leq 0$. Now the Triangle-group

$$T(m-3, 7, 2) = \langle a, b : a^{m-3} = b^7 = (ab)^2 = 1 \rangle$$

is an epimorphic image of $G(m)$. Since $|m-3| \geq 3$ it follows that $\frac{1}{|m-3|} + \frac{1}{7} + \frac{1}{2} < 1$ and we can proceed as in [**3**].

THEOREM 2. *If m is odd and $m \neq 7$, then $G(m)$ is a nontrivial free product with amalgamation.*

PROOF: If $m=5$ then the result follows from [**3**] and it is also clear that it holds for $G(3)$. Then assume that m is odd with $m \geq 9$ or $m \leq 1$. Then $m-5$ is even and $|m-5| \geq 4$. We denote $m-5=2k$. Then

$$H(k) = \langle a, b : a^3 = b^{2k} = (ab^2)^2 = 1 \rangle$$

is an epimorphic image of $G(m)$ and $H(k)$ is a nontrivial free product of H_1 and H_2 with the amalgamated subgroup H, where $H_1 = \langle b : b^{2k} = 1 \rangle$, $H_2 = \langle a, d : a^3 = d^k = (ad)^2 = 1 \rangle$ and $H = \langle b^2 : (b^2)^k = 1 \rangle \cong \langle d : d^k = 1 \rangle$. Again the rest of the proof follows as in [**3**].

THEOREM 3. *The group $G(1)$ has a subgroup of finite index mapping onto a free group of rank 2 and $G(1)$ has a free subgroup of rank 2. Furthermore, $G(1)$ is SQ-universal.*

PROOF: As in the proof of theorem 2, $G(1)$ has

$$H(-2) = \langle a, b : a^3 = b^{-4} = (ab^2)^2 = 1 \rangle$$

as an epimorphic image. By theorem 6 of [**1**], it follows that $H(-2)$ has a free subgroup of rank 2 (notice that $H(-2) \cong H(2)$). We now consider the presentation

$$H(2) = \langle a, b : a^3 = b^4 = (ab^2)^2 = 1 \rangle.$$

The subgroup K of $H(2)$ generated by $u=a$, $v=bab^{-1}$ and $w=b^2$ has index 2 in $H(2)$. The presentation

$$K = \langle u, v, w : u^3 = v^3 = w^2 = (uw)^2 = (vw)^2 = 1 \rangle$$

has the properties of our theorem (see [**1**] and [**2**]) and hence $G(1)$ has these properties.

We shall now introduce three problems which are related to the preceding results.

1) Does theorem 1 also hold for $m=1$?

2) Does theorem 2 also hold for $m=7$?

3) Does theorem 2 3old for some even m ?

THEOMEM 4. *The groups $G(4)$ and $G(2)$ are finite.*

PROOF: It is easy to see that $G(4)$ is of order 19. Then consider

$$G = G(2) = \langle x, z : z^3x^{-2}z^3x^{-1} = z^5x^{-1}z^2x^{-1} = 1\rangle.$$

Adding the relations $x=z^2$ gives $\langle z : z^3=1\rangle$. Now denote $a=xz^{-1}$, $b=zx$, $c=z^2xz^{-1}$, $d=z^3$ and consider the epimorphism from G to $\langle z : z^3=1\rangle$. Then the Reidermeister-schreier method (with the transversal $\{1, z, z^2\}$) gives the following presentation for the kernel K of the epimorphism: K has generators a, b, c, d and relators da^{-2}, $d^2b^{-1}db^{-1}$, $d^2c^{-1}dc^{-1}$, db^{-1}, $dc^{-1}a^{-1}$, $dc^{-1}da^{-1}b^{-1}$ and $da^{-1}db^{-1}c^{-1}$. Since $d=a^2$, we have $c=a^3b^{-1}$. We substitute these values for c and d into the other defining relations and then in the presentation for K we have generators a and b and relators $a^4b^{-1}a^2b^{-1}$, $aba^{-1}b$, $b^{-1}a^2ba^{-2}$ and $a^2ba^{-2}b^{-1}$. Hence

$$K = \langle a, b : [a^2, b] = a^6b^{-2} = aba^{-1}b = 1\rangle.$$

Now the last relation gives $ba^{-1}b=a^{-1}$ and by using the second relation we get $a^{12}=1$ and $b^4=1$. Then, clearly, K has order 24 and G has order 72. The proof is complete.

References

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Department of Mathematics
Pusan National University
Pusan 609-735
Republic of Korea

Department of Mathematics
University of Oulu
SF-90570 Oulu
Finland

Fachbereich Mathematik
Universität Dortmund
Postfach 50 05 00
4600 Dortmund
Germany