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p-supersolvability of factorized finite groups

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1. Introduction

All groups we consider are finite. It is well known that the product of two supersolvable normal subgroups is not supersolvable in general (see Huppert [3]).

In [2]. Baer proved that if G is the product of two supersolvable normal subgroups and the commutator subgroup G' of G is nilpotent, then G is supersolvable.

In [1]. Asaad and Shaalan proved the following generalization of Baer's theorem:

Suppose that H and K are supersolvable subgroups of G, G' is nilpotent and $G=HK$. Suppose further that H is permutable with every subgroup of K and K is permutable with every subgroup of H. Then G is supersolvable.

Further, they proved the following result:

Suppose that H is a nilpotent, K a supersolvable subgroup of G and $G=HK$. Suppose further that H is permutable with every subgroup of K and K is permutable with every subgroup of H. Then G is supersolvable.

If H and K are subgroups of a group G such that H is permutable with every subgroup of K and K is permutable with every subgroup of H, we say that H and K are mutually permutable and we say that H and K are totally permutable if every subgroup of H is permutable with every subgroup of K.

The purpose of the present communication is the presentation of some properties of products of mutually permutable subgroups:

THEOREM A. *Let $G=HK>1$ be a group where H and K are mutually permutable. Then H or K contains a nonidentity normal subgroup of G or $\boldsymbol{F}(G)\neq 1$, where $\boldsymbol{F}(G)$ denotes the Fitting subgroup of G.*

Further we present a generalization and give independent proofs of the above mentioned results of Asaad and Shaalan in the following sense:

THEOREM B. *Let p be a prime number and $G=HK$ a group such that H and K are p-supersolvable subgroups of G. If H and K are mutually permutable and G' is p-nilpotent, then G is p-supersolvable.*

THEOREM C. *Let p be a prime number and $G=HK$ a group such that H is a p-supersolvable and K a p-nilpotent subgroup of G. If H and K are mutually permutable, then G is p-supersolvable.*

2. Preliminary results

We develop a sequence of straightforward lemmas.

2.1 LEMMA. *Let G be a p-supersolvable group for some prime number p. Then $\boldsymbol{O}_{p'}(G)\neq 1$ or $P\trianglelefteq G$, where $\boldsymbol{O}_{p'}(G)$ denotes the largest normal p'-subgroup of G and $P\in Syl_p(G)$.*

For a proof see ([4]. p. 691, Th. 6. 6).

2.2 LEMMA. *Let $G=HK>1$ be a group such that H and K are cyclic subgroups of G. If $|H|\geq|K|$, then H contains a nonidentity normal subgroup of G. Moreover, G is supersolvable.*

For a proof see ([4]. p. 722, Th. 10. 1).

2.3 LEMMA. *The p-solvable group G is p-supersolvable, if and only if, for every maximal subgroup V of G, the index $|G:V|$ is p or relatively prime to p.*

This is a famous theorem due to Huppert ([4]. p. 717, Th. 9.2/9.3).

To avoid repetitions in the subsequent proofs, we separate the following facts as common routine steps in certain induction arguments.

2.4 LEMMA. *Let p be a prime number and G a p-solvable group. Suppose that for every $1\neq S\trianglelefteq G$ we have G/S is p-supersolvable, but G itself is not p-supersolvable. Then*

(a) *G has a unique minimal normal subgroup N and N is a p-group of order greater than p.*
(b) *$\Phi(G)=1$ and there is a maximal subgroup V of G such that $G=NV$ and $N\cap V=1$, where $\Phi(G)$ is the Frattini subgroup of G.*
(c) *$N=\boldsymbol{O}_p(G)=\boldsymbol{C}_G(N)$.*
(d) *If $N\leq H\leq G$, then $\boldsymbol{O}_{p'}(H)=1$.*

PROOF.
(a) Suppose that S_1 and S_2 are minimal normal subgroups of G such that $S_1\neq S_2$. By hypothesis, the factor groups G/S_1 and G/S_2 are p-

supersolvable.
Since

$$G \cong G/(S_1 \cap S_2) \lesssim G/S_1 \times G/S_2,$$

we have that G is p-supersolvable So, we conclude that the minimal normal subgroup N of G is unique. Therefore, N is an abelian p-group of order $|N|>p$.
(b) Suppose that $\Phi(G)>1$. By hypothesis $G/\Phi(G)$ is p-supersolvable. Since every maximal subgroup of G contains $\Phi(G)$, there is an index preserving 1-1-correspondence between the maximal subgroups of $G/\Phi(G)$ whith those of G and hence G is p-supersolvablye by Lemma 2.3. So $\Phi(G)=1$. Let V be a maximal subgroup of G such that $N \not\leq V$. We have that $G=NV$ and obtain $N \cap V=1$ since N is abelian.
(c) If $N<\boldsymbol{C}_G(N)$, then $1 \neq \boldsymbol{C}_G(N) \cap V=\boldsymbol{C}_V(N) \trianglelefteq NV=G$.
Then $\boldsymbol{C}_V(N)$ contains a minimal normal subgroup of G different from N, against (a).
Since $\Phi(G)=1$, we have $\Phi(\boldsymbol{O}_p(G))=1$ (see [4]. p. 269, Th. 3. 3).
So. $\boldsymbol{O}_p(G) \leq \boldsymbol{C}_G(N)=N$. It follows $N=\boldsymbol{O}_p(G)$.
(d) Let $H \leq G$ such that $N \leq H$. If $\boldsymbol{O}_{p'}(H)>1$, then
$\boldsymbol{O}_{p'}(H) \leq \boldsymbol{C}_G(N)=N$, a contradiction. ■

REMARK 1. Let G be a p-supersolvable group, and $N \neq 1$ a normal elementary abelian p-subgroup of G. If $N=\boldsymbol{C}_G(N)$, then p is the largest prime divisor of $|G|$.

3. Permutable and subnormal subgroups

We develop a sequence of definitions, lemmas and properties concerning permutability and subnormality of subgroups.

3.1 LEMMA. *Let $G=HK$ be a group and p a prime number. Then there are subgroups $P \in Syl_p(G)$, $H_p \in Syl_p(H)$ and $K_p \in Syl_p(K)$ such that $P=H_pK_p$.*

For a proof see ([4]. p. 676, Th. 4. 7).

3.2 DEFINITION. Let G be a group and $X \leq G$.
(a) X is said to be subnormal in G ($X \trianglelefteq\trianglelefteq G$), if there is a chain of subgroups

$$X=G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_r=G.$$

(b) X is said to be quasinormal in G ($X \underset{qn}{<} G$), if $XH=HX$ holds for all subgroups $H \leq G$.

REMARK 2. It is well known that if $X \underset{qn}{<} G$, then $X \trianglelefteq \trianglelefteq G$ (see [6]. p. 213, Th. 7. 1. 2)

3. 3 LEMMA. *Let G be a group, A, B and X subgroups of G such that $X \leq A \cap B$, $X \trianglelefteq \trianglelefteq A$ and $X \trianglelefteq \trianglelefteq B$. If $AB = BA$, then $X \trianglelefteq \trianglelefteq AB$.*

For a proof see ([6]. p. 239, Th. 7. 7. 1).

3. 4 DEFINITION. Let G be a group. H and K subgroups of G.

(a) We say that H and K are mutually permutable, if H is permutable with every subgroup of K and K is permutable with every subgroup of H.

(b) We say that H and K are totally permutable, if every subgroup of H is permutable with every subgroup of K.

Certainly, if H and K are normal subgroups of G, then H and K are mutually permutable. Also, if H and K are totally permutable, then they are mutually permutable.

Let $G = \boldsymbol{S}_4$ be the symmetric group of degree 4, $H \in Syl_2(G)$ and $K = \boldsymbol{A}_4$ the alternating group. Clearly $G = HK$, and H and K are mutually permutable, but not totally permutable.

3. 5 PROPOSITION. *Let $G = HK$ be a group such that H and K are mutually permutable.*

(a) If $H \cap K \leq X \leq H$ and $Y \leq K$, then $XY = YX$.
If $H \cap K \leq Y \leq K$ and $X \leq H$, then $XY = YX$. In particular, if $H \cap K \leq X \leq H$ and $H \cap K \leq Y \leq K$, then X and Y are mutually permutable.

(b) If $H \cap K = 1$, then H and K are totally permutable.

(c) $H \cap K \underset{qn}{<} H$, $H \cap K \underset{qn}{<} K$ and $H \cap K \trianglelefteq \trianglelefteq G$.

PROOF.

(a) Let $H \cap K \leq X \leq H$ and $Y \leq K$. We have

$$XY = X(H \cap K)Y = (HY \cap XK) = (YH \cap KX) = Y(H \cap K)X = YX.$$

(b) By (a). every subgroup of H is permutable with every subgroup of K.

(c) By (a) it is clear that $H \cap K \underset{qn}{<} H$ and $H \cap K \underset{qn}{<} K$. Moreover. $H \cap K \trianglelefteq \trianglelefteq H$ and $H \cap K \trianglelefteq \trianglelefteq K$, by Remark 2. So, $H \cap K \trianglelefteq \trianglelefteq G$, by Lemma 3. 3.

3. 6 LEMMA (see Maier [7]). *Let $G = HK$ be a group such that H and K are totally permutable subgroups of G. If $|G| > 1$, then H or K*

contains a nonidentity normal subgroup of G.

PROOF. Let p denote the largest prime divisor of $|G|$. Certainly p divides at least one of $|H|$ or $|K|$. Let x be a p-element of the union set $H\cup K$ of maximal order and suppose $x\in H$, say. Let y be any q-element of K where q is a prime divisor of $|K|$. Since H and K are totally permutable, we see by Lemma 2.2 that $\langle x\rangle\langle y\rangle=\langle y\rangle\langle x\rangle$ is a supersolvable group.

If $q\neq p$, then $\langle x\rangle$ is the normal Sylow-p-subgroup of $\langle x\rangle\langle y\rangle$ since $p>q$ ([4]. p. 716. Th. 9.1). If $q=p$, then $|\langle x\rangle|\geq|\langle y\rangle|$ and there exists a nonidentity normal subgroup of $\langle x\rangle\langle y\rangle$ contained in $\langle x\rangle$ by Lemma 2.2. In any case, the unique subgroup R of order p in $\langle x\rangle$ is normalized by y. We conclude that K normalizes R. Now, the normal closure $R^G=R^{HK}=R^{KH}=R^H\leq H$ is a nonidentity normal subgroup of G.

3.7 LEMMA. *Let G be a group and $X\underset{qn}{<}G$. Then*

(a) *X/X_G is nilpotent, where $X_G=\bigcap_{g\in G}X^g$.*

(b) *If X is a p-group for some prime p, then X is normalized by $\boldsymbol{O}^p(G)$, where $\boldsymbol{O}^p(G)$ denotes the smallest normal subgroup of G with p-factor group; obviously $\boldsymbol{O}^p(G)$ is the join of all p'-elements of G.*

See (Itô-Szép [5])

With these preparations we are able to prove:

THEOREM A. *Let $G=HK>1$ be a group where H and K are mutually permutable. Then H or K contains a nonidentity normal subgroup of G or $\boldsymbol{F}(G)\neq 1$.*

PROOF. Let $D=H\cap K$. Suppose that $H_G=1=K_G$. By Lemma 3.6 and Proposition 3.5 (b) we have $D\neq 1$. Since $(D_H)^G=(D_H)^K\trianglelefteq G$ and $(D_H)^K\leq K$ we have $D_H=1$. So, by Lemma 3.7 (a) D is nilpotent and since $D\trianglelefteq\trianglelefteq G$ by Prop. 3.5 (c). we obtain $D\leq\boldsymbol{F}(G)\neq 1$. ■

COROLLARY 1. *Let p be a prime number and $G=HK$ a group such that H and K are mutually permutable. If H and K are p-solvable, then G is p-solvable.*

PROOF: By induction on $|G|$.

For all $N\trianglelefteq G$ we have $G/N=(HN/N)(KN/N)$ and HN/N and KN/N are mutually permutable subgroups of G/N. If $N\neq 1$, then G/N is p-solvable by induction. Now, by Theorem A there is $1\neq N\trianglelefteq G$, such that $N\leq H$ or $N\leq K$ or $N\leq\boldsymbol{F}(G)$. In any case, N is p-solvable. It follows G

is p-solvable. ■

REMARK 3. Let $G=HK$ be a group such that H and K are mutually permutable. Then for every subgroup L of G such that $H\leq L$(or $K\leq L$) we have $L=H(L\cap K)$ (or $L=K(L\cap H)$) where H and $L\cap K$ (or K and $H\cap L$) are mutually permutable, by 3.5 (a).

The Proofs of our Theorems B and C

THEOREM B. *Let p be a prime number and $G=HK$ a group such that H and K are p-supersolvable. If H and K are mutually permutable and G' is p-nilpotent, then G is p-supersolvable.*

PROOF. Suppose that the Theorem fails, and let G be a minimal counterexample. Lemma 2.4 is obviously applicable. So G contains a unique minimal normal subgroup N which has the properties stated in Lemma 2.4.

Since we may assume that $G'>1$, we have $N\leq G'$. Since G' is p-nilpotent and $N=\boldsymbol{C}_G(N)$ we obtain that G' is a p-group. So $G'=N\in Syl_p(G)$ and the complements of N in G are cyclic (see [4]. p.165, Th.3.8). Clearly $HN<G$ and $KN<G$. Hence HN and KN are p-supersolvable by Remark 3.

Let L be a p-complement of HN and R a p-complement of KN. It is easy to check that L and R are cyclic groups of orders dividing p-1. Hence $G/N=(LN/N)(RN/N)$ is an abelian group whose exponent divides p-1. Since G/N acts irreducibely on N we have $|N|=p$, by ([4]. p.165, Th.3.8.) which means that G is p-supersolvable. ■

As consequence of Theorem B we have

COROLLARY 2 ([1]. Th.3.7). *Suppose that H and K are supersolvable subgroups of G, G' is nilpotent and $G=HK$. Suppose further that H and K are mutually permutable. Then G is supersolvable.*

To prove the Theorem C. we develop a necessary lemma.

LEMMA (*). *Let p be a prime number and $G=HK$ a group such that H and K are p-supersolvable. If H and K are totally permutable, then G is p-supersolvable.*

PROOF. Suppose that the Lemma is false, and let G be a counterexample of smallest order. By Corollary 1 G is p-solvable and so Lemma 2.4 is applicable. So G contains a unique minimal normal subgroup N which has the properties stated in Lemma 2.4.

We can assume $N \leq H$ by Lemma 3.6.

Let $N_1 \leq N$ such that $N_1 \trianglelefteq H$ and $|N_1| = p$ and let V be a complement of N in G. We have that $U = V \cap H$ is a complement of N in H. Let $D = N \cap K \trianglelefteq K$. By hypothesis we have $DU = UD$. So, $D = D(U \cap N) = DU \cap N \trianglelefteq DU$ and $D \trianglelefteq H$. So $D \trianglelefteq G$ and $D = 1$ or $D = N$.

If $D = 1$, then $N_1 = N_1(K \cap N) = N_1K \cap N \trianglelefteq N_1K$. So $N_1 \trianglelefteq G$, a contradiction.

Suppose $D = N$. We have that $W = K \cap V$ is a complement of N in K. By hypothesis we have $N_1W = WN_1$.

So $N_1 \trianglelefteq N_1W$. It follows $N_1 \trianglelefteq K$ and $N_1 \trianglelefteq G$. ■

The above Lemma implies the following

COROLLARY 3 ([1]. Th. 3.1). *Suppose that H and K are supersolvable subgroups of G and $G = HK$. Suppose further that H and K are totally permutable. Then G is supersolvable.*

For a formation-theoretic generalization of our Lemma (*) and Corollary 3 see [7].

THEOREM C. *Let p be a prime number and $G = HK$ a group such that H is a p-supersolvable and K a p-nilpotent subgroup of G. If H and K are mutually permutable, then G is p-supersolvable.*

PROOF. Suppose that the Theorem is false and let G be minimal counterexample. By corollary 1 G is p-solvable and so Lemma 2.4 is applicable. So G contains a unique minimal normal subgroup N which has the properties stated in Lemma 2.4.

By Proposition 3.5 (b) and our Lemma (*) we have $H \cap K = D \neq 1$.

First we show that $N \not\leq K$. Suppose that $N \leq K$. Since K is p-nilpotent and $N = \boldsymbol{C}_G(N)$, K is p-group. By Proposition 3.5 (c) $D \trianglelefteq \trianglelefteq G$. So D^G is a p-subgroup and hence $D \leq D^G = N$ and $D = N \cap H \trianglelefteq H$. Let V be a complement of N in G. Then $K = NS$ and $S \cap N = 1$, where $S = K \cap V$. By Proposition 3.5 (a) $DS = SD$. Therefore $D = D(S \cap N) = SD \cap N \trianglelefteq DS$ and hence $D \trianglelefteq K$. Thus $N = D \trianglelefteq G$. Since $\boldsymbol{O}_{p'}(H) = 1$ and $H_p \trianglelefteq H$, p is the largest prime divisor of H and hence of $|G|$ by Remark 1. Let $H_{p'}$ be a Sylow p-complement of H and let x be a q-element of $H_{p'}$, where q is a prime. By hypothesis $\langle x \rangle K = K \langle x \rangle$. Since $p > q$, by Burnside's splitting theorem x normalizes K and so $H_{p'}$ normalizes K. Thus $\boldsymbol{O}_p(G) = H_pK = N = D$ and hence $G = H$ which is a contradiction.

Secondly we show that $D_H = 1$. If $D_H > 1$, then $(D_H)^G = (D_H)^{HK} = (D_H)^K \leq K$. Thus $N \leq (D_H)^G \leq K$ which is a contradiction.

Since $D_H=1$, D is nilpotent by Lemma 3.7 (a). Since $D \trianglelefteq \trianglelefteq G$ by Proposition 3.5 (c). D^G is nilpotent. Thus $D \leq D^G = N$. Since $D \leq H \cap K \cap N$, we have $HN \neq G \neq KN$. So HN and KN are p-supersolvable, by Remark 3. Since $C_G(N)=N$, by Lemma 2.1 $H_p \trianglelefteq H$ and $K_p \trianglelefteq K$. By Lemma 3.7 (b) D is normalized by every p'-element of H and K.

Since D is a p-group, we have $HK_p \neq G$ or $KH_p \neq G$. Suppose $KH_p \neq G$. By Remark 3 KH_p is p-supersolvable and since $N \leq KH_p$, $O_{p'}(KH_p)=1$. By Lemma 2.1 $H_pK_p \trianglelefteq KH_p$.

For $g=hk \in G$, $h \in H$, $k \in K$ we have $(H_p)^g=(H_p)^{hk}=(H_p)^k \leq (H_pK_p)^k = H_pK_p$. Therefore, $H_p \leq (H_p)^G = N$. So H_p is abelian and hence $D \trianglelefteq H$. So $N=(D)^G=(D)^K \leq K$. This is a contradiction.

Thus $HK_p \neq G$, $KH_p = G$ and $H=H_p(H \cap K)=H_p$. So HK_p is a Sylow p-subgroup of G.

For any $g=kh \in G$, $k \in K$, $h \in H$, $K_p^g = K_p^{kh} = K_p^h \leq HK_p$. Therefore $K_p \leq (K_p)^G = N$. So K_p is abelian and hence $D \trianglelefteq K$ and $N \leq H$. Let V be a complement of N in G. Then $H=UN$ and $U \cap N = 1$, where $U = H \cap V$. Further $D=D(N \cap U)=N \cap DU \trianglelefteq DU$. So $D \trianglelefteq H$ and $N=D$, the final contradiction. ■

Our Theorem C contains

COROLLARY 3 ([1]. Th. 3.2). *Suppose that H is a nilpotent, K a supersolvable subgroup of G and $G=HK$. Suppose further that H and K are mutually permutable. Then G is supersolvable.*

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