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Toeplitz and Hankel operators on Bergman one space

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Abstract

This note provides necessary and sufficient conditions for Toeplitz and Hankel operators with harmonic symbols to boundedly map the Bergman one space to the Lebesgue one space.

1 Introduction

We begin by recalling some standard notations and definitions. Let dA denote the Lebesgue area measure on the unit disc D of the complex plane C. For $1\leq p<\infty$ and for a Lebesgue measurable function $f: D\rightarrow C$, let

$$\|f\|_{p}=\left(\int|f|^{p}\,dA\right)^{\frac{1}{p}}.$$

Here and elsewhere unless otherwise stated all integrals are taken over the unit disc. For $1\leq p<\infty$, the Bergman space L_{a}^{p} is the set of all those analytic functions $f: D\rightarrow C$ such that $\|f\|_{p}<\infty$. As usual the space of bounded analytic functions will be denoted by H^{∞} and the subspace of functions vanishing at the origin will be denoted by H_{0}^{∞}.

The Bergman space L_{a}^{2} is of course a functional Hilbert space, and the reproducing kernel at a point $w\in D$ is

$$k_{w}(z)=\pi^{-1}(1-\bar{w}z)^{-2},\ z\in D. \tag{1}$$

There is an explicit formula for the orthogonal projection (Bergman projection) P from the Lebesgue space $L^{2}(D, dA)$ onto the Bergman space L_{a}^{2}:

$$p(g)(z)=\int g(w)(1-\bar{w}z)^{-2}\frac{dA(w)}{\pi},\ g\in L^{2}(D, dA) \tag{2}$$

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and for all $z\in D$. The integral in equation (2) makes sense when $g\in L^{p}(D, dA)$ for all $1\leq p<\infty$ so we can use (2) to define P on $L^{p}(D, dA)$ for $1\leq p<\infty$. Then $P: L^{p}(D, dA)\rightarrow L_{a}^{p}$ is bounded for $1<p<\infty$ (this was first proved by Zaharjuta and Judovič; Axler ([3], Theorem 1.10) gives a proof using the Schur criterion for boundedness) and unbounded for $p=1$. However, we may note that there are bounded projections from $L^{1}(D, dA)$ onto L_{a}^{1} ([7], Theorem 1 (iv)).

For $v\in L^{1}(D, dA)$ and $f\in H^{\infty}$ let

$$T_{v}(f)=P(vf) \quad \text{and}$$
$$H_{v}(f)=(I-P)(vf)=vf-P(vf).$$

Since P does not map $L^{1}(D, dA)$ into L_{a}^{1} boundely; it is of interest to find the necessary and sufficient conditions on v, so that the Toeplitz operator $T_{v}: L_{a}^{1}\rightarrow L_{a}^{1}$, respectively, the Hankel operator $H_{v}: L_{a}^{1}\rightarrow L^{1}(D, dA)$ as densely defined operators (H^{∞} is dense in L_{a}^{1}) are bounded.

The main result of section 3, Proposition 8, characterizes Toeplitz operators with real-valued harmonic symbols which are bounded on L_{a}^{1}. Proposition 10 in section 4 provides necessary and sufficient conditions for a Hankel operator with a conjugate analytic symbol to boundedly map L_{a}^{1} into $L^{1}(D, dA)$.

In a 1972 paper Stegenga [8] characterized bounded Toeplitz operators on the Hardy space H^{1} in the case when the symbol is either a real-valued function or the conjugate of an analytic function. In a more recent paper Cima and Stegenga [4] proved that the Hankel operator $H_{f}: H^{1}\rightarrow H^{1}$, with an analytic symbol f (see their paper for the definition of this Hankel operator and other details) is bounded if and only if

$$\sup_{I}\frac{(\log|I|)^{2}}{|I|}\int_{S(I)}|f'(z)|^{2}\log\frac{1}{|z|}dA(z)<\infty. \tag{3}$$

Here I denotes a subarc of the unit circle, $|I|$ is the arc-length measure of I, and $S(I)$ is the Carleson square with I as the base. We may note that the condition on f' in Proposition 10 can be viewed in the form of (3), provided that the Carleson square $S(I)$ is replaced by the "half" Carleson square$=\{z\in S(I):|z|\leq 1-|I|/2\}$.

Throughout this note the letter c will be used as a generic notation for a constant.

2 Bloch space and dual of L_{a}^{1}

An analytic function $f: D\rightarrow C$ is called a Bloch function if $\sup_{z\in D}|f'(z)|(1-|z|^{2})<\infty$. Let B denote the space of Bloch functions. For

$f\in B$, the Bloch norm $\|f\|_B$ is defined by

$$\|f\|_B=|f(0)|+\sup_{z\in D}|f'(z)|(1-|z|^2). \tag{4}$$

For $f\in B$, it follows by integration that

$$|f(z)-f(0)|\leq\frac{1}{2}\|f-f(0)\|_B\log\left(\frac{1+|z|}{1-|z|}\right),\ z\in D, \tag{5}$$

so $f\in L_a^p$ for $0<p<\infty$. A very useful property of Bloch functions is the Möbius invariance of the Bloch norm, more precisely, if $f\in B$ then

$$\|f\circ\phi_w-f(w)\|_B=\|f-f(0)\|_B \tag{6}$$

for every Möbius map ϕ_w (to recall the definition of ϕ_w see (7)).

The dual of L_a^1 can be identified with the Bloch space B. There are many versions of this identification in the literature; see for example [1], Theorem 2.4; [3], Theorem 2.6 or [5], Lemma 5.1. Here we include an identification with the pairing that will be used in this note.

PROPOSITION 1. *Let $f\in B$. Then the pairing*

$$\langle g,f\rangle=\int g(z)\bar{f}'(z)(1-|z|^2)dA(z),\ g\in L_a^1$$

defines a bounded linear functional on L_a^1. Furthermore, given $\psi\in(L_a^1)^*$, there exists $f\in B$, unique up to a constant, such that

$$\psi(g)=\langle g,f\rangle\ g\in L_a^1 \text{ and}$$
$$\frac{1}{10}\|f\|_B\leq\|\psi\|\leq\|f\|_B,$$

where $\|\psi\|$ is the operator norm of ψ.

3 Bounded Toeplitz operators

In Lemma 2 we note a formula for a "differentiating" kernel in L_a^2. The corollary following the lemma is used to evaluate an integral during the course of the proof of Proposition 8.

LEMMA 2. *Let $h\in L_a^2$, $w\in D$ and $l_w(z)=2\pi^{-1}z(1-\bar{w}z)^{-3}$, $z\in D$. Then*

$$h'(w)=\int h\bar{l}_w\,dA.$$

PROOF. Let $h\in L_a^2$ and let k_w be the reproducing kernel (1) in L_a^2. Write

$$h(w)=\int h\bar{k}_w\, dA$$

and differentiate. □

COROLLARY 3. *If* $h\in L_a^2$ then

$$\int h|l_w|^2\, dA=\frac{h'(w)}{\pi(1-|w|^2)^3}2w + \frac{h(w)}{\pi}\left(\frac{6|w|^2}{(1-|w|^2)^4}+\frac{2}{(1-|w|^2)^3}\right).$$

PROOF. Note that $\int h|l_w|^2\, dA=\int hl_w\bar{l}_w\, dA$ and apply Lemma 2. □

The estimation given below of the integral in Lemma 4 is standard; the calculations presented will also be used in other instances. See [9], Lemma 4.2.2, page 53 and Lemma 4.2.8, page 57, for more general versions of Lemmas 4 and 5.

LEMMA 4. *Let* $w\in D$. *Then*

$$\int|1-\bar{w}z|^{-3}\, dA(z)\leq 2\pi(1+|w|)(1-|w|^2)^{-1}.$$

PROOF. Let $\phi_w: D\to D$ be the Möbius map

$$\phi_w(t)=(w-t)(1-\bar{w}t)^{-1},\ t\in D. \tag{7}$$

We change the variable in the integral by writing $z=\phi_w(t)$. Then

$$\begin{gathered}(1-\bar{w}z)=(1-|w|^2)(1-\bar{w}t)^{-1} \text{ and}\\ dA(z)=|\phi_w'(t)|^2\, dA(t)=(1-|w|^2)^2|1-\bar{w}t|^{-4}\, dA(t),\end{gathered} \tag{8}$$

so

$$\begin{aligned}\int|1-\bar{w}z|^{-3}\, dA(z)&=(1-|w|^2)^{-1}\int|\phi_w(t)||w-t|^{-1}\, dA(t)\\ &\leq(1-|w^2|)^{-1}\int|w-t|^{-1}\, dA(t).\end{aligned}$$

Integrating over the disc with center w and radius $(1+|w|)$ (so this disc contains D) and using polar coordinates with the pole at w, we obtain

$$\int|w-t|^{-1}\, dA(t)\leq 2\pi(1+|w|).$$

Result follows. □

The following lemma essentially shows that the hyperbolic derivative of a function in L_a^2 is projected back to itself by the Bergman projection.

LEMMA 5. *Let $g \in L_a^2$ with $g(0)=g'(0)=0$. Then*

$$P((1-|w|^2)(\bar{w})^{-1}g'(w))(z)=g(z),\ z\in D,$$

where P is the Bergman projection defined in (2).

PROOF. Writing $g(w)=\sum_{n=0}^{\infty} a_n w^n$ and $w=re^{i\theta}$, $0\le\theta<2\pi$, and doing a standard integration involving orthogonal functions, we have

$$\begin{aligned}\int |g'(w)|^2(1-|w|^2)^2\,dA(w) &= 2\pi\sum_0^{\infty} n^2|a_n|^2\int_0^1 r^{2n-1}(1-r^2)^2\,dr\\ &= 2\pi\sum_{n=1}^{\infty}\frac{n}{(n+1)(n+2)}|a_n|^2\\ &\le 2\pi\sum_1^{\infty}\frac{|a_n|^2}{n+1}=2\int |g|^2\,dA,\end{aligned}$$

so, clearly the function $(1-|w|^2)\ (\bar{w})^{-1}g'(w)$, $w\in D$ is in $L^2(D, dA)$. Fix $z\in D$. Then

$$\begin{aligned}&P((1-r^2)\textstyle\sum_{n=1}^{\infty} na_n r^{n-2}e^{ni\theta})\ (z)\\ &=\pi^{-1}\int_0^1\int_0^{2\pi}(1-r^2)(\sum_{n=1}^{\infty} na_n r^{n-2}e^{ni\theta})(\sum_{n=0}^{\infty}(n+1)r^n e^{-ni\theta}z^n)\ rdrd\theta\\ &=2\int_0^1\sum_{n=1}^{\infty} n(n+1)a_n z^n r^{2n-1}(1-r^2)dr\\ &=g(z)\end{aligned}$$

as desired. □

Let $\frac{\partial}{\partial z}$ denote the usual operator (defined on continuously differentiable functions on D)

$$\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left\{\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right\}.$$

If f is an analytic function on D, it immediately follows from Cauchy-Riemann equations that

$$\frac{\partial f}{\partial \bar{z}}=0 \text{ and } \frac{\partial \bar{f}}{\partial \bar{z}}=\bar{f}'.$$

On several occasions, we will make use of the following application of Green's theorem.

LEMMA 6. *Let u be a (complex-valued) continuously differentiable function on D. Suppose both u and $\frac{\partial u}{\partial \bar{z}}(1-|z|^2)$ are integrable on D.*

Then

$$\int \frac{\partial}{\partial \bar{z}}(u(z)(1-|z|^2))dA(z)=0.$$

PROOF. Let $0<r<1$ and $rD=\{z\in D: |z|<r\}$. Apply Green's theorem to $u(z)(r^2-|z|^2)$ on rD to obtain

$$\int_{rD} \frac{\partial}{\partial \bar{z}}(u(z)(r^2-|z|^2))dA(z)=0, \text{ i. e.,}$$
$$\int_{rD} \frac{\partial u}{\partial \bar{z}}(r^2-|z|^2)dA(z)-\int_{rD} u(z)zdA(z)=0.$$

Notice that for $z\in rD$, $|\frac{\partial u}{\partial \bar{z}}|(r^2-|z|^2)\leq|\frac{\partial u}{\partial \bar{z}}|(1-|z|^2)$. Let $r\to 1-$ and apply Lebesgue Dominated Convergence Theorem to get

$$\int \frac{\partial u}{\partial \bar{z}}(1-|z|^2)dA(z)-\int u(z)zdA(z)=0,$$

which is the desired result. □

We now prove a simple necessary condition for a Toeplitz operator T_v with a harmonic symbol to be bounded on L_a^1.

LEMMA 7. *Let $v\in L^1(D, dA)$ be a real-valued harmonic function on D and suppose that the Toeplitz operator $T_v: L_a^1\to L_a^1$ is bounded. Then v is the real part of a Bloch function. Thus in particular $v\in L^p(D, dA)$ for all $0<p<\infty$.*

PROOF. Since $T_v: L_a^1\to L_a^1$ is bounded,

$$f\to \int T_v(f)dA, \quad f\in L_a^1$$

is a bounded linear functional on L_a^1 so by the Hahn-Banach theorem can be extended to a linear functional on $L^1(D, dA)$. Identifying the dual of $L^1(D, dA)$ as $L^\infty(D, dA)$ we have

$$\int T_v(f)dA=\int f\bar{g}dA, \quad f\in L_a^1$$

for some $g\in L^\infty(D, dA)$. The left-hand side integral is $\pi T_v(f)(0)$, which is $\int vfdA$ for $f\in H^\infty$, so

$$\int vfdA=\int f\bar{g}dA$$

$$= \int P(f)\bar{g}dA$$
$$= \int f\overline{P(g)}dA, \quad f \in H^{\infty}. \tag{9}$$

In deducing the last integral we used the orthogonality of the Bergman projection on $L^2(D, dA)$. Pick an analytic function h such that $v = h + \bar{h}$. Then $h \in L_a^1$ ([3], Theorem 1.21) and

$$\int vfdA = \int f\bar{h}dA, \quad f \in H_0^{\infty}.$$

Hence from (9)

$$\int f\bar{h}dA = \int f\overline{P(g)}dA, \quad f \in H_0^{\infty}.$$

Replacing f by $f(z) = z^n$, $n = 1, 2, \ldots$ we deduce that h and $P(g)$ differ at most by a constant. However, $P(g) \in B$([5], Theorem V′), so the result follows. □

We are ready to prove the main theorem of this section.

PROPOSITION 8. *Let v be a real-valued harmonic function in $L^1(D, dA)$. Then the Toeplitz operator $T_v: L_a^1 \to L_a^1$ is bounded if and only if*

$$\sup_D |v| < \infty \text{ and } \sup_{z \in D} |\nabla(v)(z)|(1-|z|^2)\log\frac{1}{1-|z|^2} < \infty.$$

PROOF. Suppose $T_v: L_a^1 \to L_a^1$ is bounded. If $g \in B$ with $g(0) = g'(0) = 0$ then $g'(w)(\bar{w})^{-1}(1-|w|^2)$, $w \in D$ is bounded and so there exists a constant c such that

$$\left|\int T_v(f)(w)\bar{g}'(w)w^{-1}(1-|w|^2)dA(w)\right| \leq c\|f\|_1\|g\|_B, \text{ i.e.,}$$
$$\left|\int P(vf)(w)\bar{g}'(w)w^{-1}(1-|w|^2)dA(w)\right| \leq c\|f\|_1\|g\|_B, \quad f \in H^{\infty}. \tag{10}$$

Using Fubini's theorem and Lemma 5, we have

$$\left|\int vf\bar{g}dA\right| \leq c\|f\|_1\|g\|_B, f \in H^{\infty} \text{ and } g \in B \text{ with } g(0) = g'(0) = 0.$$

The use of Fubini's theorem in (10) is justified since both f and $g'(w)w^{-1}(1-\bar{w}z)$, $w \in D$ are bounded, $|v(z) - v(0)| \leq c\log(1-|z|)^{-1}$, $z \in D$ (Lemma 7 and use inequality (5)) and $\log(1-|z|)|1-\bar{w}z|^{-2}$, $(z, w) \in D \times D$ is integrable over $D \times D$, which can be verified by a direct calculation. Moreover, for $f \in H^{\infty}$ and $g \in B$

$$
\begin{aligned}
&|\int vf\overline{(g(0)+g'(0)z)}dA| \\
&\qquad\qquad =|\int P(vf)\overline{(g(0)+g'(0)z)}dA| \\
&\qquad\qquad \leq\|T_v\|\|f\|_1\|g(0)+g'(0)z\|_\infty \\
&\qquad\qquad \leq c\|f\|_1\|g\|_B.
\end{aligned}
$$

Thus

$$|\int vf\bar{g}dA|\leq c\|f\|_1\|g\|_B,\quad f\in H^\infty \text{ and } g\in B. \tag{11}$$

To deduce that v is bounded, we replace f and g by suitable kernel functions: fix $w\in D$ and put $f(z)=g(z)=z(1-\bar{w}z)^{-3}$, $z\in D$ in (11). Then the $\|g\|_B\leq c(1-|w|^2)^{-3}$ and the $\|f\|_1$ is estimated in Lemma 4, so

$$|\int v(z)|z|^2|1-\bar{w}z|^{-6}\, dA(z)|\leq c(1-|w|^2)^{-4},$$

or as in the notation of Lemma 2,

$$(1-|w|^2)^4|\int v|l_w|^2\, dA|\leq c. \tag{12}$$

Now let $h\in L_a^2$ be an analytic function such that $v=h+\bar{h}$. Then by Lemma 7, $h'(w)(1-|w|^2)$ is bounded. Taking real parts in the formula for $(1-|w|^2)^4\int h|l_w|^2\, dA$ in Corollary 3 and using (12) we deduce that v is bounded.

Replacing f by zf in (11) and noting that $\|zf\|_1\leq\|f\|_1$ we have

$$|\int vzf\bar{g}dA|\leq c\|f\|_1\|g\|_B,\quad f\in H^\infty \text{ and } g\in B. \tag{13}$$

Applying Lemma 6 for the function $u=vf\bar{g}$ (Lemma 7 is used to verify that the hypothesis of Lemma 6 is satisfied) and using the Cauchy-Riemann equations, we deduce that

$$\int\frac{\partial v}{\partial\bar{z}}f\bar{g}(1-|z|^2)dA=\int vzf\bar{g}dA-\int vf\bar{g}'(1-|z|^2)dA. \tag{14}$$

Since we now know that v is bounded; from (13) and (14) we have

$$|\int\frac{\partial v}{\partial\bar{z}}f\bar{g}(1-|z|^2)dA|\leq c\|f\|_1\|g\|_B,\quad f\in H^\infty \text{ and } g\in B. \tag{15}$$

Write $v=h+\bar{h}$ for some $h\in L_a^2$. Then $\frac{\partial v}{\partial\bar{z}}=\bar{h}'$. Fix $w\in D$ and as before we replace f and g by suitable kernel functions; let $f(z)=(1-\bar{w}z)^{-3}$, $z\in D$ and $g(z)=\log(1-\bar{w}z)$, $z\in D$. Then the $\|g\|_B\leq 2$ and the $\|f\|_1$ is estimated in

Lemma 4, therefore, from (15)

$$|\int h'(z)\log(1-\bar{w}z)(1-w\bar{z})^{-3}(1-|z|^2)dA(z)|\leq c(1-|w|^2)^{-1}. \qquad (16)$$

But then for $f\in L_a^1(D, (1-|z|^2)dA)$,

$$2\pi^{-1}\int f(z)(1-w\bar{z})^{-3}(1-|z|^2)dA(z)=f(w),\ w\in D \qquad (17)$$

([7], Theorem 1 (iv)). Since $h\in B$ (Lemma 7), the hypothesis of (17) is trivially satisfied by h', so from (16) it follows that

$$h'(w)\log(1-|w|^2)(1-|w|^2),\quad w\in D$$

is bounded as desired.

Conversely, suppose v is a real-valued harmonic function on D such that both v and $|\nabla(v)(z)|(1-|z|^2)\log(1-|z|^2)$ are bounded. Fix $f\in H^\infty$ and $g\in B$. Then equation (14) still holds and we may rewrite it as:

$$\int vzf\bar{g}dA=\int\frac{\partial v}{\partial\bar{z}}f\bar{g}(1-|z|^2)dA+\int vf\bar{g}'(1-|z|^2)dA. \qquad (18)$$

Note that $|\nabla v|=\frac{1}{2}|h'|$ where $v=h+\bar{h}$ and h is analytic. Also $\frac{\partial v}{\partial\bar{z}}=\bar{h}'$. Now to estimate the second integral in (18), use the hypothesis on $|\nabla v|$ and the standard point estimate for a Bloch function g (5):

$$\begin{aligned}|g(z)|&\leq|g(0)|+\|g-g(0)\|_B\log(1-|z|)^{-1}\\&\leq\|g\|_B(1+\log(1-|z|)^{-1}),\quad z\in D.\end{aligned}$$

Then from (18)

$$|\int v\tilde{f}\bar{g}dA|\leq c\|f\|_1\|g\|_B, \qquad (19)$$

where $\tilde{f}$ is the function zf. Since $v\tilde{f}\in L^2(D, dA)$, $g\in L_a^2$ and P is the orthogonal projection from $L^2(D, dA)$ onto L_a^2, the integral in (19) is equal to $\int P(v\tilde{f})\bar{g}dA$, so

$$|\int T_v(\tilde{f})\bar{g}dA|\leq c\|f\|_1\|g\|_B. \qquad (20)$$

It is not hard to see that $P(\bar{z}g)\in B$. In fact a direct calculation shows that, if $g(z)=\sum_0^\infty a_nz^n$, $z\in D$ then $P(\bar{z}g)(w)=\sum_1^\infty a_nw^{n-1}$, $w\in D$. Thus

$$|\int T_v(\tilde{f})(z)z\bar{g}(z)dA(z)|=|\int T_v(\tilde{f})\overline{P(\bar{z}g)}dA|$$

$$\leq c\|f\|_1\|P(\bar{z}g)\|_B(\text{from (20)}) \tag{21}$$
$$\leq c\|f\|_1\|g\|_B. \tag{22}$$

By an application of Lemma 6 to $u=T_v(\tilde{f})\bar{g}$ and the estimate in (21) show that

$$\left|\int T_v(\tilde{f})(z)g'(z)(1-|z|^2)dA(z)\right|\leq c\|f\|_1\|g\|_B$$
$$\leq c\|\tilde{f}\|_1\|g\|_B,$$

whence $|\langle T_v(\tilde{f}), g\rangle|\leq c\|f\|_1\|g\|_B$. The pairing $\langle\,,\rangle$ was defined in Proposition 1. Therefore, for $f\in H_0^\infty$

$$|\langle T_v(f), g\rangle|\leq\|f\|_1\|g\|_B.$$

Since the dual of L_a^1 is the Bloch space (Proposition 1), it follows that for $f\in H_0^\infty$

$$\|T_v(f)\|_1\leq c\|f\|_1.$$

Thus $T_v: L_a^1\to L_a^1$ is bounded. □

4 Bounded Hankel operators

Let $f\in L^1(D, dA)$ and $g\in H^\infty$. Let us recall the definition of $H_f(g)$:

$$H_f(g)=(I-P)(fg)=fg-P(fg).$$

Using $g=P(g)$, we get the following well-known formula for a Hankel operator:

$$H_f(g)(z)=\int\frac{f(z)-f(w)}{(1-\bar{w}z)^2}g(w)\frac{dA(w)}{\pi}\quad\text{for almost all } z\in D. \tag{23}$$

Formula (23) for a Hankel operator suggests that we investigate the growth of

$$\int\frac{|f(z)-f(w)|}{|1-\bar{w}z|^2}dA(w). \tag{24}$$

Lemma 9 provides a growth condition for (24) when $f\in B$.

LEMMA 9. *Let $f\in B$. Then there exists a constant c such that*

$$\int\frac{|f(z)-f(w)|}{|1-\bar{w}z|^2}dA(z)\leq c\|f-f(0)\|_B\log^2\frac{c}{(1-|w|^2)},\qquad w\in D.$$

PROOF. Let us change the variable in the integral by writing $z=\phi_w(t)$ (see (7) for the definition of $\phi_w(t)$ and also (8));

$$\int\frac{|f(z)-f(w)|}{|1-\bar{w}z|^2}dA(z)=\int\frac{|f\circ\phi_w(t)-f(w)|}{|1-\bar{w}t|^2}dA(t). \tag{25}$$

From (5) and (6)

$$\begin{aligned}|f\circ\phi_w(t)-f(w)|&=|f\circ\phi_w(t)-f\circ\phi_w(0)|\\&\le\|f\circ\phi_w-f\circ\phi_w(0)\|_B\log(1-|t|)^{-1}\\&=\|f-f(0)\|_B\log(1-|t|)^{-1}.\end{aligned}$$

Thus from (25)

$$\begin{aligned}\int\frac{|f(z)-f(w)|}{|1-\bar{w}z|^2}dA(z)&\le\|f-f(0)\|_B\int\frac{-\log(1-|t|)}{|1-t\bar{w}|^2}dA(t)\\&=2\pi\|f-f(0)\|_B\int_0^1\frac{-\log(1-t)}{(1-t^2|w|^2)}tdt\\&\le2\pi\|f-f(0)\|_B\int_0^1\frac{-\log(1-t)}{(1-t|w|)}dt.\end{aligned}$$

Put $g(x)=\int_0^1-\log(1-t)(1-tx)^{-1}dt,\ 0\le x<1$. Then

$$|g'(x)|\le\int_0^1-\log(1-t)(1-tx)^{-2}\,dt.$$

View $(1-tx)^{-2}dt$ as $x^{-1}\ d(1-tx)^{-1}$ and evaluate the improper integral by doing an integration by parts, to get

$$|g'(x)|\le-\log(1-x)x^{-1}(1-x)^{-1},\quad 0<x<1.$$

Since $-\log(1-x)x^{-1}$ is an increasing function of x on $0<x<1$ we have, for $0<x<1$

$$|g'(x)|\le\begin{cases}4\log2 & \text{if } 0<x\le\frac{1}{2}\\ -2\log(1-x)(1-x)^{-1} & \text{if otherwise.}\end{cases}$$

Thus for some constant c, $|g'(x)|\le c-c\log(1-x)(1-x)^{-1}$, $0<x<1$. Hence

$$|g(x)-g(0)|\le cx+\frac{1}{2}\log^2(1-x),\quad 0\le x<1,$$

from which the desired result follows. □

PROPOSITION 10. *For $f\in L_a^2$, the Hankel operator $H_f: L_a^1\to L^1(D, dA)$ is bounded if and only if*

$$\|f\|_{LB}=\sup_{z\in D}|f'(z)|(1-|z|^2)\log\frac{1}{1-|z|^2}<\infty. \tag{26}$$

Note that we do not assume f to be bounded.

PROOF. Suppose (26) holds. Then trivially $\|f\|_B<\infty$. Fix $h\in$

$L^{\infty}(D, dA)$. We begin by showing that the function defined by:

$$H(w)=\int(f(z)-f(w))(1-w\bar{z})^{-2}h(z)dA(z), \quad w\in D$$

is a Bloch function. Indeed;

$$\begin{aligned}H'(w)(1-|w|^2)&=\int -f'(w)(1-|w|^2)(1-w\bar{z})^{-2}h(z)dA(z)\\&\quad+2(1-|w|^2)\int(f(z)-f(w))(1-w\bar{z})^{-3}\bar{z}h(z)dA(z)\\&=I_1(w)+2(1-|w|^2)I_2(w).\end{aligned}$$

Now

$$\int|1-w\bar{z}|^{-2}\,dA(z)=\pi|w|^{-2}\log(1-|w|^2)^{-1}, \quad w\in D \tag{27}$$

(the limit as $w\to 0$ of the right-hand side of (27) clearly exists); whence by (26) I_1 is bounded on D. To show that $(1-|w|^2)I_2(w)$ is bounded, it is sufficient to show that $(1-|w|^2)^2I_2'(w)$ is bounded ([6], Theorem 5.5). Indeed

$$\begin{aligned}(1-|w|^2)^2I_2'(w)&=(1-|w|^2)\int -f'(w)(1-|w|^2)(1-w\bar{z})^{-3}\bar{z}h(z)dA(z)\\&\quad+3(1-|w|^2)^2\int(f(z)-f(w))(1-w\bar{z})^{-4}\bar{z}^2h(z)dA(z)\\&=J_1(w)+J_2(w).\end{aligned}$$

Then from Lemma 4

$$|J_1|\le 4\pi\|f\|_B\|h\|_\infty$$

and the fact that

$$|J_2|\le c\|f\|_B\|h\|_\infty$$

follows from [2], Theorem 1(B), see also equation (14), page 327 of the same reference. Thus H is a Bloch function and

$$\|H\|_B\le c\|h\|_\infty,$$

so here the constant c depends on f.

In view of the following well-known identity (which also follows from an application of Lemma 6)

$$\int g(w)w\bar{H}(w)dA(w)=\int g(w)\bar{H}'(w)(1-|w|^2)dA(w), \quad g\in H^\infty \tag{28}$$

we have

$$|\int g(w)w\bar{H}(w)dA(w)|\le c\|g\|_1\|h\|_\infty,\quad g\in H^\infty.$$

Applying Fubini's Theorem

$$|\int\Big(\int\frac{\bar{f}(z)-\bar{f}(w)}{(1-\bar{w}z)^2}g(w)wdA(w)\Big)\bar{h}(z)dA(z)|\le c\|g\|_1\|h\|_\infty,\quad g\in H^\infty.$$

So

$$|\int H_{\bar{f}}(\tilde{g})\,\bar{h}dA|\le c\|\tilde{g}\|_1\|h\|_\infty,\quad g\in H^\infty \text{ and } h\in L^\infty(D,dA),$$

where $\tilde{g}=wg$. Hence

$$\int|H_{\bar{f}}(g)|dA\le c\|g\|_1,\quad g\in H_0^\infty.$$

It follows that $H_{\bar{f}}$ is bounded.

To prove the converse, suppose $f\in L_a^2$ and

$$\|H_{\bar{f}}(g)\|_1\le c\|g\|_1,\quad g\in H^\infty.$$

Then

$$|\int H_{\bar{f}}(g)\,\bar{h}dA|\le c\|g\|_1\|h\|_\infty,\quad g\in H^\infty \text{ and } h\in L^\infty(D,dA). \tag{29}$$

Let $h\in H_0^\infty$. Then $P(\bar{h})=0$. Clearly for all $g\in H^\infty$, $\bar{f}g\in L^2(D,dA)$. Recalling that P: $L^2(D,dA)\to L_a^2$ is the orthogonal projection;

$$\begin{aligned}\int H\bar{f}(g)hdA&=\int\bar{f}ghda-\int P(\bar{f}g)hdA\\&=\int\bar{f}ghdA-\int\bar{f}g\overline{P(\bar{h})}dA\\&=\int\bar{f}ghdA.\end{aligned}\tag{30}$$

Likewise we can show that,

$$\int H_{\bar{f}}(g)\,\bar{h}dA=0,\quad g \text{ and } h\in H^\infty. \tag{31}$$

Replacing the function h in (30) by $h(z)=z$, writing $\tilde{g}$ for the function $\tilde{g}(z)=zg(z)$, $z\in D$ and using (29) we have

$$|\int\bar{f}\tilde{g}dA|\le c\|g\|_1\le c\|\tilde{g}\|_1,\quad \text{i.e.,}$$

$$|\int\bar{f}\tilde{g}dA|\le c\|g\|_1,\quad g\in H_0^\infty.$$

Now by an argument similar to that of Lemma 7 we deduce that $f\in B$.

From (29) and (31) we have

$$|\int H_f(g)\bar{h}dA|\leq c\|g\|_1 \operatorname{dist}(h, H^\infty),\ g\in H^\infty \tag{32}$$

and $h\in L^\infty(D, dA)$, where dist (h, H^∞) is the $L^\infty(D, dA)$ distance from h to H^∞. Fix $w\in D$. Then $\operatorname{dist}(\overline{\log(1-\bar{w}z)}, H^\infty)=2\operatorname{dist}(\operatorname{Im}\overline{\log(1-\bar{w}z)}, H^\infty)\leq 4\pi$, so replacing h in (32) by the function $\overline{\log(1-\bar{w}z)}$, $z\in D$ and using (30) we have from (32)

$$|\int \bar{f}(z)g(z)\log(1-\bar{w}z)dA(z)|\leq c\|g\|_1,\quad g\in H^\infty. \tag{33}$$

Replacing g by zg in (33) and then using identity (28) (with of course f instead of H), we have

$$|\int \bar{f}'(z)g(z)\log(1-\bar{w}z)(1-|z|^2)dA(z)|\leq c\|g\|_1,\quad g\in H^\infty.$$

Now since $f\in B$ and the argument of $\log(1-\bar{w}z)$ is bounded (independent of w and z, and we may assume that neither w nor z is 0)

$$|\int \bar{f}'(z)g(z)\overline{\log(1-\bar{w}z)}(1-|z|^2)dA(z)|\leq c\|g\|_1,\quad g\in H^\infty.$$

Put $g(z)=(1-\bar{w}z)^{-3}$, $z\in D$. Then by Lemma 4, $\|g\|_1\leq 4\pi(1-|w|^2)^{-1}$, so

$$|\int f'(z)\log(1-\bar{w}z)(1-w\bar{z})^{-3}(1-|z|^2)dA(z)|\leq c(1-|w|^2)^{-1}.$$

By (17) we get

$$f'(w)(1-|w|^2)\log(1-|w|^2)^{-1},\quad w\in D$$

to be bounded. □

COROLLARY 11. *Suppose v is a (complex-valued) harmonic function on D such that both v and $\frac{\partial v}{\partial\bar{z}}(1-|z|^2)\log(1-|z|^2)$ are bounded on D. Then the Toeplitz operator*

$$T_v:\ L_a^1\rightarrow L_a^1$$

is bounded.

PROOF. Write $v=f+\bar{g}$ where f and g are integrable analytic functions on D. Since v is bounded $v\in L^2(D, dA)$, so $f+\bar{g}(0)=P(v)\in L_a^2$; consequently $g\in L_a^2$. Also the hypothesis on $\frac{\partial v}{\partial\bar{z}}$ implies that g satisfy the hypothesis of Propostion 10, thus the Hankel operator

$$H\bar{g}: L_a^1 \to L^1(D, dA)$$

is bounded. Since v is bounded, M_v, the multiplication operator by v on $L_a^1 \to L^1(D, dA)$ is also bounded. Note that $M_v = T_v + H_v$ and $H_v = H_{\bar{g}}$. Thus the Toeplitz operator

$$T_v: L_a^1 \to L_a^1$$

is bounded.

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