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An elementary and unified approach to the Mathieu-Witt systems II: The uniqueness of W_{22}, W_{23}, W_{24}

Dedicated to Professor Tosiro Tsuzuku
on his sixtieth birthday

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Abstract.

In this paper we present a new proof of the uniqueness of the large Witt systems W_{22}, W_{23}, W_{24}. Their uniqueness is (almost) simultaneously proved by the same and simple method, and their existence is also shown.

1. Introduction

Although self-contained, this paper is a continuation of our previous article [3], which was intended as the title shows, but made no mention of the uniqueness of the Witt systems. Nowadays not a few proofs of their uniqueness are known (see, e.g. [1], [2], [4], [5, Chap. 20]). The purpose of this paper is to present an alternative, simple, elementary and unfied proof of the uniquenesss of the large Witt systems W_{22}, W_{23}, W_{24}. 'Simple, elementary' means that our proof uses only block intersection property BIP (mentioned later) which is easily shown, not using any knowledge of finite geometry such as projective planes, coding theory, etc. 'Simple, unified' means that the uniqueness of the three systems can be (almost) simultaneously proved by the same method. We note also that our uniqueness proof shows the existence of the three systems (see Remark 3).

DEFINITIONS AND NOTATION. Let Ω be a set of v points and $\mathfrak{B}$ a collection of k-subsets (called blocks) of Ω. The pair $D=(\Omega, \mathfrak{B})$ is called a *t-design* with *parameters* $t, v, k, \lambda (v>k>t>0$ and $\lambda>0)$ or, briefly a *t-(v, k, λ) design* if any t-subset of Ω is contained in exactly λ blocks of $\mathfrak{B}$. If $D=(\Omega, \mathfrak{B})$ is a t-(v, k, λ) design, then, for any $s\leq t$, the number of blocks containing any s-subset of Ω is equal to $\lambda_s=\lambda\binom{v-s}{t-s}\Big/\binom{k-s}{t-s}$, and in particular, $|\mathfrak{B}|=\lambda_0=\lambda\binom{v}{t}\Big/\binom{k}{t}$. Two t-designs with the same parameters, $\boldsymbol{D}=(\Omega, \mathfrak{B})$ and $\boldsymbol{D}'=(\Omega', \mathfrak{B}')$ are said to be *isomorphic* if there is a bijection σ from Ω onto Ω' such that $\mathfrak{B}^{\sigma}(:=\{B^{\sigma}|B\in\mathfrak{B}\})=\mathfrak{B}'$. A t-$(v, k, 1)$ design $\boldsymbol{D}=(\Omega, \mathfrak{B})$ (namely, t-design with $\lambda=1$) is called a Steiner system, and for any t-subset $\{a_1,\cdots,a_t\}$ of Ω, the block containing it is uniquely determined. This unique block is called the *block defined by* $a_1,\cdots,a_t$ and denoted by

$$<a_1,\cdots,a_t>.$$

Throughout this paper we fix the following notation. $\boldsymbol{W}_{22}$, $\boldsymbol{W}_{23}$, $\boldsymbol{W}_{24}$ denote *any* 3-(22, 6, 1), 4-(23, 7, 1), 5-(24, 8, 1) designs, respectively. They are called the *large Witt systems*, and their existence is proved by quite a few authors. For $v=22, 23$ or 24, let

$$\boldsymbol{W}_v=(\Omega_v, \mathfrak{B}^{(v)}).$$

For s-subset $S=\{a_1,\cdots,a_s\}$ of Ω_v, where $s\leq 3, 4$ or 5 according as $v=22, 23$ or 24, respectively, we set

$$\mathfrak{B}_S=\mathfrak{B}_{a_1,\cdots,a_s}=\{B\in\mathfrak{B}^{(v)}|S\subset B\}.$$

The following well-known property can be easily shown only by using the parameters of $\boldsymbol{W}_v$ (see, e. g. [1, next to the last rows of Figs. 2, 6, 8], [2, the last row of Fig. 5 and p. 30], [5, the last row of Fig. 2. 14 and p. 641]).

Block Intersection Property (*BIP*): *For any distinct blocks* B, C *of* $\boldsymbol{W}_v$, *we have*

$$|B\cap C|=0 \text{ or } 2\,;\ 1 \text{ or } 3\,;\ 0, 2 \text{ or } 4$$

according as $v=22, 23, 24$, *respectively*.

Only by using BIP, we are going to prove

THEOREM. *Large Witt systems* $\boldsymbol{W}_{22}$, $\boldsymbol{W}_{23}$, $\boldsymbol{W}_{24}$ *are unique up to isomorphism*.

2. Proof of Theorem.

We begin with the following general proposition.

PROPOSITION. *Suppose that for any* t-(v, k, λ) *design* $\boldsymbol{D}$ *with given parameters* t, v, k, λ *we have a settled method for labelling all the points and the blocks of* $\boldsymbol{D}$, *more precisely, a settled method according to which we can label all the points of* $\boldsymbol{D}$ 1, 2,$\cdots$, v *and we can explicitly write down* *all the blocks of* $\boldsymbol{D}$ *by means of* $1, 2, \cdots, v$. *Then, a* t-(v, k, λ) *design (, if exists,) is unique up to isomorphism.*

PROOF. Let $\boldsymbol{D}=(\Omega, \mathfrak{B})$ and $\boldsymbol{D}'=(\Omega', \mathfrak{B}')$ be any two t-(v, k, λ) designs. According to the method we have, we label the points and the blocks of both designs. We denote by a_i (resp., a_i') the point itself of Ω (resp., Ω') labelled i, and by B_I (resp., B_I') the block itself of $\mathfrak{B}$(resp., $\mathfrak{B}'$) expressed as $I=\{i_1, i_2, \cdots, i_k\}$. The bijection $\sigma: \Omega \longrightarrow \Omega'$ defined by $a_i^\sigma=a_i'$ gives an isomorphism from $\boldsymbol{D}$ onto $\boldsymbol{D}'$, since $B_I=\{a_{i_1}, a_{i_2}, \cdots, a_{i_k}\}$, $B_I'=\{a_{i_1}', a_{i_2}', \cdots, a_{i_k}'\}$ and $B_I^\sigma=B_I'$. □

Our proof of Theorem is based on the above proposition, and consists of two parts: What we have to do for the proof of Theorem is that for *any* lage Witt systems $\boldsymbol{W}_v$ we present *a settled method* (I) *for labelling all the points of* $\boldsymbol{W}_v$ and *a settled method* (II) *for writing explicitly down all the blocks of* $\boldsymbol{W}_v$.

In order to present such methods we use only BIP, and methods (I), (II) will be given in Lemma 1; Corollaries 1, 2, 3, respectively.

LEMMA 1. *We have a settled method by which we can label all the points of*:

(i) $\boldsymbol{W}_{22}$ $1, 2, \cdots, 22$ *and explicitly know* $\mathfrak{B}_1$ (*i.e, the* 21 *blocks containing* 1) *and a block* A *not containing* 1;

(ii) $\boldsymbol{W}_{23}$ $0, 1, 2, \cdots, 22$ *and explicitly know* $\mathfrak{B}_{0,1}$ (*i.e., the* 21 *blocks containing* 0 *and* 1) *and two blocks* A, A' *satisfying* $0 \in A \not\ni 1$ *and* $1 \in A' \not\ni 0$;

(iii) $\boldsymbol{W}_{24}$ $\infty, 0, 1, 2, \cdots, 22$ *and explicitly know* $\mathfrak{B}_{\infty,0,1}$ (*i.e.*, the 21 *blocks containing* $\infty, 0$ *and* 1) *and three blocks* A, A', A'' *satisfying* $\{\infty, 0\} \subset A \not\ni 1$, $\{1, \infty\} \subset A' \not\ni 0$ *and* $\{0, 1\} \subset A'' \not\ni \infty$.

PROOF. Our proof is tabulated in Tables 1 and 2. Three cases $\boldsymbol{W}_{22}$, $\boldsymbol{W}_{23}$, $\boldsymbol{W}_{24}$ can be treated similarly and the case $\boldsymbol{W}_{24}$ is mainly illustrated in the following.

Choose any block of $\boldsymbol{W}_{24}$ (resp., $\boldsymbol{W}_{23}$, $\boldsymbol{W}_{22}$) and call it B_1, and choose any five (resp., four, three) points of B_1 and label them pointwise '∞', '0', '1', '2', '3' (resp., '0', '1', '2', '3'; '1', '2', '3'). (In the case $\boldsymbol{W}_{23}$ (resp., $\boldsymbol{W}_{22}$) delete or ignore ∞ (resp., ∞, 0) throughout the proof.) Label the remaining three points of B_1 '4, 5, 6' setwise. Choose any point outside B_1 and label it '7'. Set

$$B_2=<\infty, 0, 1, 2, 7>.$$

By BIP we have $B_2 \cap B_1 = \{\infty, 0, 1, 2\}$ and label setwise the three points in $B_2 \backslash \{\infty, 0, 1, 2, 7\}$, '8, 9, 10'. Set

$$B_3 = <\infty, 0, 1, 3, 7>.$$

By BIP we have $B_3 \cap B_1 = \{\infty, 0, 1, 3\}$ and $B_3 \cap B_2 = \{\infty, 0, 1, 7\}$, and so the three points in $B_3 \backslash \{\infty, 0, 1, 3, 7\}$ are outside $B_1 \cup B_2$ and we label them '11, 12, 13' setwise. Set

$$A = <\infty, 0, 2, 3, 7>$$

and

$$A' = <\infty, 1, 2, 3, 7>.$$

(In the case $\boldsymbol{W}_{23}$, of course, we set $A' = <1, 2, 3, 7>$, but in the case $\boldsymbol{W}_{22}$ we do not consider A'.) In the same way as above, the three points in $A \backslash \{\infty, 0, 2, 3, 7\}$(resp., $A' \backslash \{\infty, 1, 2, 3, 7\}$) are outside $B_1 \cup B_2 \cup B_3$ (resp., $B_1 \cup B_2 \cup B_3 \cup A$) and we label them '14, 15, 16' (resp., '17, 18, 19') setwise. Label any one of the three points $\{11, 12, 13\}$ '11' and set

$$B_4 = <\infty, 0, 1, 2, 11>.$$

Since $B_4 \cap B_1 = B_4 \cap B_2 = \{\infty, 0, 1, 2\}$ and $B_4 \cap B_3 = \{\infty, 0, 1, 11\}$ by BIP, B_4 does not contain any point of $B_1 \backslash \{\infty, 0, 1, 2\} = \{3, 4, 5, 6\}$, $B_2 \backslash \{\infty, 0, 1, 2\} = \{7, 8, 9, 10\}$ and $B_3 \backslash \{\infty, 0, 1, 11\} = \{3, 7, 12, 13\}$. By BIP we can write $B_4 \cap A = \{\infty, 0, 2, x\}$. Since $A = \{\infty, 0, 2, 3, 7, 14, 15, 16\}$ and B_4 contains neither 3 nor 7, x must be one of 14, 15, 16 and we label x '14'. Similarly, we can write $B_4 \cap A' = \{\infty, 1, 2, y\}$ and y must be one of 17, 18, 19 and we label y '17'. Therefore we can write $B_4 = \{\infty, 0, 1, 2, 11, 14, 17, z\}$ where $z \notin B_1 \cup B_2 \cup B_3 \cup A \cup A'$ and we label z '20'. (In the case $\boldsymbol{W}_{22}$, we can write $B_4 = \{1, 2, 11, 14, y, z\}$ where neither y nor z is contained in $B_1 \cup B_2 \cup B_3 \cup A$ and we label the one of y, z '17' and the other '20'). Set

$$B_5 = <\infty, 0, 1, 3, 17>.$$

As in the case B_4, considering the intersections of B_5 and B_1, B_3, B_4, A', we obtain that B_5 does not contain any point of
$(B_1 \cup B_3 \cup B_4 \cup A') \backslash \{\infty, 0, 1, 3, 17\} = \{2, 4, 5, 6, 7, 11, 12, 13, 14, 20, 18, 19\}$.

(In the case $\boldsymbol{W}_{22}$, A' and so 18, 19 vanish.)

We can write $B_5 \cap B_2 = \{\infty, 0, 1, x\}$ where x is a point of $B_2 \backslash \{\infty, 0, 1\} = \{2, 7, 8, 9, 10\}$, and so x is one of 8, 9, 10 and we label x '8'. Also, $B_5 \cap A (\supset \{\infty, 0, 3\})$ must contain a point of $A \backslash \{\infty, 0, 3\} = \{2, 7, 14, 15, 16\}$ and so either of 15, 16 and we label the point in (resp., outside) $B_5 \cap A$ '15' (resp., '16'). Hence we can write $B_5=\{\infty, 0, 1, 3, 8, 15, 17, y\}$ where $y \notin B_1 \cup B_2 \cup B_3 \cup B_4 \cup A \cup A'$. We label y '21'. Set

$$B_6=<\infty, 0, 1, 7, 17>.$$

As usual, considering the intersections of B_6 and B_2, B_3, B_4, B_5, A', we obtain that B_6 does not contain any point of $(B_2 \cup B_3 \cup B_4 \cup B_5 \cup A')\backslash\{\infty, 0, 1, 7, 17\}(\supset\{2, 3, 14, 15\})$. We can write $B_6 \cap B_1=\{\infty, 0, 1, x\}$ where x is a point of $B_1\backslash\{\infty, 0, 1\}=\{2, 3, 4, 5, 6\}$ and so we label x '4'. Also, we can write $B_6 \cap A=\{\infty, 0, 7, y\}$ where y is a point of $A\backslash\{\infty, 0, 7\}=\{2, 3, 14, 15, 16\}$, so that y must be 16. Therefore we can write $B_6=\{\infty, 0, 1, 4, 7, 16, 17, z\}$ where $z \notin B_1 \cup \cdots \cup B_5 \cup A \cup A'$, and we label z '22'. Setting

$$B_7=<\infty, 0, 1, 2, 15>,$$
$$B_8=<\infty, 0, 1, 3, 16>,$$
$$B_9=<\infty, 0, 1, 7, 14>$$

and continuing similar arguments, we can label the unlabelled points and determine the remaining points contained in these blocks. (See Table 1. In the case $\boldsymbol{W}_{22}$, we label the unique point in $B_7\backslash(B_1 \cup \cdots \cup B_6 \cup A)$ (resp., outside $B_1 \cup \cdots \cup B_7 \cup A$) '18' (resp., '19').) Thus we have been able to label all the points and determine explicitly the blocks $B_1, B_2, \cdots, B_9, A, A'$ simultaneously. By BIP, these blocks generate blocks $B_{10}, \cdots, B_{21}, A''$ successively and automatically (see Table 2). □

Table 1

	∞	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	BIP
B_1	$\underline{\bigcirc}$	$\underline{\bigcirc}$	$\underline{\bigcirc}$	$\underline{\bigcirc}$	$\underline{\bigcirc}$	○	○	○	$\underline{\ }$																
B_2	●	●	●	●	×	×	×	×	●	○	○	○													× - B_1
B_3	●	●	●	×	●	×	×	×	●	×	×	×	$\underline{\bigcirc}$	○	○										× - B_1,B_2
A	●	●	×	●	●	×	×	×	●	×	×	×	×	×	×	○	○	○							× - B_1,B_2,B_3
A'	●	×	●	●	●	×	×	×	●	×	×	×	×	×	×	×	×	×	○	○	○				× - B_1,B_2,B_3,A
B_4	●	●	●	●	×	×	×	×	×	×	×	×	●	×	×	$\underline{\blacktriangle}$	△	△	($\underline{\blacksquare}$	□	□)	$\underline{\bigcirc}$			× - B_1,B_2,B_3 △ - A ; □ - A'
B_5	●	●	●	×	●	×	×	×	×	$\underline{\blacktriangle}$	△	△	×	×	×	×	$\underline{\blacksquare}$	$\underline{\square}$	●	(×	×)	×	$\underline{\bigcirc}$		× - B_1,B_3,B_4,A' △ - B_2; □ - A
B_6	●	●	●	×	×	$\underline{\blacktriangle}$	△	△	●	×	×	×	×	×	×	×	×	▼	●	(×	×)	×	×	$\underline{\bigcirc}$	× - B_2,B_3,B_4,B_5,A' ▼ - A; △ - B_1
B_7	●	●	●	●	×	×	×	×	×	×	×	×	×	$\underline{\blacktriangle}$	$\underline{\triangle}$	×	●	×	×	($\underline{\blacksquare}$	$\underline{\square}$)	×	×	▼	× - B_1,B_2,B_4,B_5,A ▼ - B_6; △ - B_3; □ - A'
B_8	●	●	●	×	●	×	×	×	×	×	$\underline{\blacktriangle}$	$\underline{\triangle}$	×	×	×	×	×	●	×	▼		▼	×	×	× - B_1,B_3,B_5,B_6,A ▼ - B_4,B_7; △ - B_2
B_9	●	●	●	×	×	×	$\underline{\blacktriangle}$	$\underline{\triangle}$	●	×	×	×	×	×	×	●	×	×	×	▼		×	▼	×	× - B_2,B_3,B_4,B_6,A ▼ - B_5,B_7; △ - B_1

Each block of $\boldsymbol{W}_{24}$ (resp., $\boldsymbol{W}_{23}$, $\boldsymbol{W}_{22}$) except B_1 is defined by five (resp., four, three) points '●' (resp., deleting '∞', '∞,0'), and the remaining three points of the block are denoted by '○','▲','■' or '▼'.

'—' indicates that the labelling of the point is finished.

'×' indicates that the block does not contain the point.

'( )' indicates the influence of the intersection of the block and A' (in the case $\boldsymbol{W}_{22}$, A' is unnecessary and deleted).

Table 2

	∞	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	BIP
B_{10}	●	●	●	●	×	×	×	×	×	×	×	×	×	×	▼	×	×	▼	×	×	▼	×	▼	×	×-B_1,B_2,B_4,B_7
B_{11}	●	●	●	×	●	×	×	×	×	×	×	▼	×	×	×	▼	×	×	×	×	▼	×	×	▼	×-B_1,B_3,B_5,B_8
B_{12}	●	●	●	×	×	×	×	▼	●	×	×	×	×	×	×	×	▼	×	×	×	▼	▼	×	×	×-B_2,B_3,B_6,B_9
B_{13}	●	●	●	×	×	▼	×	×	×	×	▼	×	×	×	▼	●	●		×	×	×	×	×	×	×-$B_4,B_5,B_7,B_9B_{11},B_{12}$ ▼-B_1,B_2,B_3
B_{14}	●	●	●	×	×	×	×	▼	×	▼	×	×	×	▼	×	●	×	▼	×	×	×	×	×	×	×-B_4,B_9,B_{11},B_{13}
B_{15}	●	●	●	×	×	×	▼	×	×	×	×	▼	▼	×	×	×	●	▼	×	×	×	×	×	×	×-B_5,B_7,B_{12},B_{13}
B_{16}	●	●	●	×	×	●	×	×	×	●	×	×	▼	×	×	×	×	×	×	▼	▼		×	×	×-$B_1,B_2,B_5,B_6B_{13},B_{14}$ ▼-B_3,B_7,B_{10}
B_{17}	●	●	●	×	×	●	×	×	×	×	×	▼	×	▼	×	×	×	×	×	×	×	▼	▼	×	×-B_1,B_6,B_{13},B_{16}
B_{18}	●	●	●	×	×	×	▼	×	×	●	×	×	×	×	▼	×	×	×	×	×	×	▼	×	▼	×-B_2,B_5,B_{14},B_{16}
B_{19}	●	●	●	×	×	×	●	×	×	×	▼	×	×	▼	×	×	×	×	▼	×	▼	×	×	×	×-B_1,B_9,B_{15},B_{18}
B_{20}	●	●	●	×	×	×	×	▼	×	×	●	×	▼	×	×	×	×	×	×	×	×	×	▼	▼	×-B_2,B_8,B_{13},B_{19}
B_{21}	●	●	●	×	×	×	×	●	×	×	×	▼	×	×	▼	×	×	×	▼	▼	×	×	×	×	×-B_1,B_{12},B_{14},B_{20}
A''	×	●	●	●	●	×	×	×	●	×	×	×	×	×	×	×	×	×	×	×	×	▼	▼	▼	×-B_1,B_2,B_3,A,A'

Each block of $\boldsymbol{W}_{24}$ (resp., $\boldsymbol{W}_{23}$, $\boldsymbol{W}_{22}$) is defined as one containing five or four (resp., four or three, three or two) points '●' (resp., deleting '∞', '∞, 0'), and the remaining three or four points of the block are denoted by '▼'.
'×' indicates that the block does not contain the point.
(A'' is a block only for $\boldsymbol{W}_{24}$)

REMARK 1. There are various ways to label the points belonging to blocks B_1, $B_2,\cdots,B_9$, A, A'. Relabelling in the following (from the upper row to the lower), we obtain the labelling in $\boldsymbol{W}_{24}$ in many papers such as [2], [3], [6]:

∞	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
∞	0	1	2	3	17	14	5	4	13	16	22	19	11	20	21	8	9	6	18	15	7	10	12

NOTATION. In the following Lemma 2 and Corollary 1, let I denote a $(v-22)$-subset of Ω_v (in particular, $I=\phi$ for $v=22$). Let $p\in\Omega_v\backslash I$ and set $\mathfrak{B}_{I,p}=\mathfrak{B}_{I\cup\{p\}}$.

LEMMA 2. *Suppose that we finish labelling all the points of* Ω_v *and that we explicitly know* $\mathfrak{B}_{I,p}$ (*i. e., the* 21 *blocks containing* I *and* p) *and a* *block A containing I but not containing p. Then, for any two distinct j_1, $j_2 \in A \backslash I$ and any $k \in \Omega_v \backslash (I \cup \{j_1, j_2\})$ we can explicitly know the block $<I, j_1, j_2, k>$, namely, the remaining three points of the block.*

PROOF. Set

$$\begin{aligned} X &= <I, j_1, j_2, k>, \\ B_0 &= <I, p, j_1, j_2,>, \\ B_1 &= <I, p, j_1, k>, \\ B_2 &= <I, p, j_2, k>. \end{aligned}$$

We want to determine explicitly the remaining three points of X and our proof is tabulated in Table 3. Since the blocks B_0, B_1, B_2 belong to $\mathfrak{B}_{I,p}$ and are explicitly known by assumption, we may assume that X is different from these blocks and $B_0 \not\ni k$, $B_1 \not\ni j_2$, $B_2 \not\ni j_1$ and B_0, B_1, B_2 are distinct. Similarly, we may assume that $X \neq A$ and so that $k \notin A$ and blocks X, A, B_0, B_1, B_2 are all distinct.

By BIP we have $B_0 \cap A = I \cup \{j_1, j_2\}$ and we can write

$$\begin{aligned} A &= I \cup \{j_1, j_2\} \cup \{a_0, a_1, a_2, a_3\}, \\ B_0 &= I \cup \{p, j_1, j_2\} \cup \{b_0, c_0, d_0\}. \end{aligned}$$

By BIP we have $B_1 \cap A = I \cup \{j_1, a\}$ where a is one of a_0, a_1, a_2, a_3, and so we may assume $a = a_1$. As $B_1 \cap B_0 = I \cup \{p, j_1\}$, we can write

$$B_1 = I \cup \{p, j_1, k, a_1\} \cup \{b_1, c_1\}$$

where b_1, c_1 are outside $A \cup B_0$. Similarly, we can write

$$B_2 = I \cup \{p, j_2, k, a_2\} \cup \{b_2, c_2\}$$

where b_2, c_2 are outside $A \cup B_0 \cup B_1$.

Now, since $X \cap A = X \cap B_0 = I \cup \{j_1, j_2\}$, $X \cap B_1 = I \cup \{j_1, k\}$ and $X \cap B_2 = I \cup \{j_2, k\}$, it follows that X does not contain any point of $(A \cup B_0 \cup B_1 \cup B_2) \backslash (I \cup \{j_1, j_2, k\}) = \{p, a_0, a_1, a_2, a_3, b_0, b_1, b_2, c_0, c_1, c_2, d_0\}$. We set

$$B_3 = <I, p, k, a_3>.$$

As $B_3 \cap B_1 = B_3 \cap B_2 = I \cup \{p, k\}$, B_3 does not contain any point of $(B_1 \cup B_2) \backslash (I \cup \{p, k\}) = \{j_1, a_1, b_1, c_1, j_2, a_2, b_2, c_2\}$. Hence $B_3 \cap A = I \cup \{a_3, a_0\}$ and we may write $B_3 \cap B_0 = I \cup \{p, b_0\}$. Thus $B_3 \cap (A \cup B_0 \cup B_1 \cup B_2) = I \cup \{p, k, a_3, a_0, b_0\}$ and we set

$$\{x_3\} = B_3 \backslash (A \cup B_0 \cup B_1 \cup B_2).$$

Since X does not contain any one of p, a_3, a_0, b_0, we have $X \cap B_3 = I \cup \{k, x_3\}$ by BIP. Next, we set

$$B_4 = <I, p, x_3, c_0>,$$
$$B_5 = <I, p, x_3, d_0>.$$

As usual, considering the intersections of B_4 and B_0, B_3, by BIP we may write $B_4 \cap B_1 = I \cup \{p, d_1\}$, $B_4 \cap B_2 = I \cup \{p, d_2\}$ where d_i is one of a_i, b_i, c_i $(i=1, 2)$ and $B_4 \cap A = I$ or $I \cup \{a_1, a_2\}$. Thus $B_4 \cap (A \cup B_0 \cup B_1 \cup B_2 \cup B_3) = I \cup \{p, x_3, c_0, d_1, d_2\}$ and we set

$$\{x_4\} = B_4 \backslash (A \cup B_0 \cup B_1 \cup B_2 \cup B_3).$$

On the other hand, since X does not contain any one of p, c_0, a_1, b_1, c_1, a_2, b_2, c_2, we obtain $X \cap B_4 = I \cup \{x_3, x_4\}$ by BIP. Similarly, we have $B_5 \cap (A \cup B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4) = I \cup \{p, x_3, d_0, e_1, e_2\}$ where e_i is one of a_i, b_i, c_i $(i=1, 2)$ and we set

$$\{x_5\} = B_5 \backslash (A \cup B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4),$$

obtaining $X \cap B_5 = I \cup \{x_3, x_5\}$. Thus we have determined the three points x_3, x_4, x_5 of $X \backslash (I \cup \{j_1, j_2, k\})$ only by using A and $B_0, B_1, \cdots, B_5$ belonging to $\mathfrak{B}_{I,p}$. □

Table 3

	I	p	j_1	j_2	k	a_0	a_1	a_2	a_3	b_0	c_0	d_0	b_1	c_1	b_2	c_2	x_3	x_4	x_5	BIP
A	(○ ○)		○	○		○	○	○	○											
B_0	(● ●)	●	●	●		×	×	×	×	○	○	○								×-A
B_1	(● ●)	●	●	×	●	△	▲	△	△	×	×	×	○	○						×-B_0 ; △-A
B_2	(● ●)	●	×	●	●	△	×	▲	△	×	×	×	×	×	○	○				×-B_0,B_1 ; △-A
B_3	(● ●)	●	×	×	●	▼	×	×	●	▲	△	△	×	×	×	×	○			×-B_1,B_2 ; ▼-A,△-B_0
B_4	(● ●)	●	×	×	×	×	△	□	×	×	●	×	△	△	□	□	●	○		×-B_0,B_3 ; △-B_1,□-B_2
B_5	(● ●)	●	×	×	×	×	△	□	×	×	×	●	△	△	□	□	●	×	○	×-B_0,B_3,B_4 ; △-B_1 ; □-B_2
X	(● ●)	×	●	●	●	×	×	×	×	×	×	×	×	×	×	×	▼	▼	▼	×-A,B_0,B_1,B_2 ; ▼-B_3,B_4,B_5

Each block of $\boldsymbol{W}_{24}$ (resp., $\boldsymbol{W}_{23}$, $\boldsymbol{W}_{22}$) except A is defined by five (resp., four, three) points'●', and the remaining three points of the block are denoted by '○', '▲', '▼' or one of '△', '□' in the cases B_4, B_5.
'×' indicates that the block does not contain the point.

REMARK 2. Lemma 2 asserts that from the 21 blocks belonging to $\mathfrak{B}_{I,p}$ and a block A with $A=<I, j_1, j_2, *>\not\ni p$, we can explicitly know new blocks $<I, j_1, j_2, k>$ for any $k\in\Omega_v\backslash(I\cup\{j_1, j_2\})$. We say that such new blocks are (explicitly) known or generated from (primary blocks) $\mathfrak{B}_{I,p}$ and (an auxiliary block) A.

COROLLARY 1. *Assume that we finish labelling all the points of* Ω_v *and that we explicitly know* $\mathfrak{B}_{I,p}$ *for some* I, p *and a block* A *satisfying* $I\subset A\not\ni p$. *Then, for any distinct points* k, l, m *of* $\Omega_v\backslash I$, *we can explicitly know the block* $<I, k, l, m>$, *and hence* $\mathfrak{B}_I$ (*i.e., the* 77 *blocks containing* I)——*we describe this by saying that* $\mathfrak{B}_I$ *is* (*explicitly*) *known or generated from* $\mathfrak{B}_{I,p}$ *and* A——*in particular, all the blocks of* $\mathbf{W}_{22}$.

PROOF. Set $X=<I, k, l, m>$.

Case (i) $|\{k, l, m\}\cap A|=3$ or 2: Since we may assume $A=<I, k, l, *>\not\ni p$, by Remark 2 we can know X from $\mathfrak{B}_{I,p}$ and A.

Case (ii) $|\{k, l, m\}\cap A|=1$: We may assume that $k\in A$ and $\{l, m\}\cap A=\phi$. Let $a\in A\backslash\{k\}$. From $\mathfrak{B}_{I,p}$ and $A=<I, k, a, *>$ we can know $A_1=<I, k, a, l>$ and $A_2=<I, k, a, m>$ by Remark 2. If $A_1\not\ni p$ (resp., $A_2\not\ni p$), then we can know X from $\mathfrak{B}_{I,p}$ and A_1 (resp., A_2). If $A_1\ni p$ and $A_2\ni p$, then $A_1=<I, k, a, p>=A_2$ and so $X=A_1$ is explicitly known.

Case (iii) $|\{k, l, m\}\cap A|=0$: Let j_1, j_2 be two distinct points of A and set

$$A_1=<I, j_1, j_2, k>, A_2=<I, j_1, j_2, l>, A_3=<I, j_1, j_2, m>.$$

Then, from $\mathfrak{B}_{I,p}$ and A we can know A_1, A_2 and A_3. If at least one of A_1, A_2, A_3, say A_1 dose not contain p, then we can know X from $\mathfrak{B}_{I,p}$ and A_1 (note that $|\{k, l, m\}\cap A_1|\geq 1$ and replace A with A_1 in cases (i), (ii)). If all A_1, A_2, A_3 contain p, then $A_1=A_2=A_3=<I, j_1, j_2, p>$ and so $X=A_1$ is explicitly known. □

COROLLARY 2. *Let* $v=24$ *or* 23 *and* $i_2\in\Omega_v$. *In the case* $v=24$, *let* i_1 *be a point of* $\Omega_{24}\backslash\{i_2\}$ *and in the case* $v=23$, *let* $i_1=\phi$. *Also, let* $p\in\Omega_v\backslash\{i_1, i_2\}$. *Assume that we finish labelling all the points of* Ω_v *and that we explicitly know* $\mathfrak{B}_{i_1, i_2, p}$ (*i.e., the* 21 *blocks containing* i_1, i_2, p) *and two blocks* A, A' *satisfying* $\{i_1, i_2\}\subset A\not\ni p$ *and* $\{i_1, p\}\subset A'\not\ni i_2$. *Then, for any four distinct points* j, k, l, m *of* $\Omega_v\backslash\{i_1\}$, *we can explicitly know the block* $<i_1, j, k, l, m>$, *and hence* $\mathfrak{B}_{i_1}$ (*i.e., the* 253 *blocks containing* i_1)——*we describe this by saying that* $\mathfrak{B}_{i_1}$ *is* (*explicitly*) *known or generated from* $\mathfrak{B}_{i_1, i_2, p}$ *and* A, A'——*in particular, all the blocks of* $\mathbf{W}_{23}$.

PROOF. Set $X=<i_1, j, k, l, m>$. By Corollary 1, we can know $\mathfrak{B}_{i_1, i_2}$ (resp. $\mathfrak{B}_{i_1, p}$) from $\mathfrak{B}_{i_1, i_2, p}$ $(=\mathfrak{B}_{i_1, p, i_2})$ and A (resp., A'). In particular, we explicitly know $\mathfrak{B}_{i_1, j, p}$ $(\subset\mathfrak{B}_{i_1, p})$. If $i_2\in X$ or $p\in X$, then $X\in\mathfrak{B}_{i_1, i_2}$ or $X\in\mathfrak{B}_{i_1, p}$, and so X is explicitly known. Thus we may assume $X\cap\{i_2, p\}=\phi$, in particular, $\{j, k, l, m\}\cap\{i_2, p\}=\phi$. Set

$$A_1=<i_1, j, k, l, i_2>,\ A_2=<i_1, j, k, m, i_2>.$$

Both are contained in $\mathfrak{B}_{i_1, i_2}$ and so explicitly known. If $A_1\ni p$ and $A_2\ni p$, then $A_1=<i_1, j, k, p, i_2>=A_2\ni m$ and so $A_1=X$, which is a contradiction, for $i_2\in A_1$ and $i_2\notin X$. Thus $A_1\not\ni p$ or $A_2\not\ni p$. If $A_1\not\ni p$ (resp., $A_2\not\ni p$), then by Remark 2 we can know X from $\mathfrak{B}_{i_1, j, p}$ and A_1 (resp., A_2). □

COROLLARY 3. *Let i_1, i_2, p be the fixed three distinct points of Ω_{24}. Assume that we finish labelling all the points of Ω_{24} and that we explicitly know $\mathfrak{B}_{i_1, i_2, p}$ (i. e., the 21 blocks containing i_1, i_2, p) and three blocks A, A', A'' satisfying $\{i_1, i_2\}\subset A\not\ni p$, $\{p, i_1\}\subset A'\not\ni i_2$, $\{i_2, p\}\subset A''\not\ni i_1$. Then, for any five distinct points j, k, l, m, n of Ω_{24}, we can explicitly know the block $<j, k, l, m, n>$, and hence all the blocks of $\boldsymbol{W}_{24}$.*

PROOF. Set $X=<j, k, l, m, n>$. By Corollary 2, we can know $\mathfrak{B}_{i_1}$ (resp., $\mathfrak{B}_p$) from $\mathfrak{B}_{i_1, i_2, p}$ $(=\mathfrak{B}_{p, i_1, i_2})$ and A, A' (resp., A', A''). In particular, we explicitly know $\mathfrak{B}_{j, k, p}$ $(\subset\mathfrak{B}_p)$. As in the proof of Corollary 2, we may assume $X\cap\{i_1, p\}=\phi$ and we set

$$A_1=<j, k, l, m, i_1>,\ A_2=<j, k, l, n, i_1>,$$

having $A_1\not\ni p$ or $A_2\not\ni p$ and being able to know X from $\mathfrak{B}_{j, k, p}$ and A_1 or A_2. □

PROOF OF THEOREM. Set $v=22, I=\phi, p=1$ in Corollary 1; $v=23, i_1=\phi, i_2=0, p=1$ in Corollary 2; $v=24, i_1=\infty, i_2=0, p=1$ in Corollary 3. Then, the assumptions (and so the conclusions) of Corollaries 1, 2, 3 hold by Lemma 1. Thus the desired methods (I), (II) are presented and the proof of Theorem is complete. □

REMARK 3. Our proof of the uniqueness of the large Witt systems $\boldsymbol{W}_{22}$, $\boldsymbol{W}_{23}$, $\boldsymbol{W}_{24}$ presented above also shows the existence of them. In fact, for instance, the existence of $\boldsymbol{W}_{22}$ is shown as follows. Set $\Omega=\{1, 2,\cdots,22\}$, and make 6-subsets of Ω, $B_1, B_2,\cdots, B_9, A$ given explicitly in Table 1 and (automatically) $B_{10},\cdots, B_{21}$ given explicitly in Table 2. Then, as seen in Corollary 1, for any three distinct points k, l, m of Ω, there is a unique $<k, l, m>$, the 6-subset of Ω containing k, l, m (this uniqueness follows from the fact that all the $<k, l, m>$ satisfy BIP). Letting $\mathfrak{B}$ denote the set of all the $<k, l, m>$, we see that $(\Omega, \mathfrak{B})$ is a 3-(22, 6, 1) design.

REMARK 4. We have seen in Table 2 that $B_1, \cdots, B_9, A, (A')$ generate $B_{10}, \cdots, B_{21}, (A'')$, and in Lemma 1 and Corollaries 1, 2, 3 that $B_1, \cdots, B_{21}, A, (A', A'')$ generate all the blocks. In conclusion, the blocks $B_1, B_2, \cdots, B_9, A$ in $\boldsymbol{W}_{22}$, added A' in $\boldsymbol{W}_{23}$ and $\boldsymbol{W}_{24}$, enable us to label all the points and generate all the blocks, and so we may call these blocks *labelling-blocks*, *generating blocks*, *propagating blocks*, *basis blocks* or *determining-blocks* of $\boldsymbol{W}_v$, $v=22, 23, 24$. We note also that even only $B_1, \cdots, B_9$ enable us to label all the points and generate $B_{10}, \cdots, B_{21}$——this shows the uniqueness of the 2-(21, 5, 1) design if we ignore $\infty, 0, 1$——, but do not generate the other blocks. When heterogeneous A, A' are added, generative power heightens extremely.

References

[1] D. J. BERGSTRAND, New uniqueness proofs for the (5, 8, 24), (5, 6, 12) and related Steiner systems, J. Comb. Th. Ser. A 33 (1982), 247-272.

[2] R. T. CURTIS, A new combinatorial approach to M_{24}, Math. Proc. Camb. Phil. Soc. 79 (1976), 25-42.

[3] S. IWASAKI, An elementary and unified approach to the Mathieu-Witt systems, J. Math. Soc. Japan 40 (1988) 393-414.

[4] W. JÓNSSON, On the Mathieu groups M_{22}, M_{23}, M_{24} and the uniqueness of the associated Steiner systems, Math. Z. 125 (1972), 193-214.

[5] F. J. MACWILLIAMS, and N. J. A. SLOANE, The theory of error-correcting codes, North Holland, Amsterdam-New York-Oxford, 1977.

[6] J. A. TODD, A representation of the Mathieu group M_{24} as a collineation group, Ann. Mat. Pura Appl. Ser. 4, 71 (1966), 199-238.

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