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Expansive foliations

Dedicated to Professor Haruo Suzuki on his sixtieth birthday

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Introduction

Expansiveness of homeomorphisms and flows has been studied by various authors in the field of dynamical systems. In this note we introduce this notion into the foliation theory and examine its influence on the topology of leaves. In § 1 we give a precise definition of expansive foliations. In § 2 we restrict our attention to the case of codimension one foliations and show that in this case topological structures of foliations completely characterize the expansiveness. As corollaries of this result we obtain that the geometric entropy ([GLW]) of a codimension one expansive foliation is positive and that the fundamental group of a manifold admitting a codimension one expansive foliation has exponential growth. In § 3 we define another notion called strong expansiveness. We show that for strongly expansive foliations results similar to (although somewhat weaker than) those obtained in § 2 hold in all codimensions.

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1. Definition

First we treat the case of codimension one. Let M be a closed C^{∞} Riemannian manifold and $\mathscr{F}$ a codimension one C^{r}, $r\geq 0$, foliation on M. Fix a one dimensional foliation $\mathscr{T}$ transverse to $\mathscr{F}$. Throughout this note we assume that all leaves of $\mathscr{F}$ and $\mathscr{T}$ are of class C^{1}. A curve (resp. an embedded curve) contained in a leaf of $\mathscr{F}$ (resp. $\mathscr{T}$) is called an *$\mathscr{F}$-curve* (resp. a *$\mathscr{T}$-arc*). A continuous map $F:[0,1]\times[0,1]\rightarrow M$ is called a *fence* if $F|[0,1]\times\{t\}$ is a C^{1} $\mathscr{F}$-curve for all $t\in[0,1]$ and $F|\{s\}\times[0,1]$ is a $\mathscr{T}$-arc for all $s\in[0,1]$. $F|[0,1]\times\{t\}$ is called a *horizontal curve* (the *lower side* if $t=0$, the *upper side* if $t=1$) of F and $F|\{s\}\times[0,1]$ is called a *vertical arc* (the *left side* if $s=0$, the *right side* if $s=1$) of F. The lower, upper, left or right side of F is denoted by $l(F)$, $u(F)$, $\lambda(F)$

or $\rho(F)$ respectively. The *holonomy map* h_F associated with F is the diffeomorphism from $\lambda(F)$ onto $\rho(F)$ defined by $h_F(F(0, t))=F(1, t)$. (Here and hereafter we often confuse a curve with its image.)

Now we propose the following.

DEFINITION 1.1. $\mathcal{F}$ is *expansive* if there exists $\delta>0$ with the property that for any compact $\mathcal{T}$-arc J there is a fence F such that $\lambda(F)=J$ and Length $(\rho(F))\geq\delta$.

This definition is independent of the Riemannian metric and the transverse foliation $\mathcal{T}$ because M is compact.

The constant δ is called an *expansive constant* of $\mathcal{F}$.

Next we consider the case of general codimension. In this case there does not necessarily exist a transverse foliation of complementary codimension. So we are obliged to use a system of transverse disks as below in place of a transverse foliation. Let M be a closed C^∞ Riemannian manifold and $\mathcal{F}$ a codimension q foliation on M. We assume that $\mathcal{F}$ has C^1 leaves. Let ν be the subbundle of TM orthogonal to the leaves of $\mathcal{F}$ and let $\nu_r\subset\nu$ be the associated bundle by closed disks of radius r. For each $x\in M$ let $D_r(x)$ be the image of the q-disk $\nu_r(x)$ under the exponential map $\exp: TM\rightarrow M$. If r is sufficiently small, then each $D_r(x)$ is an embedded disk transverse to $\mathcal{F}$. We will define a fence for codimension q foliations. Let $\alpha:[0,1]\rightarrow M$ be a C^1 $\mathcal{F}$-curve and let $r>0$. Let N be a compact neighborhood of 0 in $\boldsymbol{R}^q$. A continuous map $F:[0,1]\times N\rightarrow M$ is called a *fence* along α if $F|\{t\}\times N$ is an embedding into $D_r(\alpha(t))$ for all $t\in[0,1]$, if $F|[0,1]\times\{x\}$ is a C^1 $\mathcal{F}$-curve for every $x\in N$ and if $F|[0,1]\times\{0\}=\alpha$. $F(\{0\}\times N)$ (resp. $F(\{1\}\times N)$) is called the *left side* (resp. the *right side*) of F and is denoted by $\lambda(F)$ (resp. $\rho(F)$). We denote by $\operatorname{Int}\rho(F)$ the interior of $\rho(F)$ in $D_r(\alpha(1))$. The diffeomorphism h_F from $\lambda(F)$ onto $\rho(F)$ defined by $h_F(F(0, x))=F(1, x)$ is called the *holonomy map* associated with F.

DEFINITION 1.2. $\mathcal{F}$ is *expansive* if there exists sufficiently small $\delta>0$ (called an *expansive constant*) that the following holds: For any $x\in M$ and any $y\in D_\delta(x)-\{x\}$, one can find an $\mathcal{F}$-curve α with $\alpha(0)=x$ and a fence F along α such that $y\in\lambda(F)\subset D_\delta(x)$ and that $h_F(y)$ does not belong to $D_\delta(\alpha(1))$.

It can be seen that for a codimension one foliation, Definitions (1.1) and (1.2) are equivalent.

Orbits of a nonsingular flow φ form a one dimensional expansive foliation if and only if φ is expansive as a flow (for definition see [KS]).

Weakly stable foliations of Anosov flows are expansive.

2. Expansive foliations of codimension one

In this section we concern ourselves exclusively with codimension one foliations. Let M be a closed C^∞ Riemannian manifold and $\mathcal{F}$ a codimension one C^r, $r\geq 0$, foliation on M. As in § 1, fix a one dimensional foliation $\mathcal{T}$ transverse to $\mathcal{F}$. An open saturated subset U of $\mathcal{F}$ is an *open local minimal set* of $\mathcal{F}$ if all leaves of $\mathcal{F}|U$ are dense in U. An open local minimal set is *nontrivial* if it contains a leaf with nontrivial holonomy (*i,e*., there exists a fence F whose image is contained in U such that $l(F)$ is a loop and that the germ of h_F at $l(F)(0)$ is not the identity).

The following is the main result of this section.

THEOREM 2.1. *Let M be a closed manifold and $\mathcal{F}$ a codimension one C^r, $r\geq 0$, foliation on M. Then the following are equivalent.*

(i) *$\mathcal{F}$ is expansive.*

(ii) *There exist finitely many nontrivial open local minimal sets of $\mathcal{F}$ whose union is dense in M.*

REMARK. If $\mathcal{F}$ is of class C^2, the condition (ii) implies that $\mathcal{F}$ has finite level (see [CC] for difinition of level). But we can construct an example of a C^0 expansive foliation which has leaves at infinite level.

REMARK. As a direct corollary of (2.1), we have that every codimension one Anosov flow has a locally dense weakly stable manifold. This fact also holds for nonsingular expansive flows on closed 3-manifolds (see the end of this section).

By passing to a suitable double cover if necessary, we assume that $\mathcal{F}$ is transversely orientable. First we recall some basic definitions. An $(\mathcal{F}, \mathcal{T})$-*chart* (W, φ) is a pair of an embedding $\varphi: D^{n-1}\times D^1\to M$ and the set $W=\text{Image}\,\varphi$ such that $\varphi|D^{n-1}\times\{t\}$ is an embedding into a leaf of $\mathcal{F}$ for each $t\in D^1$ and $\varphi|\{x\}\times D^1$ is a $\mathcal{T}$-arc for each $x\in D^{n-1}$. An atlas $\mathcal{W}=\{(W_i, \varphi_i)\}_{i=1}^m$ on M is a *biregular cover* of M if it satisfies the following conditions: (1) Each (W_i, φ_i) is an $(\mathcal{F}, \mathcal{T})$-chart. (2) $\{\text{Int}\, W_i\}_{i=1}^m$ is an open cover of M. (3) If $W_i\cap W_j\neq\emptyset$, then there is an $(\mathcal{F}, \mathcal{T})$-chart (W, φ) such that $W_i\cup W_j\subset\text{Int}\, W$. $R_i=\varphi_i(\{0\}\times D^1)$ is called the *axis* of W_i. We fix such $\mathcal{W}$ hereafter.

An open saturated subset U of M is a *foliated product* if every leaf of $\hat{i}^*\mathcal{T}$ is a compact arc, where $\hat{i}$ denotes the canonical immersion from the metric completion $\hat{U}$ of U into M. The *thickness* of a foliated product U (resp. an $(\mathcal{F}, \mathcal{T})$-chart (W, φ)) is the maximum length of leaves of $\hat{i}^* \mathcal{T}$ (resp. $\mathcal{T}|W$).

Now we will show the implication (i)$\Rightarrow$(ii) of Theorem 2.1. Suppose $\mathcal{F}$ is expansive with expansive constant δ. The following is obvious by the definition of expansiveness.

LEMMA 2.2. *$\mathcal{F}$ does not possess a foliated product of thickness less than δ.*

By taking a refinement if necessary, we may assume hereafter that each element W_i of $\mathcal{W}$ has thickness less than $\delta/3$.

LEMMA 2.3. *There exist finitely many open local minimal sets of $\mathcal{F}$ whose union is dense in M.*

PROOF. Put $W_i(t)=\varphi_i(D^{n-1}\times[-t, t])$, $P_{2i-1}(t)=\varphi_i(D^{n-1}\times\{-t\})$ and $P_{2i}(t)=\varphi_i(D^{n-1}\times\{t\})$. By the definition of the biregular cover, there is a constant $0<c<1$ such that if $c<t\leq 1$ then $\bigcup_{i=1}^{m} W_i(t)=M$. Denote by $L_j(t)$ the leaf of $\mathcal{F}$ which contains $P_j(t)$, $1\leq j\leq 2m$, and put $S(t)=\bigcup_{j=1}^{2m} L_j(t)$. Fix t_0 such that $c<t_0<1$. We will claim that $S(t_0)$ is dense in M. In fact, otherwise one easily sees that each connected component of $M-\overline{S(t_0)}$ is a foliated product of thickness $<\delta/3$, contradicting (2.2). Next, we will claim that there exist j_1, $1\leq j_1\leq 2m$, and t_1, t_2, $c<t_1<t_0<t_2<1$, such that $L_{j_1}(t)$ is locally dense for each t, $t_1<t<t_2$. Indeed if this claim is not true, then, for each j, $1\leq j\leq 2m$, one can find s_j arbitrarily near t_0 such that $L_j(s_j)$ is not locally dense. But the set $\bigcup_{j=1}^{2m} L_j(s_j)$ must be dense in M by the same reason as in the preceding claim. This contradiction proves the claim. Now, obviously, $\bigcup_{t_1<t<t_2} L_{j_1}(t)$ is an open local minimal set. We denote this set by U_1. If $\overline{U_1}=M$, then we are done. Otherwise we can show that there exists j_2, $1\leq j_2\leq 2m$, suct that j_2 satisfies the property similar to the one for j_1 in the second claim and that $L_{j_2}(t_0)\subset M-\overline{U_1}$. Hence we find another open local minimal set U_2, which contains $L_{j_2}(t_0)$. Repeating this procedure, we attain the desired conclusion. (2.3) is proved.

LEMMA 2.4. *Every open local minimal set of $\mathcal{F}$ is nontrivial.*

PROOF. Let U be an open local minimal set of $\mathcal{F}$. First we see that there is some i, $1\leq i\leq m$, such that $W_i\subset U$. In fact, if we take an arbitrary compact $\mathcal{T}$-arc J contained in U, then by the expansiveness of $\mathcal{F}$, there exists a fence F such that $\lambda(F)=J$ and Length$(\rho(F))\geq\delta$. By the choice of $\mathscr{W}$, $\rho(F)$ penetrates some W_i. Thus $W_i \subset U$. Now let W_{i_0} be a chart contained in U. Then by an argument similar to the above, there is a holonomy map h_1 such that the image $h_1(R_{i_0})$ of the axis R_{i_0} by h_1 penetrates some W_{i_1}. We may assume without loss of generality that $\mathrm{Int}\, h_1(R_{i_0}) \supset R_{i_1}$. Iterating this procedure, we obtain a sequence of holonomy maps $\{h_k\}_{k=1}^{\infty}$ and a sequence of axes $\{R_{i_k}\}_{k=1}^{\infty}$ such that Int $h_k(R_{i_{k-1}}) \supset R_{i_k}$ for all k. Since the number of axes are finite, there exist $p < q$ such that $i_p = i_q$. Then for the composite map $h = h_q \circ \cdots \circ h_{p+1}$ we have that $\mathrm{Int}\, h(R_{i_p}) \supset R_{i_p}$. Hence h has a fixed point in R_{i_p}. This means that U contains a leaf with nontrivial holonomy, as desired.

The proof of the implication (i)$\Rightarrow$(ii) is complete.

To show that (ii) implies (i), let us suppose that there exist finitely many nontrivial open local minimal sets of $\mathscr{F}$ whose union is dense in M.

LEMMA 2.5. *Let U be an open local minimal set of $\mathscr{F}$. Then $\mathscr{F}|U$ is expansive.*

PROOF. By the hypothesis of $\mathscr{F}$, we can choose a leaf L in U which has *expanding holonomy*. That is, there exists a fence F_0 such that $l(F_0)$ is a loop on L based at, say, z and that the length of the $\mathscr{T}$-arc $[z, h_{F_0}(x)]$ is greater than that of $[z, x]$ for all $x \in \lambda(F_0)$. We set $\delta_U = \lambda(F_0)/2$ and will claim that δ_U is an expansive constant for $\mathscr{F}|U$. Let J be any compact $\mathscr{T}$-arc in U. It suffices to show that there exists a fence F such that $\lambda(F) \subset J$ and Length $(\rho(F)) \geq \delta_U$. Since L is dense in U, we can find a point w in $\mathrm{Int} J \cap L$. Take any $\mathscr{F}$-curve α which joins w to z. Clearly, there exists a fence F_1 such that $\lambda(F_1) \subset J$, $\rho(F_1) \subset \lambda(F_0)$ and $l(F_1) = \alpha$. By the choice of F_0 and δ_U, $\mathrm{Length}(h_{F_0}^N(\rho(F_1))) \geq \delta_U$ for some N. Then the fence F with $\lambda(F) = \lambda(F_1)$ and $l(F) = \alpha * l(F_0)^N$ satisfies Length $(\rho(F)) \geq \delta_U$, as desired. Lemma is proved.

Now let J_0 be any compact $\mathscr{T}$-arc in M. Since the union of all open local minimal sets is dense in M, there exists a compact subarc J of J_0 which is contained in some open local minimal set, say U. By (2.5), we find a fence F such that $\lambda(F) = J$ and Length $(\rho(F)) \geq \delta_U$, where δ_U is an expansive constant for $\mathscr{F}|U$. Thus, if we put $\delta = \min\{\delta_U \mid U$ is an open local minimal set of $\mathscr{F}\} > 0$, we see that $\mathscr{F}$ is expansive with expansive constant δ. This proves (ii)$\Rightarrow$(i). The proof of Theorem 2.1 is complete.

Recall that a leaf L of $\mathscr{F}$ is *resilient* if there exists a fence F with the following properties: $u(F) \subset L$, $l(F)$ is a loop in L based at, say, x and h_F is a contraction to x. Since a nontrivial open local minimal set clearly contains a resilient leaf, by (2.1) we have the following.

COROLLARY 2.6. *Let M be a closed manifold and $\mathscr{F}$ a codimension one C^r, $r\geq 0$, expansive foliation on M. Then $\mathscr{F}$ has a resilient leaf.*

COROLLARY 2.7. *Let M be a closed 3-manifold. If M admits a codimension one C^r, $r\geq 0$, expansive foliation, then $\pi_1(M)$ has exponential growth.*

PROOF. Let $\mathscr{F}$ be an expansive foliation on M. By (2.1) $\mathscr{F}$ contains no Reeb components, hence, by [N] $\mathscr{F}$ admits no null homotopic closed transversals. Then, as is well-known (see *e.g.*, the proof of [Pl, Lemma 7.2]), the growth of $\pi_1(M)$ dominates the growth of each leaf of $\mathscr{F}$. On the other hand, it is well-known that a resilient leaf has exponential growth (see *e.g.*, [HH, Chapter 9, 2.1.8]). From these facts and (2.6) we have the desired conclusion.

Since a transversely real analytic foliation admits no null homotopic closed transversals ([Ha]), by the same argument as above we also obtain the following.

COROLLARY 2.8. *Let M be a closed manifold. If M admits a codimension one real analytic expansive foliation, then $\pi_1(M)$ has exponential growth.*

A notion of geometric entropy for foliations has been introduced by Ghys, Langevin and Walczak [GLW]. In [GLW], it is shown, among others, that if a codimension one foliation $\mathscr{F}$ on a closed Riemannian manifold (M, g) has a resilient leaf, then the geometric entropy $h(\mathscr{F}, g)>0$. Thus by (2.6) we have the following.

COROLLARY 2.9. *Let M be a closed manifold with a Riemannian metric g and $\mathscr{F}$ a codimension one C^r, $r\geq 0$, expansive foliation on M. Then $h(\mathscr{F}, g)>0$.*

We conclude this section by remarking briefly that most of the results obtained in this section can be extended to foliations with circle prong singularities (see [IM] for definition). Such a singular foliation naturally arises as the stable foliation of a nonsingular expansive flow on a closed 3-manifold. Let $\mathscr{F}$ be a codimension one foliation with circle prong singularities. Then (2.1) is valid for $\mathscr{F}$. In fact, the proof given there goes through almost without change. Also, (2.6) and (2.9) are valid for expansive $\mathscr{F}$ and (2.7) is valid for expansive $\mathscr{F}$ which satisfies the properties 1) and 2) in Theorem 1.6 of [IM]. Thus we reprove, by a somewhat different method, Paternain's theorem [Pa] which says that if a closed 3-manifold M admits a nonsingular expansive flow, then $\pi_1(M)$ has exponential growth.

3. Strongly expansive foliations of arbitrary codimension

Foliations considered in this section may have arbitrary codimension. Let M be a closed C^∞ Riemannian manifold and $\mathscr{F}$ a codimension q foliation on M with C^1 leaves. As in § 1, we consider a system of transverse disks $\{D_r(x)\}_{x\in M}$ as a substitute for a transverse foliation in codimension one.

We introduce the following.

DEFINITION 3.1. $\mathscr{F}$ is *strongly expansive* if there exists $\delta>0$ (called a *strongly expansive constant*) with the following property: for any $x\in M$ and any $\epsilon>0$, there are an $\mathscr{F}$-curve α with $\alpha(0)=x$ and a fence F along α such that $\lambda(F)\subset D_\epsilon(x)$ and that $\mathrm{Int}\,\rho(F)\supset D_\delta(\alpha(1))$.

Clearly, strong expansiveness implies expansiveness.

Weakly stable foliations of Anosov flows are strongly expansive.

A *distinguished chart* (W, φ) is a pair of an embedding $\varphi: D^{n-q}\times D^q\to M$ and the set $W=\mathrm{Image}\,\varphi$ such that $\varphi|D^{n-q}\times\{y\}$ is an embedding into a leaf of $\mathscr{F}$ for each $y\in D^q$ and that $\varphi|\{x\}\times D^q$ is contained in the disk $D_{r_x}(\varphi(x,0))$ for some $r_x>0$. $\mathrm{Max}\,\{r_x\,|\,x\in D^{n-q}\}$ is called a *transverse width* of (W, φ). An atlas $\mathscr{W}=\{(W_i, \varphi_i)\}_{i=1}^m$ on M is a *distinguished cover* of M if it satisfies the following two conditions: (1) Each (W_i, φ_i) is a distinguished chart. (2) $\{\mathrm{Int}\,W_i\}_{i=1}^m$ is an open cover of M. Each of the sets $\varphi_i(D^{n-q}\times\{y\})$, $1\leq i\leq m$, $y\in D^q$ is called a *plaque*. From now on we fix such $\mathscr{W}$.

Let L be a leaf of $\mathscr{F}$ and $x\in L$. The *growth function* of L at x is defined by $g(r)=$(the number of distinct plaques which can be joined to x by $\mathscr{F}$-curves of length $\leq r$). Let G be a finitely generated group and G^1 a finite set of generators for G. The *growth function* of G relative to G^1 is defined by $g(r)=$(the number of distinct elements of G which have word -length $\leq r$). L or G is said to have *exponential growth* (resp. *quasi -exponential growth*) if its growth function g satisfies that $\liminf_{r\to\infty}(1/r)\log g(r)>0$ (resp. $\limsup_{r\to\infty}(1/r)\log g(r)>0$).

THEOREM 3.2. *Let M be a closed manifold and $\mathscr{F}$ a codimension q strongly expansive foliation on M with C^1 leaves. Then $\mathscr{F}$ has a leaf with quasi-exponential growth.*

The idea of proof is due to Plante-Thurston ([PT]) and Paternain

([Pa]).

Assume that $\mathscr{F}$ is strongly expansive with strongly expansive constant δ. First we prepare the following lemma, which is a version of [KS, Corollary (2.11)].

LEMMA 3.3. *For any ϵ, $0<\epsilon<\delta$, there exists $l=l(\epsilon)>0$ with the following property : for any $x\in M$, there are an $\mathscr{F}$-curve α_x and a fence F_x along α_x such that* (1) $\alpha_x(0)=x$, (2) Length $\alpha_x\leq l$, (3) $\lambda(F_x)\subset D_\epsilon(x)$ *and* (4) Int $\rho(F_x)\supset D_\delta(\alpha_x(1))$. *Furthermore, we may assume that there is $\zeta>0$ with the property that for any $x\in M$ and any $y\in\lambda(F_x)$*, Length $(F_x|[0,1]\times\{y\})/\text{Length }\alpha_x<\zeta$.

PROOF. Suppose the first statement does not hold. Then there are a constant ϵ, $0<\epsilon<\delta$, a divergent sequence $\{l_n\}$ of positive numbers and a sequence $\{x_n\}$ of points of M such that if α is an $\mathscr{F}$-curve with $\alpha(0)=x_n$ and with Length $\alpha\leq l_n$ and if F is a fence along α satisfying that $\lambda(F)\subset D_\epsilon(x_n)$, then $D_\delta(\alpha(1))$ is not a subset of $\text{Int}(\rho(F))$. By choosing a subsequence if necessary we may assume that $\{x_n\}$ converges to a point x_∞ of M. Then for any $\mathscr{F}$-curve α with $\alpha(0)=x_\infty$ and for any fence F along α satisfying $\lambda(F)\subset D_\epsilon(x_\infty)$, $D_\delta(\alpha(1))$ is not a proper subset of $\text{Int}(\rho(F))$. This contradicts the strong expansiveness of $\mathscr{F}$.

The second statement easily follows from the compactness of M and the following standard fact: Let $x\in M$ and let F_x be a fence with the properties (1) to (4) of (3.3). Then there exists a neighborhood U of x in M such that for each $y\in U$, we can choose as F_y a fence whose image is close to that of F_x. (3.3) is proved.

Taking δ smaller if necessary we may assume the following: for any $x\in M$, any $y\in D_\delta(x)$ and any sufficiently small $\epsilon>0$, define $B_\epsilon(y)$ by $B_\epsilon(y)=\{z\in D_\delta(x)\,|\,\text{dist}(y,z)\leq\epsilon\}$, where dist means the distance induced from the Riemannian metric on $D_\delta(x)\subset M$. Let $\eta_{xy}: B_\epsilon(y)\to D_\delta(y)$ be the embedding which is uniquely determined by requiring that for any $z\in B_\epsilon(y)$, z and $\eta_{xy}(z)$ lie on the same plaque. Then there exists a small number $\varkappa=\varkappa(\epsilon)>0$ not depending on x and y such that for any $z\in B_\epsilon(y)$, z and $\eta_{xy}(z)$ can be joined by an $\mathscr{F}$-curve of length $<\varkappa$.

PROOF OF (3.2). Put $P_i=\varphi_i(D^{n-q}\times\{0\})$. By taking a refinement of $\mathscr{W}$ if necessary, we assume that each distinguished chart W_i has sufficiently small transverse width that for any $x\in M$, $D_\delta(x)$ intersects at least one of the P_i's. We will construct a (formal disjoint) union A_n of compact transverse disks inductively. First, let x_0 be any point of M and put $A_0=D_\delta(x_0)$. Next, suppose that we have constructed $A_n=\coprod_{p=1}^{2^n}A_n^p$, where $A_n^p = D_\delta(x_n^p)$ for some $x_n^p \in M$. Take two points x_n^{p+} and x_n^{p-} in $D_{\delta/2}(x_n^p)$ such that dist$(x_n^{p+}, x_n^{p-}) > \delta/2$. Let $\epsilon > 0$ be sufficiently closer to 0 than δ. By (3.3) one can find a fence F_n^{p+} along an $\mathscr{F}$-curve α_n^{p+} satisfying the following property P(F_n^{p+}): $\alpha_n^{p+}(0) = x_n^{p+}$, $\lambda(F_n^{p+}) \subset D_\epsilon(x_n^{p+})$, Length $(\alpha_n^{p+}) \leq l = l(\epsilon)$, $\rho(F_n^{p+}) = D_\delta(\alpha_n^{p+}(1))$ and Length $(F_n^{p+}|[0,1]\times\{y\})$/Length $\alpha_n^{p+} < \zeta$ for all $y \in \lambda(F_n^{p+})$ (see (3.3)). Similarly one finds a fence F_n^{p-} along an $\mathscr{F}$-curve α_n^{p-} satisfying the property P(F_n^{p-}). Put $A_{n+1}^{2p-1} = \rho(F_n^{p-})$ and $A_{n+1}^{2p} = \rho(F_n^{p+1})$, and set $A_{n+1} = \coprod_{p=1}^{2^{n+1}} A_{n+1}^p$. Now since $A_n^p = D_\delta(x_n^p)$, by the choice of δ and the distinguished cover, A_n intersects $\bigcup_{i=1}^m P_i$ in at least 2^n points, Hence for each n there is $1 \leq i_n \leq m$ such that $A_n \cap P_{i_n}$ consists of at least $[2^n/m]$ points. Since P_i's are finite in number, one can find a divergent subsequence $\{n_k\}$ and a suffix i, $1 \leq i \leq m$, such that for each k, $A_{n_k} \cap P_i$ consists of at least $[2^{n_k}/m]$ points. Notice here that by the choice of δ and the construction of A_n, every point of A_n^p can be joined to a point of A_0 by an $\mathscr{F}$-curve of length$\leq n(\zeta l + x)$. Therefore if for each point $x \in A_0$ we choose a plaque P_x which contains x, and if we take $y \in P_i$ as a base point, we see that at least $[2^{n_k}/m]$ dictinct plaques in $\{P_x\}_{x\in A_0}$ can be joined to y by $\mathscr{F}$-curves of length$\leq n_k(\zeta l + x) + D$, where D is the diameter of P_i. This implies that the leaf of $\mathscr{F}$ which contains P_i has quasi-exponential growth. The proof of (3.2) is complete.

Let $\pi: \widetilde{M} \to M$ be the universal covering of M and $\widetilde{\mathscr{F}}$ the pulled-back foliation on $\widetilde{M}$. A distinguished chart of $\widetilde{\mathscr{F}}$ means a connected component of $\pi^{-1}(W_i)$ where W_i is a distinguished chart of $\mathscr{F}$.

COROLLARY 3.4. *Let M and $\mathscr{F}$ be as in (3.2). Suppose that $\mathscr{F}$ has the following property: For any leaf $\widetilde{L}$ of $\widetilde{\mathscr{F}}$ and for any distinguished chart $\widetilde{W}$ for $\widetilde{\mathscr{F}}$, $\widetilde{L} \cap \widetilde{W}$ consists of at most one plaque. Then $\pi_1(M)$ has exponential growth.*

PROOF. If $\mathscr{F}$ has the property in (3.4), then, as is well known (see *e.g.*, the proof of [Pl, Lemma 7.2]), the growth of $\pi_1(M)$ dominates the growth of each leaf of $\mathscr{F}$. It follows from this fact and (3.2) that $\pi_1(M)$ has quasi-exponential growth. But then $\pi_1(M)$ must necessarily have exponential growth by [HH, Chapter 9, 1.2.4].

REMARK. $\mathscr{F}$ satisfies the hypothesis of (3.4) if $\mathscr{F}$ has a structure of a foliated bundle.

COROLLARY 3.5. *Let M be a closed manifold with a Riemannian metric g and $\mathscr{F}$ a codimension q strongly expansive foliation on M. Then the geometric entropy $h(\mathscr{F}, g) > 0$.*

To prove (3.5), we need to recall the entropy relative to a distinguished cover (Our definition is slightly different from that of [GLW] and may be related to Hurder's one [Hu]). Let $\mathscr{W}=\{W_i, \varphi_i)\}_{i=1}^m$ be a distinguished cover of M. We say that a subset E of M is *$(\mathscr{F}, \mathscr{W}, r, \epsilon)$-separated* if for any two distinct points x, y of E, one of the following two conditions is verified: (i) There exists no W_i such that $\{x, y\}\subset W_i$. (ii) There exist a chart W_i, an $\mathscr{F}$-curve α and a fence $F:[0,1]\times N\to M$ along α such that (a) $\lambda(F)\subset W_i$, (b) $\alpha(0)=x$, (c) y and a point, say z, of $\lambda(F)$ belong to the same plaque of W_i, (d) $\mathrm{Max}_{x\in N}(\mathrm{Length}\, F|[0,1]\times\{x\})\leq r$ and (e) $\mathrm{dist}\,(h_F(x), h_F(z))\geq\epsilon$. We denote by $N(\mathscr{F}, \mathscr{W}, r, \epsilon)$ the maximum cardinality of $(\mathscr{F}, \mathscr{W}, r, \epsilon)$-separated sets and define $h(\mathscr{F}, \mathscr{W}, \epsilon)=\limsup_{r\to\infty}(1/r)\log N(\mathscr{F}, \mathscr{W}, r, \epsilon)$. Then by [GLW, Théorème 3.4], $h(\mathscr{F}, g)>0$ if $h(\mathscr{F}, \mathscr{W}, \epsilon)>0$. Therefore in order to prove (3.5), we have only to show that $h(\mathscr{F}, \mathscr{W}, \epsilon)>0$ for some ϵ.

PROOF OF (3.5). Let δ be a strongly expansive constant for $\mathscr{F}$. Let $l, A_n, x_n^p, F_n^{p\pm}, \varkappa$ and ζ be as in the proof of (3.2). Denote by y_n^p the point of A_0 which is joined to x_n^p by an $\mathscr{F}$-curve along $\mathscr{F}$-curves in the fences $F_i^{j\pm}$. Put $E_n=\{y_n^p|1\leq p\leq 2^n\}$. Then E_n is $(\mathscr{F}, \mathscr{W}, n(\zeta l+\varkappa), \delta)$-separated and has cardinality 2^n. From this it follows that $h(\mathscr{F}, \mathscr{W}, \delta)>0$, completing the proof.

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