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Polycyclic groups of diffeomorphisms on the half-line

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Introduction

Groups of diffeomorphisms of one-dimensional manifolds are connected with codimension one foliations and present interesting facts. Polycyclic groups of diffeomorphisms of the real line are studied by J. F. Plante [**P1**], [**P3**]. Their results are applied to codimension one foliations on manifolds with solvable fundamental groups (see S. Matsumoto [**Ma**] and [**P3**]). We are interested in the case where the groups have fixed points. This case reduces to the groups of diffeomorphisms of the half-line. For these groups in case where they are abelian, several facts are already known. We are concerned with both abelian and non-abelian cases in this paper. Partial results for polycyclic groups of diffeomorphisms of the half-line are obtained by Plante [**P2**], Plante and Thurston [**P-T**]. Our results describe the classification of such polycyclic groups, that is, polycyclic groups of the diffeomorphisms on the half-line can be essentially classified into two types. The main result is the following.

THEOREM. *Let* Γ *be a polycyclic subgroup of* $\mathrm{Diff}^r[0,\infty)$, N *the nilradical of* Γ *and let* $r=2,\ldots,\infty$. *Assume that* $\mathrm{Fix}(\Gamma)$ $(=\{x\in[0,\infty)\mid f(x)=x$ *for any* $f\in\Gamma\})=\{0\}$. *Then the following hold.*

(i) *If* $\mathrm{Fix}(N)=\{0\}$, *then* $\Gamma|_{(0,\infty)}$ *is* C^r *conjugate to a subgroup of the group* $\mathrm{Aff}^+(\boldsymbol{R})$ *of the orientation preserving affine maps of the real line.*

(ii) *If* $\mathrm{Fix}(N)\neq\{0\}$, *then there exists a contraction* $f\in\mathrm{Diff}^r[0,\infty)$ *such that* Γ *is isomorphic to a semi-direct product* $N\rtimes Z_f$ *of* N *and* Z_f *where* Z_f *denotes the infinite cyclic group generated by* f.

For the detailed definitions, see Sections 1, 5 and 6. The proof of the theorem is in Section 6. Examples of the polycyclic groups are given in Section 5.

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1. Preliminary

1.A. Polycyclic groups.

A group Γ is said to be *polycyclic* if there is a finite sequence of subgroups

$$\Gamma=\Gamma_0\supset\Gamma_1\supset\cdots\supset\Gamma_n=\{e\}$$

such that for each $i=1, \cdots, n$, Γ_i is normal in Γ_{i-1} and Γ_{i-1}/Γ_i is (finite or infinite) cyclic. In particular, if each Γ_{i-1}/Γ_i is infinite cyclic, Γ is said to be *strongly polycyclic*. It follows immediately that every subgroup of a polycyclic group is again polycyclic. Clearly a polycyclic group is finitely generated. We quote from [**Ra**] several results for polycyclic groups which we need in this paper.

Let Γ be a polycyclic group.

(1.1) PROPOSITION. *Γ admits a unique maximal non-trivial normal nilpotent subgroup.*

(cf. Corollary 2 to Lemma 4.7 of [**Ra**])

The normal subgroup of Γ in the above proposition is called the *nilradical* of Γ and we denote it by N.

(1.2) PROPOSITION. *Let Γ' be a normal subgroup of Γ such that $\Gamma'\supset N$ and let N' be the nilradical of Γ'. Then $N'=N$.*

(cf. Remark 4.9 of [**Ra**])

Remark that the nilradical N of Γ does not necessarily contain the commutator subgroup $[\Gamma, \Gamma]$ of Γ, that is, Γ/N is not necessarily abelian.

(1.3) PROPOSITION. *Γ admits a normal subgroup Γ_0 of finite index such that $\Gamma_0\supset N$ and $N\supset[\Gamma_0, \Gamma_0]$ (that is, Γ_0/N is abelian).*

(cf. Corollary 4.11 of [**Ra**])

1.B. Diffeomorphisms of the half-line.

Denote by $\mathrm{Diff}^r[0, \infty)$ the group of C^r diffeomorphisms of the half-line $[0, \infty)$ where r is a positive integer or ∞. Let Γ be a subgroup of $\mathrm{Diff}^r[0, \infty)$. Then Γ acts naturally on $[0, \infty)$ by the map $(g, x)\mapsto g(x)$. For a diffeomorphism g, we define $\mathrm{Fix}(g)$ to be the set $\{x\in[0, \infty)|g(x)=x\}$ and call it the *fixed point set* of g. Also for a subgroup $\Gamma\subset\mathrm{Diff}^r[0, \infty)$, we define $\mathrm{Fix}(\Gamma)=\{x\in[0, \infty)|g(x)=x$ for every $g\in\Gamma\}$ and call it the *fixed point set* of Γ. We say that a subset S of $[0, \infty)$ is *g-invariant* if $g(S)(=\{g(x)|x\in S\})=S$. And we say that S is *Γ-invariant* if $g(S)=S$ for every $g\in\Gamma$. If Γ_0 is a normal subgroup of Γ, then $\mathrm{Fix}(\Gamma_0)$ is Γ-invariant. The following lemma is an immediate consequence of a lemma of Kopell (cf. Lemma 1 of [**Ko**]).

(1.4) LEMMA. *Assume that* Γ *is an abelian subgroup of* $\mathrm{Diff}^r[0, \infty)$ *and* $r \geq 2$. *If* $\mathrm{Fix}(g) \cap (0, \infty) \neq \emptyset$ *for some* $g \in \Gamma$ *with* $g \neq$ *identity, then* $\mathrm{Fix}(\Gamma) \cap (0, \infty) \neq \emptyset$.

A map f of $[0, \infty)$ into itself is said to be a *contraction* if $\lim_{n \to \infty} f^n(x) = 0$ for any $x \in [0, \infty)$. It follows obviously that if f is a continuous map of $[0, \infty)$ and $f(x) < x$ for any $x \in (0, \infty)$, then f is a contraction. We denote by f_k the k-times iteration of f, instead of the usual f^k in this paper. Fix a contraction f and a point $a \in (0, \infty)$ and let $a_0 = a$ and $a_j = f_j(a)$. We obtain the following three lemmas from [**C**-**C**] (cf. also [**Sa**]).

(1.5) LEMMA. *Assume that* $f \in \mathrm{Diff}^r[0, \infty)$ *is a contraction and* $r \geq 2$. *Then, for any* $x \in [a_1, a_0]$

$$e^{-c} \cdot \frac{a_j - a_{j+1}}{a_0 - a_1} \leq f_j'(x) \leq e^c \cdot \frac{a_j - a_{j+1}}{a_0 - a_1}$$

where $c = a \cdot \sup\{|f''(x)/f'(x)| \mid x \in [0, a]\}$.

This lemma follows from Lemma(2.6) of [**C**-**C**] and the mean value theorem. Next is an immediate consequence of the above lemma.

(1.6) LEMMA. *Under the same assumption as the lemma above, the sequence* $\{f_j'\}$ *converges uniformly to* 0 *on* $[a_1, a_0]$.

Take $g \in \mathrm{Diff}^r[0, \infty)$ such that $\mathrm{Fix}(g) \supset \{a_j | j \geq 0\}$ and let $h_n = f_{-n} \circ g \circ f_n$ for $n \geq 0$ (where $f_{-n} = f_n^{-1}$). Then a generalized Kopell lemma (Theorem (2.8) of [**C**-**C**]) implies the following.

(1.7) LEMMA. *Under the same assumption as Lemma* (1.5) *and the notation above, the sequence* $\{h_n\}$ *converges uniformly to the identity on* $[a_1, a_0]$.

We shall need a more general version of Lemma(1.5). Take $h_i \in \mathrm{Diff}^r[0, \infty)$ $(i = 1, \ldots, m)$ and let $g_p = h_p \circ h_{p-1} \circ \cdots \circ h_1$ $(p = 1, \ldots, m)$. Fix a compact subinterval J of $[0, a]$. Write $J_p = g_p(J)$, $J_0 = J$ and denote by $|J_p|$ the length of J_p. We assume that each J_p is subinterval of $[0, a]$ and does not meet the interior of other J_i $(i \neq p)$. Put

$$\theta = \sup\{|h_p''(x)/h_p'(x)| \mid x \in [0, a],\ p = 1, \ldots, m\} \ (>0)$$

Then, in the same way as Lemma (1.5), we have the following lemma.

(1.8) LEMMA. *Under the above assumption and notation, if* $z \in J$, *then*

$$e^{-a\theta}\frac{|J_m|}{|J|}\leq g_m'(z)\leq e^{a\theta}\frac{|J_m|}{|J|}.$$

Moreover we fix numbers λ and ν such that $\lambda>1$ and

$$0<\nu<\frac{|J|\log\lambda}{a\theta\lambda e^{a\theta}}.$$

Then, by induction, it is proved that if $x_0\in J$ and $|w-x_0|<\nu$, then

$$g_p'(w)<\lambda g_p'(x_0)$$

(cf. [**Sa**, p. 83]). Combining this fact and Lemma(1.8), we get the following fact.

(1.9) LEMMA. *Let J_ν be a ν-neighborhood of J. Then, under the above assumption and notation,*

$$\frac{1}{\lambda}e^{-a\theta}\frac{|J_m|}{|J|}\leq g_m'(z)\leq \lambda e^{a\theta}\frac{|J_m|}{|J|}$$

if $z\in J_\nu$.

We remark that ν depends only on a, θ, $|J|$ and λ.

2. The function defined by a contraction.

Let $f\in\mathrm{Diff}^r[0,\infty)$ be a contraction and $r\geq 2$. We define

$$H_k(x)=\frac{f_k''(x)}{f_k'(x)}$$

for $k\geq 1$ where $f_k=\overbrace{f\circ\cdots\circ f}^{k}$ and

$$H(x)=\lim_{k\to\infty}H_k(x).$$

The aim of this section is to prove the following fact. This fact and Corollary (2.6) are needed in next two sections.

(2.1) PROPOSITION. *H is a well-defined C^{r-2} function of* $(0,\infty)$.

In the sequel, we fix $a\in(0,\infty)$ and $a_k=f_k(a)$, and let $f_jx=f_j(x)$. For the proof, it suffices to show that $\{H_k^{(p)}\}$ converges uniformly on $[a_1, a_0]$ for each p $(0\leq p\leq r-2)$, where $H_k^{(p)}$ is the p-th derivative of H_k. We need the following lemma.

(2.2) LEMMA. *$H_k^{(p)}(x)$ is expressed in the following polynomials of* *two types*:

(i) *We have*

$$H_k^{(p)}(x)=\frac{f_k^{(p+2)}(x)}{f_k'(x)}+P_{p,k}(x)$$

where $P_{p,k}(x)$*'s are polynomials in* $\frac{f_k''(x)}{f_k'(x)}, \cdots, \frac{f_k^{(p+1)}(x)}{f_k'(x)}$ *and for fixed* p *they have the same expression as polynomials.*

(ii) *We have*

$$H_k^{(p)}(x)=\sum_{j=0}^{k-1}\frac{Q_{p,j}(x)}{\{f'(f_jx)\}^{p+1}}$$

where $Q_{p,j}(x)$ *is a polynomial in* $f'(f_jx), \cdots, f^{(p+2)}(f_jx)$ *and* $f_j'(x), \cdots, f_j^{(p+1)}(x)$ *such that for fixed* p $Q_{p,j}(x)$*'s have same expressions and the degree of each term with respect to* $f_j'(x), \cdots, f_j^{(p+1)}(x)$ *is greater than one. (For* $j=0$, *we consider that* $f_0(x)\equiv x$.)

PROOF. We fix k and prove these assertion by induction on p.

(i) When $p=0$, the assertion is clearly true. Assume that the assertion is true for an integer $p\geq 0$. Then we have

$$\begin{aligned}H_k^{(p+1)}(x)&=(H_k^{(p)}(x))'\\&=\frac{f_k^{(p+3)}(x)}{f_k'(x)}-\frac{f_k^{(p+2)}(x)\cdot f_k''(x)}{\{f_k'(x)\}^2}+(P_{p,k}(x))'\\&\equiv\frac{f_k^{(p+3)}(x)}{f_k'(x)}+P_{p+1,k}(x).\end{aligned}$$

Since we can easily see that $(P_{p,k}(x))'$ is a polynomial in $\frac{f_k''(x)}{f_k'(x)}, \cdots, \frac{f_k^{(p+2)}(x)}{f_k'(x)}$, the assertion is true for $p+1$.

(ii) This is certainly true for $p=0$. Indeed, since $f_k'(x)=\prod_{j=0}^{k-1}f'(f_jx)$ we obtain

$$H_k^{(0)}(x)=H_k(x)=(\log f_k'(x))'=\sum_{j=0}^{k-1}\log f'(f_jx))'=\sum_{j=0}^{k-1}\frac{f''(f_jx)\cdot f_j'(x)}{f'(f_jx)}.$$

Assume that the assertion is true for an integer $p\geq 0$. We have

$$\begin{aligned}\left(\frac{Q_{p,j}(x)}{\{f'(f_jx)\}^{p+1}}\right)'&=\frac{Q_{p,j}'(x)\cdot f'(f_jx)-(p+1)Q_{p,j}(x)\cdot f''(f_jx)\cdot f_j'(x)}{\{f'(f_jx)\}^{p+2}}\\&\equiv\frac{Q_{p+1,j}(x)}{\{f'(f_jx)\}^{p+2}}.\end{aligned}$$

Clearly $Q'_{p,j}(x)$ is a polynomial in $f'(f_jx), \cdots, f^{(p+3)}(f_jx)$ and $f'_j(x), \cdots, f_j^{(p+2)}(x)$. It follows immediately that the assertion is true for $p+1$. ∎

We shall prove Proposition (2.1) by induction on p and need the following definition.

DEFINITION. By assertions (A_p) and (B_p) we mean the following:
(A_p)—$\{H_k^{(p)}\}$ converges uniformly on $[a_1, a_0]$.
(B_p)—There exists a number $C_p \geq 0$ which does not depend on k such that $|f_k^{(p+2)}(x)| \leq C_p f'_k(x)$ for all $x \in [a_1, a_0]$ and $k \geq 1$.

(2.3) LEMMA. *The assertion (A_0) is valid, that is, $\{H_k\}$ converges uniformly on $[a_1, a_0]$.*

PROOF. From the equation in the proof of (ii) of Lemma (2.2), we have

$$\begin{aligned}|H_{k+l}(x)-H_k(x)| &= \left|\sum_{j=k}^{k+l-1} \frac{f''(f_jx)\cdot f'_j(x)}{f'(f_jx)}\right| \\ &\leq M\sum_{j=k}^{k+l-1}|f'_j(x)| \\ &\quad (\text{where } M=\max\{|f''(x)/f'(x)|\,|x\in[0, a_0]\}) \\ &\leq Me^c\sum_{j=k}^{k+l-1}\frac{a_j-a_{j+1}}{a_0-a_1} \\ &\quad (\text{by Lemma } (1.5)) \\ &\leq \frac{Me^c}{a_0-a_1}\cdot a_k\end{aligned}$$

for any $x \in [a_1, a_0]$ and $l \geq 1$. Since f is a contraction, the sequence $\{a_k\}$ converges to 0. Therefore we see that $\{H_k\}$ converges uniformly on $[a_1, a_0]$. ∎

(2.4) LEMMA. *If assertions $(B_0), \cdots, (B_{p-1})$ and (A_p) are valid, then (B_p) is valid.*

PROOF. For $p=0$, clearly (A_0) implies (B_0). Next fix $p \geq 1$. (We will use (i) of Lemma (2.2).) From $(B_0), \cdots, (B_{p-1})$, it follows that there exists a number $D_2 \geq 0$ such that

$$|f_k^{(i)}(x)/f'_k(x)| \leq D_2 \ (i=2, \cdots, p+1)$$

for all $x \in [a_1, a_0]$ and $k \geq 1$. Since $P_{p,k}(x)$ is a polynomial in $\frac{f''_k(x)}{f'_k(x)}, \cdots, \frac{f_k^{(p+1)}(x)}{f'_k(x)}$, there exists a number $D_3 \geq 1$ such that

$$|P_{p,k}(x)| \leq D^3$$

for all $x \in [a_1, a_0]$ and $k \geq 1$. On the other hand, by (A_p), we see that there exists a number $D_1 \geq 0$ such that

$$|H_k^{(p)}(x)| \leq D_1$$

for all $x \in [a_1, a_0]$ and $k \geq 1$. Therefore, from (i) of Lemma (2.2), it follows that there exists a number $C_p \geq 0$ such that $|f_k^{(p+2)}(x)/f_k'(x)| \leq C_p$ for all $x \in [a_1, a_0]$ and $k \geq 0$. ■

(2.5) LEMMA. *Assume that assertions* $(B_0), \cdots, (B_{p-1})$ *are valid for* $p \geq 1$. *Then also* (A_p) *is valid.*

PROOF. Fix $p \geq 1$. From (ii) of Lemma (2.2), we have

$$|H_{k+l}^{(p)}(x) - H_k^{(p)}(x)| \leq \sum_{j=k}^{k+l-1} \left| \frac{Q_{p,j}(x)}{\{f'(f_j x)\}^{p+1}} \right|$$

(where $Q_{p,j}$ is a polynomial in $f'(f_j x), \cdots, f^{(p+2)}(f_j x)$ and $f_j'(x), \cdots, f_j^{(p+1)}(x)$). We estimate $Q_{p,j}(x)$. Clearly there exists a number D_1 such that

$$|f^{(j)}(f_j x)| \leq D_1 \quad (i=1, \cdots, p+2)$$

for all $x \in [0, a_0]$ and $j \geq 0$ because $f_j(x) \leq a_0$ for $x \in [0, a_0]$. From the assumption of the lemma, it follows that there exists a number D_2 such that

$$|f_j^{(i)}(x)| \leq D_2 f_j'(x) \quad (i=2, \cdots, p+1)$$

for all $x \in [a_1, a_0]$ and $j \geq 0$. By Lemma (1.6), we can choose a sufficiently large integer L such that if $j \geq L$ then $|f_j'(x)| \leq 1$ for all $x \in [a_1, a_0]$. Therefore, if $j \geq L$ and $q \geq 1$, then we have $\{f_j'(x)\}^q \leq f_j'(x)$ for all $x \in [a_1, a_0]$. Hence, there exists a number D_3 such that if $j \geq L$, then

$$|Q_{p,j}(x)| \leq D_3 f_j'(x)$$

for all $x \in [a_1, a_0]$. Let $m = \min\{f'(x) | x \in [0, a_0]\}$. Clearly $m > 0$. Then if k, $k+l \geq L$, we have for all $x \in [a_1, a_0]$

$$\begin{aligned} |H_{k+l}^{(p)}(x) - H_k^{(p)}(x)| &\leq \frac{D_3}{m^{p+1}} \sum_{j=k}^{k+l-1} f_j'(x) \\ &\leq \frac{D_3 \cdot e^C}{m^{p+1}(a_0 - a_1)} \sum_{j=k}^{k+l-1} (a_j - a_{j+1}) \\ &\quad \text{(by Lemma (1.5))} \\ &\leq D \cdot a_k \\ &\quad \text{(where } D = D_3 \cdot e^C / m^{p+1}(a_0 - a_1)\text{)}. \end{aligned}$$

We see that this implies that $\{H_k^{(p)}(x)\}$ converges uniformly on $[a_1, a_0]$ because $\lim_{k\to\infty} a_k=0$. ■

Now we prove the proposition.

PROOF OF PROPOSITON (2.1). From lemmas (2.3), (2.4) and (2.5), by induction, we conclude that assertions (A_p) and (B_p) $(0\leq p\leq r-2)$ are valid. In particular, for each p $(0\leq p\leq r-2)$, $H^{(p)}(x)$ exists on $(0,\infty)$ and is continuous there. Therefore $H(x)$ is a C^{r-2} function of $(0,\infty)$. ■

The following is the assertion (B_p).

(2.6) COROLLARY. *Let* $f\in \mathrm{Diff}^r[0,\infty)$ $(r\geq 2)$ *be a contraction. Then for each* p $(2\leq p\leq r)$ *there exists a number* C_p *such that*

$$|f_k^{(p)}(x)|\leq C_p f_k'(x)$$

for all $x\in[a_1, a_0]$ *and* $k\geq 1$.

REMARK. In addition to the assumption of Proposition (2.1), suppose that $f'(0)\neq 1$. Then $H(0)$ exists and $H(x)$ is a C^{r-2} function of $[0,\infty)$. This fact is described in [**St**] in somewhat different fashion.

3. Abelian groups of the diffeomorphisms

Let Γ be an abelian subgroup of $\mathrm{Diff}^r[0,\infty)$ $(r=2,\cdots,\infty)$ such that $\mathrm{Fix}(\Gamma)=\{0\}$. From Lemma (1.4), it follows that $\mathrm{Fix}(f)=\{0\}$ for any $f\in\Gamma$ with $f\neq$identity. Therefore either f or f^{-1} is a contraction if $f\neq$identity.

Suppose that $f'(0)\neq 1$ for some $f\in\Gamma$. Then a theorem of Sternberg (cf. [**St**]) says that f is C^r conjugate to the linear map $x\mapsto ax$ where $a=f'(0)$. Therefore we easily see that there exists a C^r flow $\varphi:\boldsymbol{R}\times[0,\infty)\to[0,\infty)$ such that Γ is contained in the group $\{\varphi_t|t\in\boldsymbol{R}\}$.

In case where $f'(0)=1$ for all $f\in\Gamma$ and $r=\infty$, from results in [**Se**], [**Ta**] and [**Ko**], it follows there exists a C^1 flow φ such that φ is of class C^∞ on $(0,\infty)$ and Γ is contained in the group $\{\varphi_t\}$. Also for finite r, we obtain the following result.

(3.1) THEOREM. *Let* Γ *be an abelian subgroup of* $\mathrm{Diff}^r[0,\infty)$ $(r=2,\cdots,\infty)$ *such that* $\mathrm{Fix}(\Gamma)=\{0\}$. *Then there exists a* C^1 *flow* φ *on* $[0,\infty)$ *which is of class* C^r *on* $(0,\infty)$ *such that* Γ *is contained in the group* $\{\varphi_t|t\in\boldsymbol{R}\}$. *Furthermore* φ *is unique up to parameter change.*

For finite r, we don't know whether this result follows from results in [**Se**], [**Ta**] and [**Ko**]. We give here a detailed proof because we need this result in the proof of the main theorem.

We prove the theorem in case where $f'(0)=1$ for all $f\in\Gamma$. In the sequel we assume that $f'(0)=1$ for all $f\in\Gamma$. We need several lemmas for the proof.

Fix a contraction $f\in\Gamma$. Since Γ is abelian, $f_k\circ g=g\circ f_k$ for any $g\in\Gamma$. Differentiating both sides of this equation, we have $f_k'(gx)g'(x)=g'(f_kx)f_k'(x)$, namely

$$g'(x)=g'(f_kx)\cdot\frac{f_k'(x)}{f_k'(gx)}.$$

We define formally the function $H(x, y)$ by

$$H(x, y)=\lim_{k\to\infty}\frac{f_k'(y)}{f_k'(x)}=\prod_{j=0}^{\infty}\frac{f'(f_jy)}{f'(f_jx)}.$$

Taking the limit as $k\to\infty$ on the above equation, we have $g'(x)=g'(0)H(g(x), x)$, hence

$$g'(x)=H(g(x), x)$$

by the assumption $g'(0)=1$. So we can think of $g\in\Gamma$ as the solution of this differential equation (cf. [**Ko**] and [**Se**]).

Now we investigate $H(x, y)$. Let $D_n=\{(x, y)\in[0, \infty)\times[0, \infty)|f_n(x)\leq y\leq f_{-n}(x)\}$ for $n>0$.

(3.2) LEMMA. *$H(x, y)$ exists on each D_n, and it is continuous and positive on there. Moreover $H(x, y)$ is of class C^{r-1} on* int D_n *(where* int D_n *denotes the interior of D_n). Therefore $H(x, y)$ is a C^{r-1} function on $(0, \infty)\times(0, \infty)$.*

PROOF. Define $H_k(x, y)=\dfrac{f_k'(y)}{f_k'(x)}=\prod_{j=0}^{k-1}\dfrac{f'(f_jy)}{f'(f_jx)}$ for $x, y\geq 0$ and let $B_k(x, y)=\log H_k(x, y)$. Then

$$\begin{aligned}B_k(x, y)&=\log f_k'(y)-\log f_k'(x)\\&=\sum_{j=0}^{k-1}(\log f'(f_jy)-\log f'(f_jx)).\end{aligned}$$

First we show that the sequence $\{B_k(x, y)\}_{k\in N}$ converges uniformly on $D_n(a)=D_n\cap[0, a]\times[0, a]$ for any $a>0$. From the mean value theorem, it follows that for $p\geq 0$,

$$\begin{aligned}|B_{k+p}(x, y)-B_k(x, y)|&\leq\sum_{j=k}^{k+p-1}|\log f'(f_jy)-\log f'(f_jx)|\\&=\sum_{j=k}^{k+p-1}\left|\frac{f''(\xi_j)}{f'(\xi_j)}\right||f_j(y)-f_j(x)|\end{aligned}$$

where ξ_j is some value between $f_j(x)$ and $f_j(y)$. Therefore, letting $M=\sup\{|f''(x)/f'(x)|\,|0\le x\le a\}$, we have for $(x, y)\in D_n(a)$

$$\begin{aligned}
|B_{k+p}(x, y)-B_k(x, y)| &\le M\sum_{j=k}^{k+p-1}|f_j(y)-f_j(x)| \\
&\le \begin{cases} M\sum(f_j(x)-f_{j+n}(x)) & (\text{if } x\ge y) \\ M\sum(f_j(y)-f_{j+n}(y)) & (\text{if } y\ge x) \end{cases} \\
&\le M\max\left\{\sum_{j=k}^{k+p-1}(f_j(v)-f_{j+n}(v))\,\middle|\, v=x, y\right\} \\
&\le M\max\left\{\sum_{j=k}^{k+n-1}f_j(v)-\sum_{j=u}^{k+p-1}f_{j+n}(v)\,\middle|\, v=x, y\right\} \\
&\qquad (\text{where } u=k+p-n) \\
&\le M\max\left\{\sum_{j=k}^{k+n-1}f_j(v)\,\middle|\, v=x, y\right\} \\
&\le M\max\{n\cdot f_k(v)|v=x, y\} \\
&\le Mnf_k(a)
\end{aligned}$$

(where $p\ge 0$). Since f is a contraction, for any $\varepsilon>0$, there exists sufficiently large L such that if $k\ge L$ then $|f_k(a)|<\varepsilon/Mn$. Therefore, if $k, l\ge L$, then $|B_k(x, y)-B_l(x, y)|<\varepsilon$ for all $(x, y)\in D_n(a)$. Thus $\{B_k(x, y)\}$ converges uniformly on $D_n(a)$. Let

$$B(x, y)=\lim_{k\to\infty} B_k(x, y)$$

for $(x, y)\in(0, \infty)\times(0, \infty)$ or $(x, y)=(0, 0)$. Then $B(x, y)$ is a continuous function.

Next we show that $B(x, y)$ is of class C^{r-1} on $(0, \infty)\times(0, \infty)$. By the definition in Section 2, we have

$$\begin{aligned}
\frac{\partial}{\partial x}B_k(x, y) &= \frac{\partial}{\partial x}(\log f_k'(y)-\log f_k'(x)) \\
&= -\frac{f_k''(x)}{f_k'(x)} \\
&= -H_k(x)
\end{aligned}$$

and

$$\frac{\partial}{\partial y}B_k(x, y)=H_k(y).$$

From the arguments in Section 2, it follows that the sequences $\{\partial B_k/\partial x\}$ and $\{\partial B_k/\partial y\}$ converge uniformly on any compact domain in $(0, \infty)\times(0, \infty)$. Therefore the partial derivatives of B exist and

$$\frac{\partial B}{\partial x}=-H(x),\ \frac{\partial B}{\partial y}=H(y).$$

Hence, by Proposition (2.1), we see that B is a C^{r-1} function on $(0,\infty)\times(0,\infty)$. It follows that also $H=\exp B$ is C^{r-1} function on $(0,\infty)\times(0,\infty)$. ■

By this lemma, we can conclude the following lemma.

(3.3) LEMMA. *If $g\in \mathrm{Diff}^r[0,\infty)$ commutes f (that is, $f\circ g=g\circ f$), then the function $y=g(x)$ is the solution of the differential equation*

(A) $\dfrac{dy}{dx}=H(y, x).$

Let $y=g(x)$ be the solution of the equation (A) such that its domain of definition is maximal and define g at 0 by $g(0)=0$. Remark that the function $y=f_n(x)$ is the solution of this equation. We obtain the following lemma.

(3.4) LEMMA. *The map g is a C^1 diffeomorphism of $[0,\infty)$ which is of class C^r on $(0,\infty)$.*

PROOF. Fix $a\in(0,\infty)$ and let $b=g(a)$. If there is an integer m such that $b=f_m(a)$, then, from the uniqueness of solutions on initial conditions, it follows that $g(x)\equiv f_m(x)$. Therefore, in this case, the lemma follows clearly. Next we assume that $f_m(a)<b<f_{m-1}(a)$ for some m. Let α and β $(\alpha<\beta)$ be the end points of the domain of definition of g. Since f_{m-1} and f_m are the solutions, we see that $f_m(x)<g(x)<f_{m-1}(x)$ for $x\in(\alpha,\beta)$. Therefore, by the fact that $H(y,x)>0$, we can easily see that $\alpha=0$, $\beta=\infty$ and $\lim_{x\to+0} g(x)=0$ (by the standard argument on the domain of definition of the solution of an ordinary differential equation). Thus g is a continuous function of $[0,\infty)$ and clearly g is of class C^r on $(0,\infty)$ because H is of class C^{r-1} on $(0,\infty)\times(0,\infty)$. Moreover, since H is continuous on D_m, we have

$$\lim_{x\to+0} g'(x)=\lim_{x\to+0} H(g(x),x)=H(0,0)=1$$

Therefore g is of class C^1 at 0. Thus, since $g'(x)>0$ for all $x\in[0,\infty)$, g is a C^1 diffeomorphism of $[0,\infty)$. ■

Let $a, b\in \boldsymbol{R}^*=(0,\infty)$ and let $g_{a,b}$ be the solution of the equation (A) such that $g_{a,b}(a)=b$. Define the map $\Psi:\boldsymbol{R}^*\times\boldsymbol{R}^*\times[0,\infty)\to[0,\infty)$ by $\Psi(a,b,x)=g_{a,b}(x)$. From the theorems in ordinary differential equations on the dependence of solutions on initial conditions, it follows that Ψ is of class C^{r-1} on $\boldsymbol{R}^*\times\boldsymbol{R}^*\times(0,\infty)$. Furthermore Ψ is continuous on $\boldsymbol{R}^*\times\boldsymbol{R}^*\times[0,\infty)$. Let G be the set of all solutions of (A). Define the map

$\psi: G\to \boldsymbol{R}^*$ by $\psi(g)=g(1)$. From the uniqueness of solutions on initial conditions, it follows that ψ is bijective and $\psi^{-1}(t)=g_{1,t}$. We define the C^∞ structure of G by the map ψ. Then clearly G is diffeomorphic to $\boldsymbol{R}$.

(3.5) LEMMA. *G is a group under composition of maps. Moreover G is a Lie group and isomorphic to $\boldsymbol{R}$.*

PROOF. For g, $h\in G$ we have

$$\begin{aligned}(g\circ h)'(x)&=g'(hx)\cdot h'(x)\\ &=H(g(hx), h(x))\cdot H(h(x), x)\\ &=\prod_{j=0}^{\infty}\frac{f'(f_j(hx))}{f'(f_j(g\circ hx))}\cdot\prod_{j=0}^{\infty}\frac{f'(f_jx)}{f'(f_j(hx))}\\ &=\prod_{j=0}^{\infty}\frac{f'(f_jx)}{f'(f_j(g\circ hx))}\\ &=H(g\circ h(x), x)\end{aligned}$$

and

$$\begin{aligned}(g^{-1})'(x)&=\frac{1}{g'(g^{-1}(x))}\\ &=\frac{1}{H(g(g^{-1}(x)), g^{-1}(x))}\\ &=\prod_{j=0}^{\infty}\frac{f'(f_jx)}{f'(f_j(g^{-1}x))}\\ &=H(g^{-1}(x), x).\end{aligned}$$

It follows that $g\circ h$, $g^{-1}\in G$, that is, G is a group. Let $\rho: G\times G\to G$ be the group operation and $\gamma: G\to G$ the inversion. Then, for $(t, s)\in\boldsymbol{R}^*\times\boldsymbol{R}^*$, we have

$$\begin{aligned}\psi\circ\rho\circ(\psi^{-1}\times\psi^{-1})(t, s)&=\psi(g_{1,t}\circ g_{1,s})\\ &=g_{1,t}(g_{1,s}(1))\\ &=\Psi(1, t, s)\end{aligned}$$

and

$$\begin{aligned}\psi\circ\gamma\circ\psi^{-1}(t)&=\psi(g_{1,t}^{-1})\\ &=\psi(g_{t,1})\\ &=\Psi(t, 1, 1).\end{aligned}$$

Therefore ρ and γ are C^{r-1} maps. It follows that G is a topological group. Since G is a C^∞ manifold, by the theorems on Lie groups (cf. [**M-Z**]), we see that G is a Lie group. Hence G is isomorphic to $\boldsymbol{R}$ as a Lie group. ■

PROOF OF THEOREM (3. 1). Let $\iota: \boldsymbol{R} \to G$ be the isomorphism of Lie groups such that $\iota(1)=f$. Define the map $\varphi: \boldsymbol{R}\times[0, \infty)\to[0, \infty)$ by $\varphi(t, x)=(\iota(t))(x)$. Since

$$\varphi\circ((\iota^{-1}\circ\phi^{-1})\times id)(s, x)=g_{1,s}(x)=\Psi(1, s, x)$$

(where id denotes the identity map), we see that φ is a C^0 flow on $[0, \infty)$. Moreover, by the theorems on Lie groups (cf. [**M-Z**]), we see that φ is of class C^1 on $\boldsymbol{R}\times[0, \infty)$ because each φ_t is of class C^1 on $[0, \infty)$. Similarly φ is of class C^r on $\boldsymbol{R}\times(0, \infty)$ because each φ_t is of class C^r on $(0, \infty)$. Clearly Γ is contained in $G=\{\varphi_t\}$. This completes the proof. ■

REMARK. Also in case where $f'(0)\neq 1$ for some $f\in\Gamma$, we can prove the theorem in the similar way. But it is much simpler to use a theorem of Sternberg.

4. The quasi-invariant vector fields on the half-line

Let $f\in\mathrm{Diff}^{r+1}[0, \infty)$ $(r\geq 0)$ and let α be a positive number. We say that a vector field X on $[0, \infty)$ is *(f, α)-quasi-invariant* if $f_*X=\alpha X$. Fixing α with $0<\alpha<1$ and a contraction f such that $f'(0)=1$, we shall show that there exist many $(f, 1/\alpha)$-quasi-invariant C^r vector fields on $[0, \infty)$. We shall need such vector fields in order to describe an example of polycyclic subgroups of $\mathrm{Diff}^r[0, \infty)$. For the construction of these vector fields, we need the following fact.

(4. 1) PROPOSITION. *Let $f\in\mathrm{Diff}^{r+1}[0, \infty)$ $(r=1, \cdots, \infty)$ be a contraction and let X be a $(f, 1/\alpha)$-quasi-invariant C^0 vector field on $[0, \infty)$ where $0<\alpha<1$. If X is of class C^r on $(0, \infty)$ and $f'(0)=1$, then X is of class C^r at 0, therefore X is C^r vector field on $[0, \infty)$.*

In the sequel we assume that f is a contraction and X is a $(f, 1/\alpha)$-quasi-invariant C^0 vector field on $[0, \infty)$ which is of class C^r on $(0, \infty)$ and $0<\alpha<1$. Let

$$X(x)=F(x)\frac{\partial}{\partial x}.$$

Then clearly $F(x)$ is of class C^r on $(0, \infty)$. For the proof of Proposition (4. 1) it suffices to show that $F(x)$ is of class C^r at 0. First we calculate $F^{(p)}(f_kx)$. By the assumption on X, we easily see that

$$f_k'(x)\cdot F(x)=\frac{1}{\alpha^k}F(f_kx).$$

Therefore,

$$F(f_k x)=a^k f_k'(x)F(x).$$

This implies that

$$F'(f_k x)=\frac{a^k}{f_k'(x)}\{f_k''(x)F(x)+f_k'(x)F'(x)\}.$$

(4.2) LEMMA. *We have*

$$F^{(p)}(f_k x)=\frac{a^k}{\{f_k'(x)\}^{2p-1}}\cdot Q_{p,k}(x) \quad \textit{for any } x\in(0,\infty)$$

where $Q_{p,k}(x)$ *is a polynomial in* $f_k'(x),\dots,f_k^{(p+1)}(x)$ *and* $F(x),\dots,F^{(p)}(x)$ *such that the expression of* $Q_{p,k}(x)$ *does not depend on* k *and each its term contains at least one* $f_k^{(i)}(x)$ *(for some* $i=1,\dots,p+1$*).*

PROOF. We prove the lemma by induction on p. When $p=1$, this is the observation above. For some p we assume that the equation is true. Then we have

$$\begin{aligned} F^{(p+1)}(f_k x)&=\frac{a^k}{f_k'(x)}\left[\frac{(1-2p)f_k''(x)}{\{f_k'(x)\}^{2p}}\cdot Q_{p,k}(x)+\frac{1}{\{f_k'(x)\}^{2p-1}}\cdot Q_{p,k}'(x)\right] \\ &=\frac{a^k}{\{f_k'(x)\}^{2p+1}}\{(1-2p)f_k''(x)Q_{p,k}(x)+f_k'(x)Q_{p,k}'(x)\} \\ &\equiv\frac{a^k}{\{f_k'(x)\}^{2p+1}}Q_{p+1,k}(x). \end{aligned}$$

By the assumption of induction, we see immediately that $Q_{p+1,k}(x)$ has the desired property. This completes the proof. ■

(4.3) LEMMA. *Fix* $a\in(0,\infty)$ *and a positive integer* q. *Assume that* $f'(0)=1$. *Then the sequence* $\{a^k/\{f_k'(x)\}^q\}_{k\in N}$ *converges uniformly to* 0 *on the closed interval* $[0, a]$.

PROOF. From the assumption of the lemma, it follows that for any $\varepsilon>0$ there exists δ such that $f'(x)\geq 1-\varepsilon$ for all $x\in[0,\delta]$. Since f is a contraction, there exists a sufficiently large integer L such that if $k\geq L$ then $f_k(a)\leq\delta$. Therefore, if $k\geq L$, then $f'(f_k x)\geq 1-\varepsilon$ for all $x\in[0, a]$. Thus we have

$$\begin{aligned} f_k'(x)&=f'(f_{k-1}x)\cdots f'(f_L x)\cdot f'(f_{L-1}x)\cdots f'(x) \\ &\geq(1-\varepsilon)^{k-L}C^L \end{aligned}$$

for all $x\in[0, a]$ where $C=\min\{f'(x)|x\in[0, a]\}$. It follows that

$$0\leq\frac{\alpha^k}{\{f_k'(x)\}^q}\leq\frac{\alpha^k}{(1-\varepsilon)^{q(k-L)}C^{qL}}$$
$$=\frac{(1-\varepsilon)^{qL}}{C^{qL}}\left\{\frac{\alpha}{(1-\varepsilon)^q}\right\}^k.$$

Now we choose ε sufficiently small such that $0<\frac{\alpha}{(1-\varepsilon)^q}<1$ for $0<\alpha<1$. And for this ε we choose such δ and L as above. Then, by the inequality above, we see that $\{\alpha^k/\{f_k'(x)\}^q\}$ converges uniformly to 0 on $[0, a]$. ■

PROOF OF PROPOSITION (4.1). By Lemma (4.2), we have

$$F^{(p)}(f_kx)=\frac{\alpha^k}{\{f_k'(x)\}^{2p-1}}Q_{p,k}(x).$$

Fix p and $a\in(0,\infty)$. By the property of $Q_{p,k}(x)$ and Corollary (2.6), we see that there exists C_p such that

$$|Q_{p,k}(x)|\leq C_pf_k'(x)$$

for all $x\in[a_1, a_0]$ and $k\geq L$, where L is chosen so that if $k\geq L$ then $f_k'(x)\leq 1$ for all $x\in[a_1, a_0]$. It follows that

$$|F^{(p)}(f_kx)|\leq\frac{C_p\alpha^k}{\{f_k'(x)\}^{2(p-1)}}$$

for all $x\in[a_1, a_0]$. Therefore, from Lemma (4.3), the sequence of functions $\{F^{(p)}(f_kx)\}_{k\in N}$ converges uniformly to 0 on $[a_1, a_0]$. This implies that

$$\lim_{x\to+0}F^{(p)}(x)=0$$

for each p $(p=1,\cdots,r)$. By using de l'Hopital's theorem, we see that $F(x)$ is of class C^r at 0. This completes the proof. ■

From Proposition (4.1) we obtain the following result, which is the purpose of this section.

(4.4) THEOREM. *Let $f\in\mathrm{Diff}^{r+1}[0,\infty)$ $(r=1,2,\dots,\infty)$ be a contraction such that $f'(0)=1$ and let $0<\alpha<1$. Then there exist non-trivial $(f, 1/\alpha)$-quasi-invariant C^r vector fields on $[0,\infty)$.*

PROOF. There exists a C^r function F on $(0,\infty)$ such that

$$f'(x)\cdot F(x)=\frac{1}{\alpha}F(fx).$$

Indeed, fixing $a\in(0,\infty)$, we can take a function F_0 on $[a_1, a_0]$ (where $a_1=f(a)$, $a_0=a$) such that $F_0(a_1)=\alpha f'(a_0)F_0(a_0)$. Then define F by $F(x)=\alpha^k f_k'(f_k^{-1}x)F_0(f_k^{-1}x)$ for $x\in[a_{k+1}, a_k]$ (where $a_k=f_k(a_0)$). It follows easily that F is well-defined and satisfies the equation above. Reforming F_0 in neighborhoods of a_1 and a_0, we can make the function F of class C^r. We see that $F(x)$ extends a continuous function on $[0,\infty)$ by defining $F(0)=0$. Define $X(x)=F(x)\cdot\partial/\partial x$. Then, clearly X is $(f, 1/\alpha)$-quasi-invariant vector field on $[0,\infty)$ and from Proposition (4.1) it follows that X is of class C^r on $[0,\infty)$. ∎

REMARK. We must mention whether the vector field X in the above proof is complete or not, for we need a complete one to construct an example of polycyclic groups in the next section. It is sufficient for our purpose to notice that if $X(x)=0$ for some $x\in(0,\infty)$, then X is complete.

5. Examples of polycyclic groups of diffeomorphisms

We describe examples of different two types of polycyclic groups of diffeomorphisms on the half-line. First one is quoted from [**P2**]. Second one needs the result in the previous section.

EXAMPLE 1. Denote by $\mathrm{Aff}^+(\boldsymbol{R})$ the subgroup of $\mathrm{Diff}^\infty(\boldsymbol{R})$ consisting of orientation preserving affine maps ($x\mapsto ax+b$ for some $a>0$ and $b\in\boldsymbol{R}$). Then $\mathrm{Aff}^+(\boldsymbol{R})$ admits Lie group structure of dimension 2 and the natural action $\Psi:\mathrm{Aff}^+(\boldsymbol{R})\times\boldsymbol{R}\to\boldsymbol{R}$ is real-analytic. We denote this Lie group by $\mathscr{A}$. Let $\phi:\boldsymbol{R}\to\boldsymbol{R}^+(=(0,\infty))$ be a C^∞ diffeomorphism such that

$$\phi(t)=\begin{cases}\dfrac{1}{1-t} & (-\infty<t\leq 0)\\ \textit{some smooth function} & (0<t\leq 2)\\ t & (2<t<\infty).\end{cases}$$

Then there is the action Φ of $\mathscr{A}$ on $\boldsymbol{R}^+$ which is induced from Ψ by ϕ. Moreover this induced action Φ extends on $[0,\infty)$ by defining $\Phi(g,0)=0$ for $g\in\mathscr{A}$. It follows immediately that Φ is a C^∞ action on $[0,\infty)$. We denote by T_g the diffeomorphism of $[0,\infty)$ defined by $x\mapsto\Phi(g,x)$. Define the subgroup G of $\mathrm{Diff}^\infty[0,\infty)$ by $G=\{T_g|g\in\mathscr{A}\}$. Clearly G is isomorphic to $\mathscr{A}$. Since $\mathscr{A}$ has many polycyclic subgroups, so does G. We notice the following fact.

(5.1) LEMMA. *Let Γ be a polycyclic subgroup of G. Then the following hold.*

(i) *Γ is isomorphic to a semi-direct product $\boldsymbol{Z}^n\rtimes\boldsymbol{Z}^k$ of $\boldsymbol{Z}^n$ and $\boldsymbol{Z}^k$*

for some n, k.

(ii) *The orbit of the natural action of* Γ *on* $[0, \infty)$ *at* $x\in(0, \infty)$ *is dense in* $[0, \infty)$ *if* $n>1$.

EXAMPLE 2. Let $A\in SL(n, \boldsymbol{Z})$ and let $0<\alpha<1$. Assume that α is an eigenvalue of ${}^tA$ (the transposed matrix of A) and take a corresponding eigenvector ${}^t(a_1, \dots, a_n)$ where $a_i\in\boldsymbol{R}$ $(i=1, \dots, n)$. Let $f\in \mathrm{Diff}^{r+1}[0, \infty)$ $(r=2, \dots, \infty)$ be a contraction such that $f'(0)=1$. Then, from Theorem (4. 4), we can take a non-trivial $(f, 1/\alpha)$-quasi-invariant C^r vector field X such that $X(x)=0$ for some $x\in(0, \infty)$. Since X is complete, we have the C^r flow $\Phi: \boldsymbol{R}\times[0, \infty)\to[0, \infty)$ associated by X. Clearly $f\circ\Phi_t\circ f^{-1}=\Phi_{\frac{1}{\alpha}t}$, that is,

$$f^{-1}\circ\Phi_t\circ f=\Phi_{\alpha t}.$$

From these data A, α, ${}^t(a_1, \dots, a_n)$, f and Φ, we construct a polycyclic subgroup Γ_A of $\mathrm{Diff}^r[0, \infty)$ as follows.

Let $A=(m_{ij})$, $m_{ij}\in\boldsymbol{Z}$ and let $g_i=\Phi_{a_i}$ for $i=1, \dots, n$. Since ${}^t(a_1, \dots, a_n)$ is an eigenvector of ${}^tA$ in respect to eigenvalue α, we have

$$\sum_{j=1}^{n} m_{ji}a_j=\alpha a_i.$$

Therefore

$$\begin{aligned} f^{-1}\circ g_i\circ f &=\Phi_{\alpha a_i} \\ &=\Phi_{\Sigma m_{ji}a_j} \\ &=g_1^{m_{1i}}\circ\cdots\circ g_j^{m_{ji}}\circ\cdots\circ g_n^{m_{ni}}. \end{aligned}$$

Denote by N the subgroup generated by $g_1, \dots, g_n$ and let Γ_A be the subgroup generated by $g_1, \dots, g_n$ and f. Clearly N is a free abelian group and the normal subgroup of Γ_A. It follows that Γ_A is a polycyclic subgroup of rank$\leq n+1$ and N is the nilradical of Γ_A. We notice the following fact.

(5. 2) LEMMA. (i) Γ_A *is isomorphic to a semi-direct product* $\boldsymbol{Z}^m\rtimes\boldsymbol{Z}$ *for some* m.

(ii) *Let* $p\in(0, \infty)$ *be a point such that* $X(p)=0$. *Then the orbit through* p *of the natural action of* Γ_A *on* $[0, \infty)$ *is discrete in* $(0, \infty)$.

Lemmas (5. 1) and (5. 2) show the difference between Example 1 and Example 2.

6. Main theorem

Let Γ be a polycyclic subgroup of $\mathrm{Diff}^r[0,\infty)$ $(r=2,\dots,\infty)$ and let N be the nilradical of Γ. It is well known that N is a free abelian group (e.g., cf. [**P-T**]). Define $\Gamma_*=\Gamma|_{(0,\infty)}=\{f|_{(0,\infty)}\in\mathrm{Diff}^r(0,\infty)|f\in\Gamma\}$. We say that Γ_* is C^s *conjugate* to a subgroup of $\mathrm{Aff}^+(\boldsymbol{R})$ $(s\leq r)$ if there exists a C^s diffeomorphism $h:(0,\infty)\to\boldsymbol{R}$ such that $h\Gamma_*h^{-1}=\{h\circ f\circ h^{-1}\in\mathrm{Diff}^s(\boldsymbol{R})|f\in\Gamma_*\}$ is contained in $\mathrm{Aff}^+(\boldsymbol{R})$. It has been shown in [**P2**] that if $\mathrm{Fix}(N)=\{0\}$, then Γ_* is C^0 conjugate to a subgroup of $\mathrm{Aff}^+(\boldsymbol{R})$. In this section, we prove the following theorem, which is the purpose of this paper.

(6.1) THEOREM. *Let* Γ *be a polycyclic subgroup of* $\mathrm{Diff}^r[0,\infty)$ *and let* $r=2,\dots,\infty$. *Assume that* $\mathrm{Fix}(\Gamma)=\{0\}$. *Then the following hold.*

(i) *If* $\mathrm{Fix}(N)=\{0\}$, *then* $\Gamma|_{(0,\infty)}$ *is* C^r *conjugate to a subgroup of* $\mathrm{Aff}^+(\boldsymbol{R})$.

(ii) *If* $\mathrm{Fix}(N)\neq\{0\}$, *then there exists a contraction* $f\in\mathrm{Diff}^r[0,\infty)$ *such that* Γ *is isomorphic to a semi-direct product* $N\rtimes Z_f$ *of* N *and* Z_f *where* Z_f *denotes the infinite cyclic group generated by* f.

PROOF OF (i). Assume that $\mathrm{Fix}(N)=\{0\}$. Since N is abelian, from Theorem (3.1), it follows that there exists a C^1 flow φ on $[0,\infty)$ which is of class C^r on $(0,\infty)$ such that $N\subset\{\varphi_t\}$. We define the map $h:(0,\infty)\to\boldsymbol{R}$ by $h^{-1}(t)=\varphi(t,1)$. Since the flow φ has no fixed point in $(0,\infty)$, h is a well-defined C^r diffeomorphism. Denote by τ_b the translation of $\boldsymbol{R}$ by b $(t\mapsto t+b)$. Then, since for any $g\in N$ there exists b such that $g=\varphi_b$, we obtain that

$$h\circ g\circ h^{-1}(t)=h\circ\varphi_b(\varphi(t,1))=b+t=\tau_b(t).$$

That is, for any $g\in N$, the map $h\circ g\circ h^{-1}$ is a translation of $\boldsymbol{R}$. If $\mathrm{rank}(N)=1$, it follows easily that $\Gamma=N$. Then the assertion (i) of the theorem follows clearly.

Next we assume that $\mathrm{rank}(N)\geq 2$. Let $\tilde{N}=hN_*h^{-1}$ and let $B=\{b\in\boldsymbol{R}|\tau_b\in\tilde{N}\}$. It is well known that B is a dense subset of $\boldsymbol{R}$. For $f\in\Gamma\backslash N$, let $\tilde{f}=h\circ f\circ h^{-1}$. Since $\tilde{N}$ is a normal subgroup of $h\Gamma_*h^{-1}$, for any $\tau_b\in\tilde{N}$, there exists $\tau_c\in\tilde{N}$ such that $\tilde{f}\circ\tau_b=\tau_c\circ\tilde{f}$. Differentiating the both sides of the equation above, we have $\tilde{f}'(t+b)=\tilde{f}'(t)$. Applying $t=0$, we obtain $\tilde{f}'(b)=\tilde{f}'(0)$ for any $b\in B$. Since B is dense in $\boldsymbol{R}$ and $\tilde{f}'$ is continuous, $\tilde{f}'(t)$ is identically equal to $\tilde{f}'(0)$. That is, $\tilde{f}$ is an affine map. This completes the proof of (i) of the theorem. ■

For the proof of (ii) of the theorem, we assume that $\mathrm{Fix}(N)\neq\{0\}$ in the sequel. By Proposition (1.3), we can take a normal subgroup Γ_0 of finite index in Γ such that $N \subset \Gamma_0$ and $[\Gamma_0, \Gamma_0] \subset N$. From Propositon (1.2), it follows that N is also the nilradical of Γ_0. Notice that for each $f \in \Gamma_0$ the subset $\mathrm{Fix}(N)$ is f-invariant. First we shall prove (ii) for Γ_0 and next we shall prove (ii) for Γ.

The preliminary step of the proof of (ii) for Γ_0 is to show that $\mathrm{Fix}(f)=\{0\}$ for each $f \in \Gamma_0 \backslash N$.

Fix $f \in \Gamma_0 \backslash N$ and let $p \in \mathrm{Fix}(f)$. Then the following three cases are considered:

Case(a) $p \in \mathrm{Fix}(N)$ and there exists a sequence $\{p_n\}$ converging to p such that $p_n \in \mathrm{Fix}(N)$ and $p_n \notin \mathrm{Fix}(f)$.

Case(b) $p \in \mathrm{Fix}(N)$ but there exist no such sequences as in Case(a).

Case(c) $p \notin \mathrm{Fix}(N)$.

We shall show that $\mathrm{Fix}(f)=\{0\}$. To prove this fact, we prepare the following lemmas.

(6.2) LEMMA. *Let* $p \in \mathrm{Fix}(f)$ *in Case*(a). *Then* $p \in \mathrm{Fix}(g)$ *for any* $g \in \Gamma_0 \backslash N$.

PROOF. Suppose on the contrary that $p \notin \mathrm{Fix}(g)$. There exists an interval (q, q_1) containing p such that $q \in \mathrm{Fix}(g)$, $(q, q_1) \cap \mathrm{Fix}(g)=\emptyset$ and $q_1 \in \mathrm{Fix}(g)$ or $q_1=\infty$. We assume that g is a contraction of $[q, q_1)$ because we may replace g with g^{-1} if necessary. Then clearly $\lim_{n \to \infty} g_n(p)=q$ (where g_n is the n-times iteration of g). Since $p \in \mathrm{Fix}(N) \cap \mathrm{Fix}(f)$ and Γ_0/N is abelian, it follows easily that $g_n(p) \in \mathrm{Fix}(N) \cap \mathrm{Fix}(f)$. Therefore we see that $q \in \mathrm{Fix}(N) \cap \mathrm{Fix}(f)$ because $\mathrm{Fix}(N) \cap \mathrm{Fix}(f)$ is a closed set. For each positive integer n, there exists $h_n \in N$ such that $g_n^{-1} \circ f^{-1} \circ g_n = f^{-1} \circ h_n$. We consider g and f^{-1} to be restricted on the interval $[q, q_1)$. Then, applying Lemma (1.7) to p and g, $f^{-1} \in \mathrm{Diff}^r[q, q_1)$, we see that the sequence $\{f^{-1} \circ h_n\}_{n \in N}$ converges uniformly to id on $[p_1, p]$ (where $p_1 = g(p)$). In other words, $\{h_n\}$ converges uniformly to f on $[p_1, p]$. On the other hand, from the assumption of the lemma, there exists a point $p' \in [p_1, p]$ such that $p' \in \mathrm{Fix}(N)$ but $p' \notin \mathrm{Fix}(f)$. Therefore $\{h_n\}$ $(\subset N)$ can not converge to f at p'. This is a contradiction. Hence $p \in \mathrm{Fix}(g)$. ■

Lemma (6.2) implies the next lemma.

(6.3) LEMMA. *If there exists* $p \in \mathrm{Fix}(f)$ *in Case*(*a*), *then* $p=0$ *and* $\mathrm{Fix}(f)=\{0\}$.

PROOF. From Lemma (6.2) it follows that $p \in \mathrm{Fix}(\Gamma_0)$. Since

$\mathrm{Fix}(\Gamma)=\{0\}$, clearly $\mathrm{Fix}(\Gamma_0)=\{0\}$. Thus we have $p=0$.

Fix a point $q_0\in \mathrm{Fix}(N)$ such that $q_0\neq 0$ and $q_0\notin \mathrm{Fix}(f)$. This q_0 exists clearly from the assumption of the lemma. Let $p_M=\max\{x\in \mathrm{Fix}(f)|0<x<q_0\}$ and $p_m=\min\{x\in \mathrm{Fix}(f)|x>q_0\}$. Now we assume that $\mathrm{Fix}(f)\neq\{0\}$. Then either p_M or p_m exists. First suppose that p_M exists. Clearly $0<p_M<q_0$ and $(p_M, q_0]\cap \mathrm{Fix}(f)=\emptyset$. Furthermore $\{f_n(q_0)\}_{n\in Z}$ is contained in $\mathrm{Fix}(N)$ and $\lim_{n\to\infty} f_n(q_0)=p_M$ or $\lim_{n\to-\infty} f_n(q_0)=p_M$. That is, p_M is such a point as in Case(a). Therefore, by the first assertion of this lemma, we see that $p_M=0$. This contradicts $p_M>0$. Similarly, supposing that p_m exists leads to a contradiction. Hence $\mathrm{Fix}(f)=\{0\}$. ■

(6.4) LEMMA. *Let $p\in \mathrm{Fix}(f)$ in Case(c), that is, $p\notin \mathrm{Fix}(N)$. Then $p\in \mathrm{int\,Fix}(f)$.*

PROOF. Suppose that $p\notin \mathrm{int\,Fix}(f)$. From the assumption of the lemma, $p\neq 0$. Therefore there exists an open interval (q, q_0) containing p such that $(q, q_0)\cap \mathrm{Fix}(N)=\emptyset$, $q\in \mathrm{Fix}(N)$ and $q_0\in \mathrm{Fix}(N)$ or $q_0=\infty$. Notice that $q\in \mathrm{Fix}(f)$. In fact, if $q\notin \mathrm{Fix}(f)$, then $\{f_n(q)|n\in \boldsymbol{Z}\}\cap(q, q_0)\neq\emptyset$. Since $\{f_n(q)\}\subset \mathrm{Fix}(N)$, we see that $(q, q_0)\cap \mathrm{Fix}(N)\neq\emptyset$. This contradicts the choice of (q, q_0). Hence $q\in \mathrm{Fix}(f)$. Moreover we have the following fact.

CLAIM. $q\in \mathrm{Fix}(g)$ for any $g\in\Gamma_0\backslash N$.

PROOF. Suppose that $q\notin \mathrm{Fix}(g)$. Then neither $g^{-1}(q)$ nor $g(q)$ can be contained in (q, q_0) because $g^{-1}(q)$, $g(q)\in \mathrm{Fix}(N)$. Since $q_0\leq g^{-1}(q)<\infty$ or $q_0\leq g(q)<\infty$, we have $q_0<\infty$ and $q_0\in \mathrm{Fix}(N)$. Therefore, by the same argument as above, we see that $q_0\in \mathrm{Fix}(f)$. And there exists the interval (u, v) which contains $[q, q_0]$ such that $(u, v)\cap \mathrm{Fix}(g)=\emptyset$, $u\in \mathrm{Fix}(g)$ and $v\in \mathrm{Fix}(g)$ or $v=\infty$. Without loss of generality, we assume that g is a contraction of $[u, v)$. Then, in the same way as in the proof of Lemma (6.2), (applying Lemma (1.7) to these g, f and q_0), we see that there exists a sequence $\{h_n\}\subset N$ such that $\{h_n\}$ converges uniformly to f on $[g(q_0), q_0]$. Remarking that $g(q_0)\leq q$, we see that $\{h_n\}$ converges uniformly to f on $[q, q_0]$. Restrict N on $[q, q_0)$. Since $(q, q_0)\cap \mathrm{Fix}(N)=\emptyset$, we can apply the proof of (i) of Theorem (6.1) to N. By this observation, we see that N is conjugate to a subgroup of translation group of $\boldsymbol{R}$. Therefore $f=\lim h_n$ must be the identity on $[q, q_0)$ because $\emptyset\neq \mathrm{Fix}(f)\cap(q, q_0)\ni p$. That is, $p\in(q, q_0)\subset \mathrm{Fix}(f)$. This contradicts our assumption that $p\notin \mathrm{int\,Fix}(f)$. Therefore $q\in \mathrm{Fix}(g)$. This completes the proof of Claim.

The above claim implies that $q\in \mathrm{Fix}(\Gamma_0)$. Since $\mathrm{Fix}(\Gamma_0)=\{0\}$, it follows that $q=0$. Therefore, we see that $q_0<\infty$ and $q_0\in \mathrm{Fix}(N)$ by the assumption $\mathrm{Fix}(N)\neq\{0\}$ and the choice of (q, q_0). Moreover $q_0\in\mathrm{Fix}(g)$ for any $g\in\Gamma_0\backslash N$. Indeed, if $q_0\notin\mathrm{Fix}(g)$, then $\emptyset\neq\{g_n(q_0)|n\in\boldsymbol{Z}\}\cap(q, q_0)\subset \mathrm{Fix}(N)\cap(q, q_0)$. This contradicts the choice of (q, q_0). Thus $q_0\in\mathrm{Fix}(g)$ for any $g\in\Gamma_0\backslash N$. Therefore $q_0\in\mathrm{Fix}(\Gamma_0)=\{0\}$. This contradicts the choice of q_0. Hence we conclude that $p\in\mathrm{int}\ \mathrm{Fix}(f)$. ■

Now we can show the following.

(6.5) LEMMA. $\mathrm{Fix}(f)=\{0\}$ *for any* $f\in\Gamma_0\backslash N$.

PROOF. Suppose that $\mathrm{Fix}(f)\neq\{0\}$ for $f\in\Gamma_0\backslash N$. By Lemma (6.3) and (6.4), we observe that if $p\in\mathrm{Fix}(f)\backslash\{0\}$ and $p\notin\mathrm{int}\ \mathrm{Fix}(f)$, then p is a point in Case(b), that is, $p\in\mathrm{Fix}(N)$ and there is no sequence such as $\{p_n\}$ where $\lim p_n=p$, $p_n\in\mathrm{Fix}(N)$ but $p_n\notin\mathrm{Fix}(f)$. Let $J=(a, b)$ be a component of $(0,\infty)\backslash\mathrm{Fix}(f)$. Then the above observation implies that $a\in\mathrm{Fix}(f)$ and $a\in\mathrm{Fix}(N)$, and if $b<\infty$, then $b\in\mathrm{Fix}(f)$ and $b\in\mathrm{Fix}(N)$. And $J\cap \mathrm{Fix}(N)=\emptyset$. In fact, if $q\in J\cap\mathrm{Fix}(N)\neq\emptyset$, then a and b (if $b<\infty$) are adherent points of the set $\{f_n(q)|n\in\boldsymbol{Z}\}$ ($\subset\mathrm{Fix}(N)$). That is, a and b (if $b<\infty$) are points in Case(a). Therefore, from Lemma (6.3), it follows that $a=0$ and $b=\infty$, contradicting the assumption that $\mathrm{Fix}(f)\neq\{0\}$. Thus we see that $J\cap\mathrm{Fix}(N)=\emptyset$. Now we have the following.

CLAIM. For any $h\in N$, $f\circ h=h\circ f$ on J.

PROOF. Denote by Γ_f the subgroup of Γ_0 generated by f and N. We restrict Γ_f to $[a, b)$, and denote by Γ_f^* the restriction of Γ_f to (a, b). Since $\mathrm{Fix}(N)\cap[a, b)=a$ and clearly $\mathrm{Fix}(\Gamma_f)\cap[a, b)=\{a\}$, by applying (i) of the theorem, we see that Γ_f^* is C^r conjugate to a subgroup of $\mathrm{Aff}^+(\boldsymbol{R})$. The conjugation maps N into the translation group since N is abelian and $\mathrm{Fix}(N)\cap(a, b)=\emptyset$. Furthermore $f|_{(a,b)}$ is also conjugate to a translation because $\mathrm{Fix}(f)\cap(a, b)=\emptyset$. Therefore Γ_f^* must be conjugate to a subgroup of the translation group. It follows that Γ_f^* is abelian. This completes the proof of Claim.

This implies that $f\circ h=h\circ f$ on $[0,\infty)$ for any $h\in N$. That is, Γ_f is an abelian group. And Γ_f is a normal subgroup of Γ_0 because $\Gamma_f\supset N\supset [\Gamma_0,\Gamma_0]$. This contradicts the maximality of N. Therefore $\mathrm{Fix}(f)=\{0\}$. ■

The following lemma completes the proof of (ii) for Γ_0.

(6.6) LEMMA. Γ_0/N *is a free abelian group of rank* 1 *and has a generator which is represented by a contraction.*

PROOF. On the contrary, assume that Γ_0/N is of rank ≥ 2. Then there exist contractions f, $g\in\Gamma_0$ such that the subgroup of Γ_0/N generated by the elements which are represented by f and g is of rank 2. Let Λ be the subgroup of Γ_0 generated by f and g. Notice that $\mathrm{Fix}(N)$ is unbounded since $\mathrm{Fix}(N)$ is f-invariant and $\mathrm{Fix}(N)\neq\{0\}$. Notice that $\mathrm{Fix}(N)\neq[0,\infty)$. (Otherwise $N=\{id\}$ and $\Gamma_0\cong\Gamma_0/N$ is abelian, which contradicts the maximality of N.) Let $a\in\mathrm{Fix}(N)$ $(a\neq 0)$ be the upper endpoint of a certain component of $[0,\infty)\backslash\mathrm{Fix}(N)$. We consider the orbit through a of the natural action of Λ on $[0,\infty)$. Denote this orbit by $\mathcal{O}$ and denote its closure by $\bar{\mathcal{O}}$. Since $\bar{\mathcal{O}}\subset\mathrm{Fix}(N)$, there exists $c\in\bar{\mathcal{O}}$ such that $(c,a)\cap\bar{\mathcal{O}}=\emptyset$. Letting $J=[c,a]$, we see that if h_1, $h_2\in\Lambda$ and $h_1(a)\neq h_2(a)$, then $\mathrm{int}(h_1(J)\cap h_2(J))=\emptyset$. Furthermore we see that $c\notin\mathcal{O}$. Indeed, if $c\in\mathcal{O}$ and $c=h(a)$ for some $h\in\Lambda$, then $\mathcal{O}=\{h^n(a)|n\in\boldsymbol{Z}\}=\bar{\mathcal{O}}\cap(0,\infty)$. It follows that $f(a)=h^i(a)$ for some $i\in\boldsymbol{Z}$. Since $h^{-i}\circ f(a)=a\neq 0$, by Lemma (6.5), we have that $h^{-i}\circ f\in N$. In a similar way, we obtain that $h^{-j}\circ g\in N$ for some j. These imply that the subgroup of Γ_0/N generated by the elements which are represented by f and g is generated by the element which is represented by h. This contradicts the choice of f and g. Therefore, we have $c\notin\mathcal{O}$.

It follows that there exists a strictly increasing sequence $\{a_m\}\subset\mathcal{O}$ such that $a_m<c$ and $\lim_{m\to\infty}a_m=c$. For each m, there exist m_1, $m_2\in\boldsymbol{Z}$ such that $a_m=f_{m_1}\circ g_{m_2}(a)$. Since $a_m<a$, we see that $m_1>0$ or $m_2>0$. Replacing $\{a_m\}$ with its subsequence if necessary, we can assume that $m_2>0$ without loss of generality. Let $h_m=f_{m_1}\circ g_{m_2}$ and $J_m=h_m(J)$. Since the points $a_m=h_m(a)$ are distinct, the intervals $h_m(J)$ are disjoint. Applying Lemma (1.9), we see that there exist α, $\nu>0$ such that

$$0<h_m'(z)\leq\alpha|J_m|$$

for all $z\in\hat{J}$ (where $|J_m|$ denotes the length of J_m and $\hat{J}$ denotes the ν-neighborhood of J). Since $\lim_{m\to\infty}|J_m|=0$, for sufficiently large m, $h_m'(z)$ can be made arbitrarily small uniformly for all $z\in\hat{J}$. Therefore, since $\lim_{m\to\infty}a_m=c$ and $c\in\hat{J}$, for sufficiently large m we have

$$h_m(\hat{J})\subset\hat{J}.$$

It follows that h_m has a fixed point in $\hat{J}$ (cf. [**Sa**, p. 83]). By Lemma (6.5) we see that $h_m=f_{m_1}\circ g_{m_2}\in N$. This contradicts the choice of f and g. Thus we conclude that Γ_0/N is of rank 1.

Next we show that Γ_0/N is free abelian. Since $\mathrm{Fix}(f)=\{0\}$ for any $f\in\Gamma_0\backslash N$ and $\mathrm{Fix}(f_n)=\mathrm{Fix}(f)$, we see that $\mathrm{Fix}(f_n)=\{0\}$ for all $n\in\boldsymbol{Z}$. Therefore $f_n\notin N$ for any $f\in\Gamma_0\backslash N$ and $n\in\boldsymbol{Z}$. It follows Γ_0/N has no element of finite order. Thus Γ_0/N is free abelian of rank 1 and therefore, is generated by only one generater which is represented by some f such that $\mathrm{Fix}(f)=\{0\}$. Since f or f^{-1} is clearly a contraction, the lemma follows. ■

PROOF OF (ii) OF THEOREM (6.1). We show that the same assertion as the lemma above holds for Γ/N. Since Γ/Γ_0 is polycyclic and of finite order, there exists a sequence of subgroups

$$\Gamma=\Gamma^k\supset\Gamma^{k-1}\supset\cdots\supset\Gamma^0=\Gamma_0$$

such that for each $i=1,\ldots,k$, Γ^{i-1} is normal in Γ^i and Γ^i/Γ^{i-1} is a finite cyclic group. By induction, we show that the same assertion as Lemma (6.6) for each Γ^i is valid. For $i=0$, this is Lemma (6.6). Assume that the assertion is valid for some i. That is, Γ^i/N is infinite cyclic group and generated by an element $\hat{g}$ which is represented by a contraction g. Since Γ^i/N is normal in Γ^{i+1}/N, it follows that for each $\hat{f}\in\Gamma^{i+1}/N$, $\hat{f}\circ\hat{g}\circ\hat{f}^{-1}=\hat{g}$ or $\hat{g}^{-1}$. But each $f\in\mathrm{Diff}^r[0,\infty)$ preserves the order of points in $[0,\infty)$. Therefore $\hat{f}\circ\hat{g}\circ\hat{f}^{-1}=\hat{g}$ for each $\hat{f}\in\Gamma^{i+1}/N$, which means that Γ^{i+1}/N is abelian. By Proposition (1.3) we see that N is also the nilradical of Γ^{i+1}. Since we can apply the same argument as Γ_0 to Γ^{i+1}, by Lemma (6.6), we conclude that the assertion for Γ^{i+1} is valid. This completes the proof. ■

REMARK. By Lemma (5.1) and Theorem (6.1), we notice that every polycyclic subgroup of $\mathrm{Diff}^r[0,\infty)$ $(r\geq 2)$ is strongly polycyclic.

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