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Sheaf cohomology theory for measurable spaces

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1. Introduction

Sheaf theory was mainly applied to topology, differential geometry, algebraic geometry, and so on (see G. E. BREDON [1], R. G. SWAN [2] and J. DIEUDONNÉ [7]). For example in topology, it is very useful in proving theorems such as the duality theorems of POINCARÉ, ALEXANDER, and LEFSCHETZ. It is important when we want to obtain global properties from local ones. It takes more effect when combined with homological algebra, especially cohomology theory. For instance, cohomology theory is used in defining characteristic classes and cohomology vanishing theorem is useful in calculating them.

The aim in this paper is to develop a sheaf cohomology theory for a measurable space $(\Omega, \mathfrak{A})$. To each σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ we associate an abelian group $\mathscr{F}(\mathfrak{B})$ and call the system of them *σ-sheaf* $\mathscr{F}$ over $(\Omega, \mathfrak{A})$. We formulate cohomology group with coefficients in it. We treat mainly a cohomology group with coefficients in a σ-sheaf of measurable transformation group or automorphism on $(\Omega, \mathfrak{A})$. It gives certain relation between the local characteristics and the global ones of transformation group on $(\Omega, \mathfrak{A})$. We show cohomology vanishing theorems with respect to it.

We state a summary of each section below.

In Section 2, we give the definition of σ-sheaf which plays a role of describing the local-global interplay. We construct two kinds of σ-sheaves: one is a σ-sheaf of measurable transformation group and the other is a σ-sheaf of integrable functions over a finite measure space.

Section 3 constructs a cohomology group with coefficients in a σ-sheaf $\mathscr{F}$ in the similar way to the construction of the Čech cohomology for topological space. To irrustlate cohomology group, we regard the σ-algebra $\mathfrak{A}$ and σ-subalgebras of $\mathfrak{A}$ as the domain and its sub-domains, respectively. By a *σ-covering* over $(\Omega, \mathfrak{A})$, we mean a collection of σ-subalge-

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bras $\mathfrak{A}_i$ of $\mathfrak{A}$ such that $\mathfrak{A}=\bigvee_{i\in I}\mathfrak{A}_i$. The cohomology group $H^q(\{\mathfrak{A}_i\}, \mathscr{F})$ with respect to it gives the degree of variety of the difference between the abelian groups $\mathscr{F}(\mathfrak{A}_{i0}\cap\cdots\cap\mathfrak{A}_{iq})$ and $\mathscr{F}(\mathfrak{A}_{i0}\cap\cdots\cap\mathfrak{A}_{i_{k-1}}\cap\mathfrak{A}_{i_{k+1}}\cap\cdots\cap\mathfrak{A}_{iq})$, corresponding to domain $\mathfrak{A}_{i0}\cap\cdots\cap\mathfrak{A}_{iq}$ and to its neighbor domain $\mathfrak{A}_{i0}\cap\cdots\cap\mathfrak{A}_{i_{k-1}}\cap\mathfrak{A}_{i_{k+1}}\cap\cdots\cap\mathfrak{A}_{iq}$, respectively.

In Section 4, we investigate a cohomology group with coefficients in a σ-sheaf of an automorphism $\phi:\Omega\to\Omega$. We describe the local recurrence of it by the cohomology group. The local recurrence means the occurrence that there exists an integer m such that $\phi^m A=A$ holds for every $A\in\mathfrak{A}_{i0}\cap\cdots\cap\mathfrak{A}_{iq}$. In terms of σ-sheaf, this corresponds to $\phi^m\in\mathscr{F}(\mathfrak{A}_{i0}\cap\cdots\cap\mathfrak{A}_{iq})$, where ϕ is regarded as a set map $\mathfrak{A}\to\mathfrak{A}$. We study in Theorem 4.1 the difference between the abelian groups $\mathscr{F}(\mathfrak{A}_{i0}\cap\cdots\cap\mathfrak{A}_{iq})$ and $\mathscr{F}(\mathfrak{A}_{i0}\cap\cdots\cap\mathfrak{A}_{i_{k-1}}\cap\mathfrak{A}_{i_{k+1}}\cap\cdots\cap\mathfrak{A}_{iq})$, namely, whether $\phi^m A\neq A$ for some $A\in\mathfrak{A}_{i0}\cap\cdots\cap\mathfrak{A}_{i_{k-1}}\cap\mathfrak{A}_{i_{k+1}}\cap\cdots\cap\mathfrak{A}_{iq}$ or not. It is shown that the existence of the nonzero element of cohomology group corresponds to the difference. We state the structure of the 0-th cohomology group H^0 in Theorem 4.2 and give an automorphism, named *increasing automorphism*, that the cohomology groups become zero in Theorem 4.6 and Theorem 4.7.

As the cohomology group indicates the degree of difference of local recurrence of automorphism among local parts of a given space, it will enable us to classify automorphisms on measurable spaces or measure spaces by the cohomology group.

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2. σ-Sheaf

In this section we give the definition of σ-sheaf and some examples of it.

Let $(\Omega,\mathfrak{A})$ be a measurable space. A *σ-sheaf* $\mathscr{F}$ over $(\Omega,\mathfrak{A})$ is a collection $\{\mathscr{F}(\mathfrak{B}), \rho_{\mathfrak{C},\mathfrak{B}}\}$ of abelian groups and homomorphisms which satisfies the following conditions (C1) and (C2).

(C1) An abelian group $\mathscr{F}(\mathfrak{B})$ corresponds to each σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$.

(C2) For any two σ-subalgebras $\mathfrak{B}$ and $\mathfrak{C}$ with $\mathfrak{C}\subset\mathfrak{B}$, a group homomorphism

$$\rho_{\mathfrak{C},\mathfrak{B}}:\mathscr{F}(\mathfrak{B})\to\mathscr{F}(\mathfrak{C}),$$

called the *restriction mapping*, possesses the property $\rho_{\mathfrak{B},\mathfrak{B}}=\mathrm{id}_{\mathscr{F}(\mathfrak{B})}$ and $\rho_{\mathfrak{D},\mathfrak{B}}=\rho_{\mathfrak{D},\mathfrak{C}}\circ\rho_{\mathfrak{C},\mathfrak{B}}$ for any triplet $\mathfrak{D}\subset\mathfrak{C}\subset\mathfrak{B}$ of σ-subalgebras.

Let $\mathscr{F}$ be a σ-sheaf over $(\Omega,\mathfrak{A})$ and $\mathfrak{B}$ be a σ-subalgebra of $\mathfrak{A}$. An element $s \in \mathscr{F}(\mathfrak{B})$ is called a *section* of $\mathscr{F}$ over $\mathfrak{B}$. For every $A \in \mathfrak{A}$, $\mathscr{F}(A) := \mathscr{F}\{\Omega, \phi, A, A^c\})$ is called the *stalk* of $\mathscr{F}$ at A.

At first we consider a transformation group on the measurable space $(\Omega, \mathfrak{A})$.

An abelian group G is called a *measurable transformation group* on $(\Omega, \mathfrak{A})$ if G acts on $(\Omega, \mathfrak{A})$, i.e., every element $g \in G$ is a bijection from Ω onto Ω such that

(1) g and g^{-1} are measurable.
(2) $eA = A$ and $g_1(g_2A) = (g_1g_2)A$ for every $A \in \mathfrak{A}$ and $g_1, g_2 \in G$, where e denotes the unit element of G.

Definition 2.1 We call the measurable space $(\Omega, \mathfrak{A})$ with such a group action G a *measurable G-space* $(\Omega, \mathfrak{A})$.

We construct a σ-sheaf for a measurable G-space $(\Omega, \mathfrak{A})$.

Example 2.2. (σ-sheaf of measurable transformation group) Let $(\Omega, \mathfrak{A})$ be a measurable G-space. For every σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, we define an abelian group

$$\mathscr{F}(\mathfrak{B}) = \{g \in G \mid gA = A \text{ for every } A \in \mathfrak{B}\}$$

and the inclusion mapping

$$\rho_{\mathfrak{C}, \mathfrak{B}} : \mathscr{F}(\mathfrak{B}) \to \mathscr{F}(\mathfrak{C})$$

for σ-subalgebras $\mathfrak{B}$ and $\mathfrak{C}$ with $\mathfrak{C} \subset \mathfrak{B}$. We call this σ-sheaf a *σ-sheaf of measurable transformation group G over* $(\Omega, \mathfrak{A})$.

We consider the case of measure space. A measurable mapping ϕ: $(\Omega, \mathfrak{A}, \mu) \to (\Omega, \mathfrak{A}, \mu)$ on a measure space $(\Omega, \mathfrak{A}, \mu)$ is called an *automorphism* if ϕ is measure-preserving, bijective, and ϕ^{-1} is also measure-preserving. We call a class $\{\phi_t\}_{t \in \boldsymbol{R}}$ of automorphisms on $(\Omega, \mathfrak{A}, \mu)$ a *flow* if $\phi_t \circ \phi_s = \phi_{t+s}$ holds for t, $s \in \boldsymbol{R}$ and ϕ_0 is the identity mapping.

A measurable transformation group G on a measure space $(\Omega, \mathfrak{A}, \mu)$ is called a *measure-preserving* transformation group if every element g of G is measure-preserving. It is an abstract automorphism or flow on the measure space $(\Omega, \mathfrak{A}, \mu)$. Refer to [10] for measure-preserving transformation, automorphisms and flows.

To construct a σ-sheaf for an automorphism ϕ, we associate a cyclic group

$$G_\phi = \{\phi^n \mid n \in \boldsymbol{Z}\}$$

as a measure-preserving transformation group. The same consideration can be applied to the case of flow. We will study the behavior of the automorphism ϕ in terms of the cohomology group in Section 4.

Example 2.3. (The case of measure space) We may need a σ-sheaf of transformation group handling the omittion of sets of measure zero. We introduce the way to take measure into consideration. Let G be a measure-preserving transformation group on a measure space $(\Omega, \mathfrak{A}, \mu)$. We define an equivalence relation on $\mathfrak{A}$ by $A \sim B$ iff $\mu(A \varDelta B)=0$, where $A \varDelta B$ denotes the symmetric difference $(A \backslash B) \cup (B \backslash A)$. For every σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, we set

$$\mathscr{F}(\mathfrak{B}) = \{g \in G | gA \sim A \text{ for every } A \in \mathfrak{B}\}.$$

Then we get a σ-sheaf of measure-preserving transformation group G *mod* 0 over the measure space $(\Omega, \mathfrak{A}, \mu)$. Of course we can also apply Example 2.2 to an arbitrary measure space.

We introduce a σ-sheaf of integrable functions.

Example 2.4. (σ-sheaf of integrable functions) Let $(\Omega, \mathfrak{A}, \mu)$ be a finite measure space. We denote by $\mathscr{I}$ the class of all integrable functions on $(\Omega, \mathfrak{A}, \mu)$. For every σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, we set

$$\mathscr{F}(\mathfrak{B}) = \{f \in \mathscr{I} | f \text{ is } \mathfrak{B}\text{-measurable}\}$$

and define the restriction mapping

$$\rho_{\mathfrak{C},\mathfrak{B}} : f \in \mathscr{F}(\mathfrak{B}) \mapsto E(f|\mathfrak{C}) \in \mathscr{F}(\mathfrak{C})$$

by taking the conditional expectation (see [8] and [9]). We obtain a *σ-sheaf of integrable functions* over $(\Omega, \mathfrak{A}, \mu)$.

3. Cohomology group with coefficients in σ-sheaf

In this section we formulate a cohomology group with coefficients in σ-sheaf in the similar way to the Čech cohomology theory (see [3] and [4]). We investigate a cohomology group with coefficients in a σ-sheaf of an automorphism in Section 4.

At first we give the definition of a covering over a measurable space $(\Omega, \mathfrak{A})$.

Definition 3.1 Let $(\Omega, \mathfrak{A})$ be an arbitrary measurable space. A class $\{\mathfrak{A}_i\}_{i \in I}$ of σ-subalgebras of $\mathfrak{A}$ is called a σ-covering of $\mathfrak{A}$ if $\mathfrak{A} = \bigvee_{i \in I} \mathfrak{A}_i$, where the right-hand side denotes the smallest σ-algebra that includes $\bigcup_{i \in I} \mathfrak{A}_i$.

Similarly σ-covering *mod* 0 $\{\mathfrak{A}_i\}$ of σ-algebra $\mathfrak{A}$ of a measure space

$(\Omega, \mathfrak{A}, \mu)$ can be defined by $\mathfrak{A}/\sim = \bigvee_{i\in I} \mathfrak{A}_i/\sim$ with an equivalence relation $\sim$ defined in Example 2. 3.

We get ready for constructing a cohomology group with respect to a σ-covering, in the same way as the Čech cohomology group with respect to an open covering.

Let $\{\mathfrak{A}_i\}_{i\in I}$ be a σ-covering of σ-algebra $\mathfrak{A}$ of the measurable space $(\Omega, \mathfrak{A})$ and $\mathscr{F}$ a σ-sheaf over $(\Omega, \mathfrak{A})$. For each nonnegative integer q, we put

$$\Sigma_q := \{I_q=(i_0, \cdots, i_q) | i_0, \cdots, i_q \in I\}, \ \mathfrak{A}_{I_q} := \mathfrak{A}_{i_0} \cap \cdots \cap \mathfrak{A}_{i_q}$$

and

$$C^q(\{\mathfrak{A}_i\}, \mathscr{F}) := \prod_{I_q \in \Sigma_q} \mathscr{F}(\mathfrak{A}_{I_q}).$$

It is to be noted that the elements $i_0, \cdots, i_q$ of I_q may overlap. Each element $\sigma = \{\sigma_{i_0,\cdots,i_q}\} \in C^q(\{\mathfrak{A}_i\}, \mathscr{F})$ is called a *q-cochain* of the *σ-covering* $\{\mathfrak{A}_i\}$.

We define a coboundary homomorphism

$$\delta_q : C^q(\{\mathfrak{A}_i\}, \mathscr{F}) \to C^{q+1}(\{\mathfrak{A}_i\}, \mathscr{F})$$

by

$$\{(\delta_q\sigma)_{i_0\cdots i_{q+1}}\} = \left\{\sum_{k=0}^{q+1}(-1)^k \rho_{\mathfrak{A}(I_{q+1}),\mathfrak{A}(I^k_{q+1})}(\sigma_{i_0\cdots i_{k-1} i_{k+1}\cdots i_{q+1}})\right\},$$

where $\mathfrak{A}(I_{q+1}) = \mathfrak{A}_{i_0} \cap \cdots \cap \mathfrak{A}_{i_{q+1}}$ and $\mathfrak{A}(I^k_{q+1}) = \mathfrak{A}_{i_0} \cap \cdots \cap \mathfrak{A}_{i_{k-1}} \cap \mathfrak{A}_{i_{k+1}} \cap \cdots \cap \mathfrak{A}_{i_{q+1}}$.

We often omit writing $\rho_{\mathfrak{A}(I_{q+1}), \mathfrak{A}(I^k_{q+1})}$.

Lemma 3. 2 $\delta_{q+1}\circ\delta_q = 0, \ q=0, 1, 2, \cdots$.

Proof. We assume that $\{\tau_{i_0\cdots i_{q+1}}\} = \delta_q\{\sigma_{i_0\cdots i_q}\}$. Then

$$\begin{aligned}
&\sum_{j=0}^{q+2}(-1)^j \tau_{i_0\cdots \hat{i}_j \cdots i_{q+2}} \\
&= \sum_{j=0}^{q+2}(-1)^j \left(\sum_{k=0}^{j-1}(-1)^k \sigma_{i_0\cdots \hat{i}_k \cdots \hat{i}_j \cdots i_{q+2}} + \sum_{k=j+1}^{q+2}(-1)^{k-1}\sigma_{i_0\cdots \hat{i}_j \cdots \hat{i}_k \cdots i_{q+2}}\right) \\
&= \sum_{k<j}(-1)^{j+k}\sigma_{i_0\cdots \hat{i}_k \cdots \hat{i}_j \cdots i_{q+2}} + \sum_{j<k}(-1)^{k+j-1}\sigma_{i_0\cdots \hat{i}_j \cdots \hat{i}_k \cdots i_{q+2}} = 0,
\end{aligned}$$

where $\hat{i}_k$ means that we omit this index i_k. □

Now we formulate a cohomology group with respect to a σ-covering.

Definition 3. 3 Let $\{\mathfrak{A}_i\}$ be a σ-covering of $\mathfrak{A}$ and $\mathscr{F}$ a σ-sheaf over $(\Omega, \mathfrak{A})$. We define the class of *q-cocycles* $Z^q(\{\mathfrak{A}_i\}, \mathscr{F})$ by

$$Z^q(\{\mathfrak{A}_i\}, \mathscr{F}) := \{\sigma \in C^q(\{\mathfrak{A}_i\}, \mathscr{F}) | \delta_q \sigma = 0\}$$

and the class of *q-coboundaries* $B^q(\{\mathfrak{A}_i\}, \mathscr{F})$ by

$$B^q(\{\mathfrak{A}_i\}, \mathscr{F}) := \delta_{q-1}(C^{q-1}(\{\mathfrak{A}_i\}, \mathscr{F})), \text{ if } q \geq 1, \text{ and } B^0(\{\mathfrak{A}_i\}, \mathscr{F}) := 0.$$

Since it follows from Lemma 3.2 that $B^q(\{\mathfrak{A}_i\}, \mathscr{F})$ is a subgroup of $Z^q(\{\mathfrak{A}_i\}, \mathscr{F})$, we can define a quotient group

$$H^q(\{\mathfrak{A}_i\}, \mathscr{F}) := Z^q(\{\mathfrak{A}_i\}, \mathscr{F}) / B^q(\{\mathfrak{A}_i\}, \mathscr{F}),$$

called a *cohomology group with coefficients in* $\mathscr{F}$ *with respect to* $\{\mathfrak{A}_i\}$.

To define a cohomology group for the measurable space $(\Omega, \mathfrak{A})$, we don't use an inductive limit with respect to the refinement of covering. It is due to the difference of structures between measurable spaces and topological spaces.

Definition 3.4 We define a *cohomology group of* $\mathfrak{A}$ *with coefficients in* $\mathscr{F}$ by the direct product of cohomology groups of all σ-coverings of $\mathfrak{A}$.

$$H^q(\mathfrak{A}, \mathscr{F}) := \prod_{\{\mathfrak{A}_i\}} H^q(\{\mathfrak{A}_i\}, \mathscr{F}).$$

4. Cohomology group of measurable transformation group and some cohomology vanishing theorems

In this section we state some results concerning the cohomology group with coefficients in σ-sheaf of measurable or measure-preserving transformation group. We state the significance of cohomology group of automorphism. Cohomology group of automorphism is related to the orbit structure of automorphism, especially recurrence. Refer to [5] and [6] for topological transformation groups.

Let $\{\mathfrak{A}_i\}$ be a σ-covering of $\mathfrak{A}$. We consider a σ-sheaf $\mathscr{F}$ of an automorphism ϕ on a measure space $(\Omega, \mathfrak{A}, \mu)$ and regard the ϕ as a set map $\phi: \mathfrak{A} \to \mathfrak{A}$. We study certain connection between the recurrence of ϕ with respect to a domain $\mathfrak{A}_{i0} \cap \cdots \cap \mathfrak{A}_{iq}$ and the one with respect to its neighbor domain $\mathfrak{A}_{i0} \cap \cdots \cap \mathfrak{A}_{i_{k-1}} \cap \mathfrak{A}_{i_{k+1}} \cap \cdots \cap \mathfrak{A}_{iq}$. Cohomology group $H^q(\{\mathfrak{A}_i\}, \mathscr{F})$ reflects the recurrence of an automorphism ϕ as a set map, with respect to a decomposition $\{\mathfrak{A}_i\}$ of $\mathfrak{A}$. Of course it depends on the choice of σ-covering $\{\mathfrak{A}_i\}$.

We denote by $\{\phi^{m(i_0, \cdots, i_q)}\}$ a q-cochain with respect to the σ-sheaf $\mathscr{F}$, where m is a function from Σ_q into $\boldsymbol{Z}$ and $\phi^{m(i_0, \cdots, i_q)} \in \mathscr{F}(\mathfrak{A}_{i0} \cap \cdots \cap \mathfrak{A}_{iq})$.

Theorem 4.1 *Let* ϕ *be an automorphism on a measure space* $(\Omega, \mathfrak{A}, \mu)$, $\{\mathfrak{A}_i\}_{i \in I}$ *a* σ*-covering of* $\mathfrak{A}$, *and* $\mathscr{F}$ *a* σ*-sheaf of* G_ϕ. *If there exists a non-zero element*

$$[\phi^{m(i_0,\cdots,i_q)}]\in H^q(\{\mathfrak{A}_i\},\mathscr{F})$$

with a representative element $\{\phi^{m(i_0,\cdots,i_q)}\}\in Z^q(\{\mathfrak{A}_i\},\mathscr{F})$ $(q\geq 1)$, *then for every* $k\in I$ *there exists a* $(j_0,\cdots,j_{q-1})\in\sum_{q-1}$ *such that* $\phi^{m(j_0,\cdots,j_{q-1},k)}\notin\mathscr{F}(\mathfrak{A}_{j_0}\cap\cdots\cap\mathfrak{A}_{j_{q-1}})$, *that is,* $\phi^{m(j_0,\cdots,j_{q-1},k)}$ *has difference of recurrence between* $\mathfrak{A}_{j_0}\cap\cdots\cap\mathfrak{A}_{j_{q-1}}\cap\mathfrak{A}_k$ *and* $\mathfrak{A}_{j_0}\cap\cdots\cap\mathfrak{A}_{j_{q-1}}$.

Proof. We take and fix an arbitrary index $k\in I$. Suppose that $\phi^{m(i_0,\cdots,i_{q-1},k)}\in\mathscr{F}(\mathfrak{A}_{i_0}\cap\cdots\cap\mathfrak{A}_{i_{q-1}})$ for every $(i_0,\cdots,i_{q-1})\in\sum_{q-1}$. Then we define a $(q-1)$-cochain

$$\{\phi^{n(i_0,\cdots,i_{q-1})}\}\in C^{q-1}(\{\mathfrak{A}_i\},\mathscr{F})$$

by $\phi^{n(i_0,\cdots,i_{q-1})}:=\phi^{(-1)^q m(i_0,\cdots,i_{q-1},k)}$. Since $\{\phi^{m(i_0,\cdots,i_q)}\}$ is a q-cocycle,

$$\begin{aligned}\phi^{m(i_0,\cdots,i_q)}&=(-1)^q\{\phi^{m(i_1,\cdots,i_q,k)}-\phi^{m(i_0,i_2,\cdots,i_q,k)}+\cdots+(-1)^q\phi^{m(i_0,\cdots,i_{q-1},k)}\}\\&=\phi^{n(i_1,\cdots,i_q)}-\phi^{n(i_0,i_2,\cdots,i_q)}+\cdots+(-1)^q\phi^{n(i_0,\cdots,i_{q-1})}\\&=(\delta_{q-1}\{\phi^{n(i_0,\cdots,i_{q-1})}\})_{i_0,\cdots,i_q}\end{aligned}$$

for every $(i_0,\cdots,i_q)\in\sum_q$. Hence $\{\phi^{m(i_0,\cdots,i_q)}\}\in B^q(\{\mathfrak{A}_i\},\mathscr{F})$. This contradicts our assumption. □

The order of cohomology group of automorphism indicates and measures the variety of behavior of the automorphism ϕ. If the cohomology group $H^q(\{\mathfrak{A}_i\},\mathscr{F})$ doesn't vanish for large integer q, then ϕ has different recurrence around the intersections $\mathfrak{A}_{i_0}\cap\cdots\cap\mathfrak{A}_{i_q}$ of many domains $\mathfrak{A}_i$ of σ-covering and hence the behaviors of ϕ on each $\mathfrak{A}_i$ are various. If $H^q(\{\mathfrak{A}_i\},\mathscr{F})=0$ for all $q\geq 1$, the behaviors of ϕ on each $\mathfrak{A}_i$ are alike each other.

We show the structure of 0-th cohomology group H^0.

Theorem 4.2 *Let* $(\Omega,\mathfrak{A})$ *be a measurable G-space and* $\mathscr{F}$ *a* σ-*sheaf of G. Then* $H^0(\{\mathfrak{A}_i\},\mathscr{F})\simeq\mathscr{F}(\mathfrak{A})$ *holds for every* σ-*covering* $\{\mathfrak{A}_i\}$ *of* $\mathfrak{A}$.

Proof. We define a homomorphism $f:=\mathscr{F}(\mathfrak{A})\rightarrow C^0(\{\mathfrak{A}_i\},\mathscr{F})$ by

$$g\in\mathscr{F}(\mathfrak{A})\mapsto\{\rho_{\mathfrak{A}_i,\mathfrak{A}}(g)\}\in C^0(\{\mathfrak{A}_i\},\mathscr{F}).$$

Since $\delta_0 f(g)=0$ for every $g\in\mathscr{F}(\mathfrak{A})$, we can regard f as a mapping from $\mathscr{F}(\mathfrak{A})$ into $H^0(\{\mathfrak{A}_i\},\mathscr{F})$. At first, we show that f is injective. For two elements g_1 and g_2 of $\mathscr{F}(\mathfrak{A})$, suppose that $\rho_{\mathfrak{A}_i,\mathfrak{A}}(g_1)=\rho_{\mathfrak{A}_i,\mathfrak{A}}(g_2)$ holds for every $i\in I$. Then since ρ is an inclusion mapping, $g_1=g_2$. Hence f is injective. We show the surjectiveness. For every $\sigma\in H^0(\{\mathfrak{A}_i\},\mathscr{F})$, there exists an $g_\sigma\in G$ such that $g_\sigma A=A$ for every $A\in\bigcup_{i\in I}\mathfrak{A}_i$. Since

$$\mathscr{M}=\{A\in\bigvee_{i\in I}\mathfrak{A}_i|g_\sigma A=A\}$$

is a σ-algebra including $\bigcup_{i\in I}\mathfrak{A}_i$, $\mathscr{M}=\bigvee_{i\in I}\mathfrak{A}_i$ holds. Hence $g_\sigma\in\mathscr{F}(\bigvee_{i\in I}\mathfrak{A}_i)$ and f is surjective. This completes the proof. $\square$

We can obtain the same result as above for a σ-sheaf of G mod 0. In the case of general σ-sheaf $\mathscr{F}$, the cohomology group $H^0(\{\mathfrak{A}_i\},\mathscr{F})$ isn't always isomorphic to $\mathscr{F}(\mathfrak{A})$.

Corollary 4. 3 *Let $(\Omega,\mathfrak{A})$ be a measurable G-space and $\mathscr{F}$ a σ-sheaf of G, and $\{\mathfrak{A}_i\}$ a σ-covering of $\mathfrak{A}$. Then if the order of $H^0(\{\mathfrak{A}_i\},\mathscr{F})$ is finite, it equals to the number of automorphisms $g\in G$ such that $gA=A$ holds for every $A\in\mathfrak{A}$.*

The proof follows immediately from Theorem 4. 2.

Next we shall show automorphisms whose cohomology groups vanish.

Lemma 4. 4 *Let ϕ be an automorphism on a measure space $(\Omega,\mathfrak{A},\mu)$ and $\mathscr{F}$ a σ-sheaf of G_ϕ mod 0. For every σ-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, we have*

$$\mathscr{F}(\bigvee_{n=0}^{\infty}\phi^n\mathfrak{B})=\mathscr{F}(\mathfrak{B}).$$

Proof. We take an arbitrary $g\in\mathscr{F}(\mathfrak{B})$ and prove that $g\in\mathscr{F}(\bigvee_{n=0}^{\infty}\phi^n\mathfrak{B})$. Note that

$$\mathscr{F}(\mathfrak{B})=\mathscr{F}(\phi\mathfrak{B})=\cdots=\mathscr{F}(\phi^n\mathfrak{B})$$

and $gA\sim A$ holds for every $A\in\bigcup_{n=0}^{\infty}\phi^n\mathfrak{B}$. We show that

$$\mathscr{M}=\{A\in\bigvee_{n=0}^{\infty}\phi^n\mathfrak{B}\,|\,gA\sim A\}$$

is a σ-algebra. It is clear that $\phi,\Omega\in\mathscr{M}$. If $A\in\mathscr{M}$ holds, then

$$\begin{aligned}\mu(gA^c\Delta A^c)&=\mu((gA^c\backslash A^c)\cup(A^c\backslash gA^c))\\&=\mu((gA^c\cap A)\cup(A^c\cap gA))=\mu((A\cap(gA)^c)\cup(gA\cap A^c))\\&=\mu((A\backslash gA)\cup(gA\backslash A))=\mu(gA\Delta A)=0,\end{aligned}$$

and hence $A^c\in\mathscr{M}$.

Suppose that $A_1, A_2, \cdots, A_i, \cdots\in\mathscr{M}$, then

$$\begin{aligned}\mu(g(\bigcup_{i=1}^{\infty}A_i)\Delta\bigcup_{i=1}^{\infty}A_i)&=\mu((g(\bigcup_{i=1}^{\infty}A_i)\backslash\bigcup_{i=1}^{\infty}A_i)\cup(\bigcup_{i=1}^{\infty}A_i\backslash g(\bigcup_{i=1}^{\infty}A_i)))\\&=\mu(g(\bigcup_{i=1}^{\infty}A_i)\backslash\bigcup_{i=1}^{\infty}A_i)+\mu(\bigcup_{i=1}^{\infty}A_i\backslash g(\bigcup_{i=1}^{\infty}A_i)).\end{aligned}$$

Since

$$\begin{aligned}\mu(g(\bigcup_{i=1}^{\infty}A_i)\backslash\bigcup_{i=1}^{\infty}A_i) &= \mu(g(\bigcup_{i=1}^{\infty}A_i)\cap(\bigcup_{i=1}^{\infty}A_i)^c) = \mu(g(\bigcup_{i=1}^{\infty}A_i)\cap(\bigcap_{i=1}^{\infty}A_i^c))\\ &= \mu((\bigcup_{i=1}^{\infty}gA_i)\cap(\bigcap_{i=1}^{\infty}A_i^c)) \leqq \sum_{i=1}^{\infty}\mu(gA_i\cap(\bigcap_{i=1}^{\infty}A_i^c)) \leqq \sum_{i=1}^{\infty}\mu(gA_i\cap A_i^c)\\ &= \sum_{i=1}^{\infty}\mu(gA_i\backslash A_i)\end{aligned}$$

and

$$\begin{aligned}\mu(\bigcup_{i=1}^{\infty}A_i\backslash g(\bigcup_{i=1}^{\infty}A_i)) &= \mu(\bigcup_{i=1}^{\infty}(A_i\backslash g(\bigcup_{i=1}^{\infty}A_i)))\\ &\leqq \sum_{i=1}^{\infty}\mu(A_i\backslash g(\bigcup_{i=1}^{\infty}A_i)) \leqq \sum_{i=1}^{\infty}\mu(A_i\backslash gA_i),\end{aligned}$$

it holds that

$$\begin{aligned}&\mu(g(\bigcup_{i=1}^{\infty}A_i)\varDelta\bigcup_{i=1}^{\infty}A_i)\\ &\leqq \sum_{i=1}^{\infty}(\mu(gA_i\backslash A_i)+\mu(A_i\backslash gA_i)) \leqq \sum_{i=1}^{\infty}\mu((gA_i\backslash A_i)\cup(A_i\backslash gA_i))\\ &= \sum_{i=1}^{\infty}\mu(gA_i\varDelta A_i) = 0,\end{aligned}$$

which concludes that $\bigcup_{i=1}^{\infty}A_i\in\mathscr{M}$. Hence $\mathscr{M}$ is a σ-algebra and $\mathscr{M}=\bigvee_{n=0}^{\infty}\phi^n\mathfrak{B}$. Therefore $g\in\mathscr{F}(\bigvee_{n=0}^{\infty}\phi^n\mathfrak{B})$ holds. This completes the proof. □

Definition 4.5 An automorphism ϕ on a measure space $(\Omega, \mathfrak{A}, \mu)$ is called an *increasing automorphism* if there exists a σ-subalgebra $\mathfrak{K}$ of $\mathfrak{A}$ such that

$$\mathfrak{K}\subsetneqq\phi\mathfrak{K} \quad \text{and} \quad \bigvee_{n=0}^{\infty}\phi^n\mathfrak{K}/\sim\; = \mathfrak{A}/\sim,$$

where $\sim$ is an equivalence relation defined in Example 2.3. It is to be noted that $\{\phi^i\mathfrak{K}\}$ is a σ-covering mod 0 of $\mathfrak{A}$.

An increasing automorphism ϕ on a probability space $(\Omega, \mathfrak{A}, P)$ satisfying that

$$\bigcap_{n=0}^{\infty}\phi^{-n}\mathfrak{K}/\sim\; = \{A\in\mathfrak{A}|P(A)=0 \text{ or } 1\}$$

is called a *Kolmoqorov automorphism.*

Theorem 4.6 *Let ϕ be an increasing automorphism on a measure space $(\Omega, \mathfrak{A}, \mu)$ and $\mathscr{F}$ a σ-sheaf of G_ϕ mod 0. Then $H^0(\{\mathfrak{A}_i\}, \mathscr{F})=0$ for every*

σ-covering $\{\mathfrak{A}_i\}$.

Proof. By Theorem 4.2 and Lemma 4.4,

$$H^0(\{\mathfrak{A}_i\}, \mathscr{F}) \simeq \mathscr{F}(\mathfrak{A}) = \mathscr{F}(\bigvee_{n=0}^{\infty} \phi^n \mathfrak{K}) = \mathscr{F}(\mathfrak{K}).$$

The increasing property $\mathfrak{K} \subsetneqq \phi\mathfrak{K}$ implies that $\mathscr{F}(\mathfrak{K})=\{\mathrm{id.}\}$, because if $\phi^m A = A$ for every $A \in \mathfrak{K}$ $(m>0)$ then $\phi^m\mathfrak{K}=\mathfrak{K}$, which contradicts that $\mathfrak{K} \subsetneqq \phi^m\mathfrak{K}$. Hence $H^0(\{\mathfrak{A}_i\}, \mathscr{F})=0$ holds. This completes the proof. □

Theorem 4.7 *Let ϕ be an increasing automorphism on a measure space $(\Omega, \mathfrak{A}, \mu)$ and $\mathscr{F}$ a σ-sheaf of G_ϕ mod 0. Then we obtain*

$$H^q(\{\phi^i\mathfrak{K}\}, \mathscr{F}) = 0, \quad q \geq 1.$$

Proof. Since it follows from the increasing property of ϕ that

$$\mathscr{F}(\phi^{i_0}\mathfrak{K} \cap \cdots \cap \phi^{i_q}\mathfrak{K}) = \mathscr{F}(\phi^{min\{i_0,\cdots,i_q\}}\mathfrak{K}) = \mathscr{F}(\mathfrak{K}) = \{id.\}$$

for every $(i_0, \cdots, i_q) \in \Sigma_q$, we find

$$\begin{aligned} C^q(\{\phi^i\mathfrak{K}\}, \mathscr{F}) &= \prod_{I_q \in \Sigma_q} \mathscr{F}(\phi^{i_0}\mathfrak{K} \cap \cdots \cap \phi^{i_q}\mathfrak{K}) \\ &= \prod_{I_q \in \Sigma_q} \mathscr{F}(\phi^{min\{i_0,\cdots,i_q\}}\mathfrak{K}) \\ &= \prod_{I_q \in \Sigma_q} \mathscr{F}(\mathfrak{K}) = \prod_{I_q \in \Sigma_q} \{id.\} = 0. \end{aligned}$$

Hence we obtain $H^q(\{\phi^i\mathfrak{K}\}, \mathscr{F})=0$. □

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