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Nonhomogeneity of Picard dimensions on the half ball

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We denote by H^m the upper half space $\{x=(x_1, \cdots, x_m)\in R^m : x_m>0\}$ in the Euclidean m-space $R^m (m\geq 2)$ and by $\hat{H}^m$ the closure of H^m with respect to the one point compactification of R^m. Setting $\delta H^m=\hat{H}^m\backslash H^m$, we may view $\{x\in\hat{H}^m : x_m=0\}$ as a subset of an ideal boundary δH^m of H^m and the origin $x=0$ as an ideal boundary point of H^m. Take the upper half ball $U_s^+=\{x=(x_1, \cdots, x_m)\in H^m : |x|<s\}$ $(0<s\leq 1)$ which may be regarded as a relative neighbourhood of the ideal boundary point $x=0$ of H^m. The set $\Gamma_s^+\equiv\{x\in H^m : |x|=s\}$ is a relative boundary of U_s^+ and $\gamma_s^+\equiv\{x\in\delta H^m : |x|\leq s\}$ is an ideal boundary of U_s^+. Therefore the boundary ∂U_s^+ of U_s^+ and the closure $\bar{U}_s^+$ of U_s^+ in $\hat{H}^m$ are $\Gamma_s^+\cup\gamma_s^+$ and $U_s^+\cup\Gamma_s^+\cup\gamma_s^+$, respectively. In particular we set $U_1^+=U^+$ and $\Gamma_1^+=\Gamma^+$. By a *density* $P(x)$ on U_s^+ we mean a locally Hölder continuous function $P(x)$ defined on $\bar{U}_s^+\backslash\{0\}$. Hence P may have a singularity at the ideal boundary point $x=0$.

Consider the time independent Schrödinger equation

$$L_P u(x) \equiv -\triangle u(x)+P(x)u(x) = 0 \tag{1}$$

defined on $\bar{U}_s^+\backslash\{0\}$, where $\triangle$ is the Laplacian $\triangle=\sum_{i=1}^m \partial^2/\partial x_i^2$. We are interested in the class $PP(U_s^+)$ of nonnegative solutions of (1) in U_s^+ with vanishing boundary values on $\partial U_s^+\backslash\{0\}$. The first P indicates the dependence of the class on the density P and the second P stands for the initial of the term positive (nonnegative) so that the class associated with another density Q is denoted by $QP(U_s^+)$. It is convenient to consider the subclass $PP_1(U_s^+)\equiv\{u\in PP(U_s^+) : u(x_s)=1\}$, where x_s is an arbitrary point fixed in U_s^+. Since $PP_1(U_s^+)$ is a compact and convex set with respect to almost uniform convergence on U_s^+, we can consider the set $ex.PP_1(U_s^+)$ of extreme points of $PP_1(U_s^+)$ and the cardinal number $\#(ex.PP_1(U_s^+))$ of $ex.PP_1(U_s^+)$ which will be referred to as the *Picard dimension* of (U_s^+, P) at $x=0$, $\dim(U_s^+, P)$ in notation:

$$\dim(U_s^+, P) = \#(ex.PP_1(U_s^+)).$$

In particular we say that the *Picard principle* is valid for (U_s^+, P) at $x=0$

if $\dim(U_s^+, P)=1$.

A density $P(x)$ is said to be radial if it depends only upon $|x|$. T. Tada showed in [15] $\dim(U^+, P)=1$ or c for any nonnegative radial density on U^+ if $m=2$, where c is the cardinal number of a continuum. M. Murata [10] showed $\dim(U_s^+, P)=1$ if P is locally Hölder continuous on the entire $\bar{U}_s^+$ and if there exists the Green's function of (1) on U_s^+. Also Y. Pinchover [14] showed $\dim(U_s^+, P)=1$ provided that there exists the Green's function of (1) on U_s^+ and $P(x)=O(|x|^{-2})$ as $x \to 0$.

If P is a density on U^+, then (1) is defined on $\bar{U}^+\setminus\{0\}$. In this case we will show:

Proposition *There exists a t in $(0, 1]$ such that*

$$\dim(U_s^+, P) = \dim(U_t^+, P)$$

for any s in $(0, t]$.

Hence we can define for a density P on U^+ the *Picard dimension* of P at $x=0$, $\dim P$ in notation, by

$$\dim P = \lim_{s \downarrow 0} \dim(U_s^+, P).$$

In particular we say that the *Picard principle* is valid for a density P at $x=0$ if $\dim P=1$.

In contrast with U_s^+ we take a punctured ball $U_s=\{x\in R^m : 0<|x|<s\}$ $(0<s\le 1)$ in $R^m\setminus\{0\}$ and we may regard $x=0$ as an ideal boundary component of the space $R^m\setminus\{0\}$ so that the relative boundary of U_s is $\Gamma_s\equiv\{x\in R^m : |x|=s\}$. But in this case we denote by $\bar{U}_s$ the relative closure $U_s\cup\Gamma_s$ of U_s in $R^m\setminus\{0\}$. We set $U_1=U$ and $\Gamma_1=\Gamma$. If $\tilde{P}$ is a density defined on U, i.e. a locally Hölder continuous function defined on $\bar{U}$, then Schrödinger equation

$$L_{\tilde{P}}u(x) \equiv -\triangle u(x)+\tilde{P}(x)u(x) = 0$$

is defined on $\bar{U}$. We can consider the class $\tilde{P}P(U_s)$ of nonnegative solutions of $L_{\tilde{P}}u=0$ on U_s with vanishing boundary values on Γ_s for each s in $(0, 1]$. With an arbitrary fixed point $\tilde{x}_s$ in U_s, $\tilde{P}P_1(U_s)\equiv\{u\in\tilde{P}P(U_s) : u(\tilde{x}_s)=1\}$ is a compact and convex set. The cardinal number of the set of extreme points of $\tilde{P}P_1(U_s)$ will be referred to as the *Picard dimension* of $(U_s, \tilde{P})$ at $x=0$, $\dim(U_s, \tilde{P})$ in notation (M. Nakai [11]). It was shown in M. Nakai [12], M. Murata [9] and M. Nakai and T. Tada [13] that there exists a t in $(0, 1]$ such that

$$\dim(U_s, \tilde{P}) = \dim(U_t, \tilde{P})$$

for each s in $(0, t]$. Hence by the same way as above the *Picard dimension* of the density $\tilde{P}$ on U at $x=0$ and the *Picard principle* for the density $\tilde{P}$ at $x=0$ are defined in [13] (also see M. Nakai [11] and [12]). We say that for a density $\tilde{P}$ on U the *homogeneity of Picard dimensions* holds at $x=0$ if $\dim \tilde{P} = \dim c\tilde{P}$ for any constant $c>0$ ([11], [13]). In particular we say that for a density $\tilde{P}$ on U the *homogeneity of the Picard principle* is valid at $x=0$ if $\dim \tilde{P} = \dim c\tilde{P} = 1$ for any constant $c>0$ ([13]).

It was shown in M. Kawamura and M. Nakai [7] that for nonnegative radial densities on U the homogeneity of Picard dimensions is always valid at $x=0$. The nonhomogeneity of the Picard principle for negative radial densities at $x=0$ is studied in [4] and [5]. The nonhomogeneity of Picard dimensions for signed radial densities is also studied in T. Tada [16].

In anologous to the case of the punctured ball U in which $x=0$ is an isolated ideal boundary component, we say that for a density P on U^+ the *homogeneity of Picard dimensions* holds at $x=0$ if $\dim cP = \dim P$ for any $c>0$. In particular we say that for a density P on U^+ the *homogeneity of the Picard principle* is valid at $x=0$ if $\dim cP = \dim P = 1$ for any $c>0$.

Consider the negative densities Q and R on U^+ given by

$$Q(x) \equiv -\frac{1}{4|x|^2}\left\{m^2 + \frac{1}{\left(\log\frac{\eta}{|x|}\right)^2} + \frac{1}{\left(\log\frac{\eta}{|x|}\cdot\log\log\frac{\eta}{|x|}\right)^2}\right\} \tag{2}$$

and

$$R(x) \equiv -\frac{1}{4|x|^2}\left\{m^2 + \frac{1}{\left(\log\frac{\eta}{|x|}\right)^2} + \frac{2}{\left(\log\frac{\eta}{|x|}\cdot\log\log\frac{\eta}{|x|}\right)^2}\right\}, \tag{3}$$

where η is any fixed constant with $\eta > e^e$. The purpose of this paper is to show the following result which states that the homogeneity of the Picard principle does not necessarily hold at $x=0$ for negative densities on U^+.

Theorem *The density Q given by* (2) *satisfies*

$$\dim Q = 1 \qquad \textit{but} \qquad \dim cQ = 0$$

for any $c>1$. The density R given by (3) *satisfies*

$$\dim R = 0 \qquad \textit{but} \qquad \dim cR = 1$$

for any $0<c<1$.

To show the theorem we will see that $\dim(U_s^+, Q)=1$ and $\dim(U_s^+, R)=0$ for any s in $(0, 1]$ so that we can take 1 as the value of t in the proposition for densities Q and R. But in the latter half of the theorem we will see that, whenever we select a constant $c\in(0, 1)$, we merely can take a t in $(0, 1)$ depending upon the constant c.

1. We begin with some definitions. A function u is a solution of (1) in U_s^+ if u is a C^2 function on U_s^+ which satisfies (1) in U_s^+. A lower semicontinuous, lower finite function v on U_s^+ is a supersolution of (1) in U_s^+ if $v(x)\geq u(x)$ in B whenever $v(x)\geq u(x)$ on the boundary ∂B of B for any ball B in U_s^+ with $\bar{B}\subset U_s^+$ and for any solution $u(x)$ of (1) in B continuous in $\bar{B}$. If $v(x)$ is a C^2 function on U_s^+, then $v(x)$ is a supersolution of (1) on U_s^+ if and only if $L_P v(x)\geq 0$ on U_s^+. A *potential* p of (1) on U_s^+ is a positive supersolution of (1) in U_s^+ such that, if $p\geq u$ holds on U_s^+ for some solution u of (1) in U_s^+, then $u\leq 0$ on U_s^+. We take any point y fixed in U_s^+. By the *Green's function* $G_s(x, y)$ of (1) on U_s^+(with its pole y in U_s^+) we mean, if it exists, the potential of (1) on U_s^+ satisfying $L_P G_s(x, y)=\delta_y(x)$ on U_s^+, where $\delta_y(x)$ is the Dirac measure at y. The pair $(U_s^+, \mathscr{H}_P)$ with the sheaf $\mathscr{H}_P$ of solutions of (1) on U_s^+ is a Brelot's harmonic space. There exits a potential of (1) on U_s^+ if and only if there exists the Green's function $G_s(x, y)$ of (1) on U_s^+ ([2], [6], etc.).

Choose the negative radial densities $\tilde{Q}$ and $\tilde{R}$ on U given by

$$\tilde{Q}(x) \equiv -\frac{1}{4|x|^2}\left\{(m-2)^2+\frac{1}{\left(\log\frac{\eta}{|x|}\right)^2}+\frac{1}{\left(\log\frac{\eta}{|x|}\cdot\log\log\frac{\eta}{|x|}\right)^2}\right\} \tag{4}$$

and

$$\tilde{R}(x) \equiv -\frac{1}{4|x|^2}\left\{(m-2)^2+\frac{1}{\left(\log\frac{\eta}{|x|}\right)^2}+\frac{2}{\left(\log\frac{\eta}{|x|}\cdot\log\log\frac{\eta}{|x|}\right)^2}\right\}. \tag{5}$$

We set $\log_2|x|=\log\log|x|$ and $\log_3|x|=\log\log_2|x|$. Take the functions $\tilde{p}(x)$ and $\tilde{q}(x)$ given by

$$\tilde{p}(x) \equiv |x|^{-\frac{m-2}{2}}\left\{\log\frac{\eta}{|x|}\log_2\frac{\eta}{|x|}\right\}^{\frac{1}{2}},$$

$$\tilde{q}(x) \equiv \log_3\frac{\eta}{|x|}.$$

Consider the Schrödinger equations

$$L_{\tilde{Q}}u(x) \equiv (-\triangle+\tilde{Q}(x))u(x) = 0 \tag{6}$$

$$L_{\tilde{R}}u(x) \equiv (-\triangle+\tilde{R}(x))u(x) = 0 \tag{7}$$

on $\bar{U}_s$ with $0<s<1$.

Lemma 1 ([5]) *$\bar{p}(x)$ and $\bar{p}(x)\tilde{q}(x)$ are linearly independent solutions of* (6) *on U.*

Lemma 2 ([5]) $\dim(U_s, \tilde{R})=0$ *for any s in* $(0, 1]$.

We also use the following boundary Harnack principle ([1]):

Lemma 3 *Take any r in* $(0, s)$ *and an arbitrary a in* $(0, r)$ *with* $a+r<s$. *Let u and v be any positive solutions of* (1) *on* $U_{r+a}^{+}\setminus\bar{U}_{r-a}^{+}$ *which vanish continuously on* $\partial(U_{r+a}^{+}\setminus U_{r-a}^{+})\setminus(\Gamma_{r+a}^{+}\cup\Gamma_{r-a}^{+})$. *Then there exists a positive constant* $c>1$ *such that*

$$\frac{u(x)}{u(x')} \leq c\frac{v(x)}{v(x')}$$

holds for any u, v, x and x′ in Γ_r^{+}.

2. We denote by $\omega=(\omega_1, \cdots, \omega_m)$ the coordinates of the unit sphere Γ so that the spherical coordinates of a point $x\neq 0$ can be expressed as $r\omega$ with $r=|x|$ and $\omega=x/|x|$. The Laplacian $\triangle=\triangle_x=\triangle_{r\omega}$ is decomposed into the form

$$\triangle_x = \triangle_r + r^{-2}\triangle_\omega$$

where $\triangle_r=\partial^2/\partial r^2+(m-1)r^{-1}\partial/\partial r$ and $\triangle_\omega$ is the Laplace-Beltrami operator on Γ with respect to the natural Riemannian metric on Γ induced by the Euclidean metric on R^m. Since the coordinate function ω_m is a spherical harmonic of order one, we have $\triangle_\omega\omega_m=-(m-1)\omega_m$ on Γ(cf., e.g. [8]). We consider the function $p(x)$ on U^+ given by

$$p(x) = p(r\omega) \equiv \bar{p}(r)\omega_m.$$

Then it is easy to see that

$$\triangle p(x) = (\triangle_r + \frac{1}{r^2}\triangle_\omega)\bar{p}(r)\omega_m = (\triangle_r\bar{p}(r))\omega_m + \frac{1}{r^2}\bar{p}(r)\triangle_\omega\omega_m$$

on U^+. Since $\tilde{Q}(x)-(m-1)/|x|^2=Q(x)$ on $\bar{U}^+\setminus\{0\}$, the function $p(x)$ is a solution of $L_Q u=0$ on U^+ where $Q(x)$ is the density given by (2). Since

$\tilde{p}(x)\tilde{q}(x)$ is also a solution of (6) on U, by the same computation $\tilde{q}(x)p(x)$ is also a solution of $L_Q u=0$ on U^+.

Choose any s in $(0, 1]$ and take an arbitrary t fixed in $(0, s)$. We set

$$h(x) = \frac{\tilde{q}(x)-\tilde{q}(s)}{\tilde{q}(t)-\tilde{q}(s)}p(x)$$

which is a solution of $L_Q u=0$ on $U_s^+\backslash \bar{U}_t^+$ which coincides with $p(x)$ on Γ_t^+ and 0 on Γ_s^+. Observe that

$$\frac{\tilde{q}(x)-\tilde{q}(s)}{\tilde{q}(t)-\tilde{q}(s)} > 1 \quad (<1, \text{resp.})$$

for $|x|<t(>t, \text{resp.})$. In view of this we see that

$$h(x) > p(x) \quad (h(x)<p(x), \text{resp.})$$

for $|x|<t(>t, \text{resp.})$. Consider the function $v(x)$ given by $h(x)$ on $U_s^+\backslash \bar{U}_t^+$ and $p(x)$ on $\bar{U}_t^+$. Since

$$v(x) = \min(h(x), p(x)) \quad (x\in U_s^+),$$

$v(x)$ is a positive supersolution of $L_Q u=0$ on U_s^+. The unicity theorem assures that $v(x)$ is not a solution of $L_Q u=0$ on U_s^+ by virtue of the fact that $h(x)\neq p(x)$ on U_s^+. Hence by the Riesz decomposition theorem (cf., e.g. [2], [6]) there exists a potential and thus the Green's function of $L_Q u=0$ on U_s^+. Observe that $Q(x)=O(|x|^{-2})$ as $x\to 0$. Theorem 7.1 in [14] shows that $\dim(U_s^+, P)=1$ if there exists the Green's function of (1) on U_s^+ and $P(x)=O(|x|^{-2})$ as $x\to 0$. Therefore $\dim(U_s^+, Q)=1$ for any s in $(0, 1]$. We have shown:

Assertion *Let Q be the density on U^+ given by* (2). *Then* $\dim(U_s^+, Q)=1$ *for any s in* $(0, 1]$ *and hence* $\dim Q=1$.

3. *Proof of Theorem.* For the density Q given by (2) suppose that there exists a positive solution h in $cQP(U_s^+)$ for some constant $c>1$ and some s in $(0, 1]$. Consider the function $h^*(x)$ given by

$$h^*(x) = \int_{\Gamma^+} h(r\omega)\omega_m d\omega.$$

For $x=(x_1, \cdots, x_{m-1}, x_m)$ we denote $x_1, \cdots, x_{m-1}$ by x' so that x can be expressed as (x', x_m). We also take the function $\tilde{h}(x)$ given by

$$\tilde{h}(x) = \begin{cases} h(x', x_m) & \text{if } x_m>0 \\ -h(x', -x_m) & \text{if } x_m\leq 0. \end{cases}$$

Since the density cQ is radial, we may regard cQ as a density on U_s so that $\triangle\tilde{h}(x)=cQ(x)\tilde{h}(x)$ holds on U_s. Since $\tilde{h}(x',x_m)\omega_m=\tilde{h}(x',-x_m)(-\omega_m)$ for any $x=(x',x_m)$ in U_s, we have $2h^*(x)=\int_\Gamma \tilde{h}(r\omega)\omega_m d\omega$. Since Γ is compact, the Green's formula yields that

$$\int_\Gamma (\triangle_\omega \tilde{h}(r\omega)\omega_m - \tilde{h}(r\omega)\triangle_\omega\omega_m)d\omega = 0.$$

Also we observe that

$$2\triangle_r h^*(x) = \int_\Gamma \triangle_r \tilde{h}(r\omega)\omega_m d\omega = \int_\Gamma \{(\triangle - \frac{1}{r^2}\triangle_\omega)\tilde{h}(r\omega)\}\omega_m d\omega.$$

Therefore we have

$$\int_\Gamma \triangle\tilde{h}(r\omega)\omega_m d\omega = cQ(x)\int_\Gamma \tilde{h}(r\omega)\omega_m d\omega = 2cQ(x)h^*(x)$$

and

$$\begin{aligned}-\frac{1}{r^2}\int_\Gamma \triangle_\omega \tilde{h}(r\omega)\omega_m d\omega &= -\frac{1}{r^2}\int_\Gamma \tilde{h}(r\omega)\triangle_\omega\omega_m d\omega \\ &= \frac{m-1}{|x|^2}\int_\Gamma \tilde{h}(r\omega)\omega_m d\omega = 2\frac{m-1}{|x|^2}h^*(x).\end{aligned}$$

It follows from these identities that

$$\triangle_r h^*(x) = (cQ(x)+\frac{m-1}{|x|^2})h^*(x)$$

on U_s. For any densities $\tilde{S}(x)$ and $\tilde{T}(x)$ on U we write $\tilde{S}(x)\prec\tilde{T}(x)$ if there exists an s in $(0,1]$ such that $\tilde{S}(x)<\tilde{T}(x)$ on U_s. Observe that the relation

$$\begin{aligned}&4|x|^2(\log\frac{\eta}{|x|}\log_2\frac{\eta}{|x|})^2(\tilde{R}(x)-cQ(x)-\frac{m-1}{|x|^2}) \\ &\quad = (c-1)\{m^2(\log\frac{\eta}{|x|}\log_2\frac{\eta}{|x|})^2+(\log_2\frac{\eta}{|x|})^2+1\}-1 > 0\end{aligned}$$

is valid for any constant $c>1$ where $\tilde{R}$ is the density given by (5). Therefore we have $cQ(x)+(m-1)/|x|^2\prec\tilde{R}$ for any $c>1$. Since

$$\begin{aligned}L_{\tilde{R}}h^*(x) &= (-\triangle+cQ(x)+\frac{m-1}{|x|^2})h^*(x)+(\tilde{R}(x)-cQ(x)-\frac{m-1}{|x|^2})h^*(x) \\ &= (\tilde{R}(x)-cQ(x)-\frac{m-1}{|x|^2})h^*(x) > 0,\end{aligned}$$

there exists a t in $(0,1]$ such that $L_{\tilde{R}}h^*(x)>0$ on U_s for any s in $(0,t)$ so that h^* is a positive supersolution of (7) on U_s but not a solution of (7). Therefore there exists the Green's function of (7) on U_s. It is known ([3]) that $\dim(U_s, \tilde{P})=1$ whenever $\tilde{P}(x)=O(|x|^{-2})$ as $x\to 0$ and there exists the Green's function of $L_{\tilde{P}}u=0$ on U_s. Hence $\dim(U_s, \tilde{R})=1$. But this contradicts Lemma 2. Thus $\dim(U_s^+, cQ)=0$ for any $c>1$ and every s in $(0, 1]$ and a fortiori $\dim cQ=0$ for any $c>1$. From this and the above assertion the first part of the theorem follows.

We next consider the density R on U^+ given by (3) and suppose that there exists a positive solution u in $RP(U_s^+)$ for some s in $(0, 1]$. We set

$$u^*(x) = \int_{\Gamma^+} u(r\omega)\omega_m d\omega$$

and

$$\tilde{u}(x) = \begin{cases} u(x', x_m) & \text{if } x_m>0 \\ -u(x', -x_m) & \text{if } x_m\leq 0. \end{cases}$$

Then we have

$$2\triangle_r u^*(x) = \int_\Gamma \{(\triangle - \frac{1}{r^2}\triangle_\omega)\tilde{u}(r\omega)\}\omega_m d\omega.$$

The density $R(x)$ is radial so that we may consider $R(x)$ as a radial density on U. Then by the same method as in the proof of the first part of Theorem we deduce that

$$\int_\Gamma \triangle\tilde{u}(r\omega)\omega_m d\omega = 2R(x)u^*(x)$$

and

$$-\frac{1}{r^2}\int_\Gamma \triangle_\omega\tilde{u}(r\omega)\omega_m d\omega = -\frac{1}{r^2}\int_\Gamma \tilde{u}(r\omega)\triangle_\omega\omega_m d\omega = 2\frac{m-1}{|x|^2}u^*(x).$$

Since $u^*(x)$ is radial, we may regard $u^*(x)$ as a positive radial function on U_s. Therefore we have $\triangle_r u^*(x)=(R(x)+(m-1)/|x|^2)u^*(x)$ on U_s. Since $\tilde{R}(x)=R(x)+(m-1)/|x|^2$ on U_s, $u^*(x)$ is a positive radial solution of (7) on U_s. This contradicts Lemma 2. Therefore $\dim(U_s^+, R)=0$ for any s in $(0, 1]$.

To complete the proof of Theorem we only have to show that $\dim(cR)=1$ for any c in $(0, 1)$. For any densities $S(x)$ and $T(x)$ on U^+ we also write $S(x)< T(x)$ if there exists an s in $(0, 1]$ such that $S(x)< T(x)$ on U_s^+. We observe that the following relation is valid for any c in $(0, 1)$:

$$4|x|^2(\log\frac{\eta}{|x|}\log_2\frac{\eta}{|x|})^2(cR(x)-Q(x))$$
$$=(1-c)\{m^2(\log\frac{\eta}{|x|}\log_2\frac{\eta}{|x|})^2+(\log_2\frac{\eta}{|x|})^2+1\}-c>0.$$

Therefore we have $Q(x)<cR(x)$ for any c in $(0,1)$. By the Assertion there exists a positive solution $u(x)$ in $QP(U^+)$. Since we have

$$L_{cR}u(x)=L_Q u(x)+(cR(x)-Q(x))u(x)=(cR(x)-Q(x))u(x)>0,$$

there exists a $t\in(0,1)$ such that $L_{cR}u(x)>0$ on U_s^+ for any $s\in(0,t)$ so that $u(x)$ is a positive supersolution but not a solution of $L_{cR}u=0$ in U_s^+. Also $cR(x)=O(|x|^{-2})$ as $x\to 0$. Therefore Theorem 7.1 in [14] yields that $\dim(U_s^+, cR)=1$ for any s in $(0,t)$ and a fortiori $\dim cR=1$ for any c in $(0,1)$. The proof of Theorem is herewith complete.

4. *Proof of Proposition.* The proposition will be shown by the minor modification of the method in [12] and [13] where it was shown that the existence of a t such that there exists a bijective positive linear mapping of $\tilde{P}P(U_t)$ onto $\tilde{P}P(U_s)$ for any s in $(0,t)$.

We denote by $\hat{C}(\bar{\Gamma}_t^+)$ the space of all continuous functions φ on the closure $\bar{\Gamma}_t^+$ of Γ_t^+ with $\varphi(x',0)=0$ and for each φ in $\hat{C}(\bar{\Gamma}_t^+)$ we set

$$\tilde{\varphi}(x)=\begin{cases}\varphi(x',x_m) & \text{if } x_m>0\\ -\varphi(x',-x_m) & \text{if } x_m\leq 0.\end{cases}$$

Then $\tilde{\varphi}$ is in the space $\hat{C}(\Gamma_t)$ of all continuous functions ψ on Γ_t which satisfy $\psi(x',-x_m)=-\psi(x',x_m)$ for each $x=(x',x_m)\in\Gamma_t$. Conversely if ψ is in $\hat{C}(\Gamma_t)$, then $\psi|_{\bar{\Gamma}_t^+}$ is in $\hat{C}(\bar{\Gamma}_t^+)$. Therefore $\hat{C}(\bar{\Gamma}_t^+)$ is the restriction of $\hat{C}(\Gamma_t)$ to $\bar{\Gamma}_t^+$. The space $\hat{C}(\Gamma_t)$ is a closed subspace of the Banach space $C(\Gamma_t)$ of all continuous functions on Γ_t equipped with the sup-norm on Γ_t. Therefore $\hat{C}(\bar{\Gamma}_t^+)$ may be regarded as a Banach space for any t in $(0,1]$.

If $PP(U_t^+)=\{0\}$ for any $0<t<1$, then the proposition trivially holds. If $PP(U_{t_0}^+)\neq\{0\}$ for some t_0 in $(0,1]$, then there exists a positive solution h in $PP(U_{t_0}^+)$. Take any t in $(0,t_0)$. Then we have $h>0$ on $U_t^+\cup\Gamma_t^+$. We choose any s fixed in $(0,t)$ and any r in $(0,s)$. Denote by $D_{s,r}h$ the solution of (1) on $U_s^+\backslash\bar{U}_r^+$ with boundary values h on Γ_s^+ and zero on $\partial(U_s^+\backslash U_r^+)\backslash\Gamma_s^+$. Then the minimum principle yields that $h\geq D_{s,r}h$ on $U_s^+\backslash\bar{U}_r^+$ for every r in $(0,s)$. Hence we have $h\geq D_s h\equiv\lim_{r\downarrow 0}D_{s,r}h$ on U_s^+. We also denote by $K_s h$ the solution of (1) on $U_t^+\backslash\bar{U}_s^+$ with boundary values h on Γ_s^+ and zero on $\partial(U_t^+\backslash U_s^+)\backslash\Gamma_s^+$. Then $K_s h<h$ on $U_t^+\backslash\bar{U}_s^+$. Setting $v(x)=D_s h$ on U_s^+ and $v(x)=K_s h$ on $U_t^+\backslash U_s^+$, $v(x)$ is a positive supersolution of

(1) but not a solution of (1) on U_t^+. Therefore there exists the Green's function of (1) on U_t^+ for any t in $(0, t_0)$. We fix any such t in $(0, t_0)$ and take any s in $(0, t)$.

For any u in $PP(U_t^+)$, we set

$$\tau u \equiv u - D_s u. \tag{8}$$

Then we have $u - D_s u \geq 0$ on U_s^+. The mapping τ given by (8) is a positive, homogeneous and additive operator of $PP(U_t^+)$ into $PP(U_s^+)$.

We now show that τ is injective, i.e. if $\tau u = \tau v$ on U_s^+ fo some u, v in $PP(U_t^+)$, then $w \equiv u - v = 0$ on U_t^+. For this it sufficies to show that $w = 0$ on Γ_s^+ by the minimum principle. Suppose that $w \neq 0$ on Γ_s^+. Considering $-w$ instead of w if necessary, we assume that $\sup_{\Gamma_s^+} w > 0$. Then there exists a point x_s^0 in Γ_s^+ with $w(x_s^0) > 0$. We set $c \equiv \inf\{\lambda \in R : \lambda h \geq w$ on $\Gamma_s^+\}$. Since $u + v > w$ on Γ_s^+, c is a positive finite constant by Lemma 3. Also since $ch - w \geq 0$ on $\partial(U_t^+ \backslash U_s^+)$, the minimum principle yields that $ch - w > 0$ on $U_t^+ \backslash \bar{U}_s^+$. Owing to the identity $w = D_{s,r} w$ on Γ_s^+, $ch - D_{s,r} w \geq 0$ is valid on $\partial(U_s^+ \backslash U_r^+)$ and hence on $U_s^+ \backslash \bar{U}_r^+$. As $r \to 0$ we obtain that $ch - D_s w \geq 0$ on U_s^+. Also, since $\tau u = \tau v$ on U_s^+, the identity $w = D_s w$ on U_s^+ implies that $ch - w \geq 0$ on U_s^+. Therefore $ch - w \geq 0$ on U_t^+. The minimum principle yields that $ch - w > 0$ on U_t^+. Applying Lemma 3 to solutions $ch - w$ and h, there exists a constant $c_1 > 1$ such that $(h(x)/h(x_s^0)) \leq c_1((ch(x) - w(x))/(ch(x_s^0) - w(x_s^0)))$ on Γ_s^+. Hence

$$w \leq c\left(1 - \frac{1}{c_1}\left(1 - \frac{w(x_s^0)}{ch(x_s^0)}\right)\right)h$$

on Γ_s^+. But this contradicts the definition of c. Thus we have $w(x) = 0$ on Γ_s^+ and a fortiori τ is injective.

We next show that τ is surjective. We show that there exists a function u in $PP(U_t^+)$ with $\tau u = v$ for any v in $PP(U_s^+)$. Take an r in $(0, s)$ and for a given φ in $\hat{C}(\bar{\Gamma}_r^+)$ consider the solution $K\varphi$ of (1) on $U_t^+ \backslash \bar{U}_r^+$ with boundary values φ on Γ_r^+ and zero on $\partial(U_t^+ \backslash U_r^+) \backslash \Gamma_r^+$. Then K is a linear and order-preserving mapping of $\hat{C}(\bar{\Gamma}_r^+)$ into the class of solutions of (1) on $U_t^+ \backslash \bar{U}_r^+$ with boundary values zero on $\partial(U_t^+ \backslash U_r^+) \backslash \Gamma_r^+$.

For any φ in $\hat{C}(\bar{\Gamma}_r^+)$ we consider the operator T given by

$$T\varphi = D_s(K\varphi|_{\Gamma_s^+}).$$

Then T is a linear operator of $\hat{C}(\bar{\Gamma}_r^+)$ into itself which is also order-preserving.

We fist suppose that the equation

$$\varphi - T\varphi = v \quad \text{on} \quad \Gamma_r^+ \tag{9}$$

is solved by a function φ in $\hat{C}(\bar{\Gamma}_r^+)$ with $\varphi \geq 0$ on Γ_r^+ for a given v in $PP(U_s^+)$. We set

$$u = \begin{cases} K\varphi & \text{on} \quad U_t^+ \backslash U_r^+ \\ D_sK\varphi + v & \text{on} \quad \bar{U}_s^+. \end{cases}$$

We observe that $K\varphi - (D_sK\varphi + v)$ is equal to $\varphi - (T\varphi + v) = 0$ on Γ_r^+ in view of (9) and is equal to $K\varphi - (K\varphi + 0) = 0$ on Γ_s^+. Therefore $K\varphi - (D_sK\varphi + v)$ is a solution of (1) on $U_s^+ \backslash \bar{U}_r^+$ with boundary values zero on $\partial(U_s^+ \backslash U_r^+)$. Hence $K\varphi = D_sK\varphi + v$ on $U_s^+ \backslash \bar{U}_r^+$. Therefore u is a well defined solution of (1) on U_t^+. Since $K\varphi = u$ on Γ_s^+, $D_sK\varphi = D_su$ on U_s^+. Thus we have $u - D_su = v$, i.e. $\tau u = v$ on U_s^+. Hence τ is surjective.

It remains to solve the integral equation (9) for a given $v \in PP(U_s^+)$. We set $c = \inf\{c_0 > 0 : c_0h \geq v \text{ on } \Gamma_r^+\}$ which is finite and positive by Lemma 3. Then $ch \geq v$ on Γ_r^+. Since $h > 0$ on Γ_t^+, $h > Kh$ on $U_t^+ \backslash \bar{U}_r^+$ in view of the minimum principle. In particular we have $h > Kh$ on Γ_s^+. This inequality yields that $h \geq D_sh > D_sKh$ on U_s^+. Again applying Lemma 3 to solutions $h - D_sKh$ and h, there exists a constant $c_1 > 1$ such that $h \leq c_1(h - Th)$ on Γ_r^+. Therefore $Th \leq (1 - 1/c_1)h$ on Γ_r^+ and a fortiori we have

$$q \equiv \sup_{\Gamma_r^+} \frac{Th(x)}{h(x)} < 1.$$

From this it follows that $q^nh \geq T^nh$ on Γ_r^+ for any positive integer n. Also T is order-preserving so that the inequality $ch \geq v$ on Γ_r^+ implies that $cTh \geq Tv$ on Γ_r^+. Therefore the inequalities $q^nc\|h\| \geq q^nch \geq cT^nh \geq T^nv$ are valid where $\|\cdot\|$ is the sup-norm on Γ_r^+. This implies that $\|T^nv\| \leq c\|h\|q^n$. Therefore $\varphi = \sum_{n=0}^{\infty} T^nv$ has $\sum_{n=0}^{\infty} c\|h\|q^n$ as its majorant series and a fortiori $\varphi \in \hat{C}(\bar{\Gamma}_r^+)$ with $\varphi \geq 0$ on Γ_r^+.

References

[1] A. ANCONA, *Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschitzien,* Ann. Inst. Fourier, Grenoble, **28**(1978), 169-213.

[2] C. CONSTANTINESCU and A. CORNEA, *Potential Theory on Harmonic Spaces*, Springer-Verlag, 1972.

[3] H. IMAI, *Picard principle for linear elliptic differential operators*, Hiroshima Math. J., **14**(1985), 527-535.

[4] H. IMAI, *On Picard dimensions of nonpositive densities in Schrödinger equations,* Complex Variables, (to appear).
[5] H. IMAI, *Nonhomogeneity of Picard dimensions for negative radial densities*, Hiroshima Math. J., (to appear).
[6] F.-Y. MAEDA, *Dirichlet Integral on Harmonic Spaces*, Lecture Notes in Math., **803**, Springer-Verlag, 1980.
[7] M. KAWAMURA and M. NAKAI, *A test for Picard principle for rotation free densities, II*, J. Math. Soc. Japan, **14**(1976), 323-341.
[8] C. MÜLLER, *Spherical Harmonics*, Lecture Notes in Math., **17**, Springer-Verlag, 1966.
[9] M. MURATA, *Isolated singularities and positive solutions of elliptic equations in* R^n, Preprint Series, **14**(1986/1987), 1-39. Matematisk Institut, Aarhus Univerisitet.
[10] M. MURATA, *On construction of Martin boundaries for second order elliptic equations*, Publ. RIMS, Kyoto Univ., **26**(1990), 585-627.
[11] M. NAKAI, *A test for Picard principle,* Nagoya Math. J., **56**(1974), 105-119.
[12] M. NAKAI, *Picard principle and Riemann theorem*, Tôhoku Math. J., **28**(1976), 277-292.
[13] M. NAKAI and T. TADA, *Monotoneity and homogeneity of Picard dimensions for signed radial densities*, NIT Sem. Rep. Math., **99**(1993), 1-51.
[14] Y. PINCHOVER, *On positive Liouville theorems and asymptotic behavior of solutions of Fuchsian type elliptic operators*, Preprint.
[15] T. TADA, *The Martin boundary of the half disk with rotation free densities*. Hiroshima Math. J., **16**(1986), 315-325.
[16] T. TADA, *Nonhomogeneity of Picard dimensions of rotation free hyperbolic densities*, Hiroshima Math J., (to appear).

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