Hokkaido Mathematical Journal Vol. 42 (2013) p. 1–9

A note on extreme norms on $\mathbb{R}^2$

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(Received January 13, 2011; Revised January 13, 2012)

Abstract. We denote by AN_2 the set of all absolute normalized norms on $\mathbb{R}^2$. It is known that the set AN_2 and the set of all continuous convex functions ψ on $[0,1]$ with $\max\{1-t,t\} \le \psi(t) \le 1$ for $t \in [0,1]$ (denoted by Ψ_2) are in a one to one correspondence under the equation $\psi(t) = \|(1-t,t)\|$. Recently, we characterized extreme points of AN_2 by considering Ψ_2. In this paper we give another proof of this result.

Key words: absolute normalized norm, extreme point.

1. Introduction and preliminaries

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *absolute* if $\|(x_1,x_2)\| = \|(|x_1|,|x_2|)\|$ for all $x_1,x_2 \in \mathbb{R}$, and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p$ $(1 \le p \le \infty)$ are basic examples:

$$\|(x_1,x_2)\|_p = \begin{cases} (|x_1|^p + |x_2|^p)^{1/p} & \text{if} \quad 1 \le p < \infty, \\ \max\{|x_1|,|x_2|\} & \text{if} \quad p = \infty. \end{cases}$$

Let AN_2 be the family of all absolute normalized norms on $\mathbb{R}^2$.

Let Ψ_2 be the set of all continuous convex functions on the interval $[0,1]$ satisfying $\max\{1-t,t\} \le \psi(t) \le 1$ for $t \in [0,1]$. Then by [1], AN_2 and Ψ_2 are in a one-to-one correspondence with $\psi(t) = \|(1-t,t)\|$ for $t \in [0,1]$ and

$$\|(x_1,x_2)\|_\psi = \begin{cases} (|x_1|+|x_2|)\psi\left(\frac{|x_2|}{|x_1|+|x_2|}\right) & \text{if} \quad (x_1,x_2) \ne (0,0), \\ 0 & \text{if} \quad (x_1,x_2) = (0,0) \end{cases}$$

(see also [5], [7], [6]). For ℓ_p-norm $\|\cdot\|_p$, the corresponding convex function

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2000 Mathematics Subject Classification : 46B20, 46B25.

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The first author was supported in part by Grant-in-Aid for Scientific Research (C), 23540189, Japan Society for the Promotion of Science.

ψ_p is

$$\psi_p(t) = \begin{cases} ((1-t)^p + t^p)^{1/p} & \text{if} \quad 1 \le p < \infty, \\ \max\{1-t, t\} & \text{if} \quad p = \infty. \end{cases}$$

We consider the convex structure of the set AN_2 in the sense that

$$\|\cdot\|,\ \|\cdot\|' \in AN_2,\ \lambda \in [0,1] \Rightarrow (1-\lambda)\|\cdot\| + \lambda\|\cdot\|' \in AN_2.$$

Note that Ψ_2 also has its own convex structure, and the correspondence $\psi \to \|\cdot\|_\psi$ preserves the operation to take a convex combination. A norm $\|\cdot\| \in AN_2$ is an *extreme point* of AN_2 if

$$\|\cdot\| = \frac{1}{2}(\|\cdot\|' + \|\cdot\|''),\ \|\cdot\|', \|\cdot\|'' \in AN_2 \Rightarrow \|\cdot\|' = \|\cdot\|''.$$

The definition of extreme point of Ψ_2 is similar to that of AN_2. It is easy to see that ψ is an extreme point of Ψ_2 if and only if $\|\cdot\|_\psi$ is an extreme point of AN_2 (see [4]).

As in [4], we determined the set of all extreme points of AN_2 by considering the set Ψ_2. After that, the authors were informed by Professor P. N. Dowling about the result of R. Grząślewicz [2], which solved a problem posed by Professor A. Pietsch at the Winter School on Functional Analysis in January 1978 (cf. [8]). The method in [4] is different from that of R. Grząślewicz [2].

The main results are stated as follows. Let $\psi'_L(1/2)$ (resp. $\psi'_R(1/2)$) be the left (resp. the right) derivative of ψ at $t = 1/2$.

Theorem 1 *Let $\psi \in \Psi_2$. We define a function φ as $\varphi = 2\psi - \psi_\infty$.*

(i) *If $\psi'_R(1/2) \ge \psi'_L(1/2) + 1$, then $\varphi \in \Psi_2$ and*

$$\psi = \frac{\varphi + \psi_\infty}{2}.$$

(ii) *Let $\psi'_R(1/2) < \psi'_L(1/2) + 1$. Then $\varphi \notin \Psi_2$. However, we can find a function $\varphi_0 \in \Psi_2$ with*

$$\varphi_0(t) = \begin{cases} \varphi(t) & \text{if } \ t \in [0, s_0] \cup [t_0, 1], \\ \frac{\varphi(t_0)-\varphi(s_0)}{t_0-s_0} t + \frac{\varphi(s_0)t_0-\varphi(t_0)s_0}{t_0-s_0} & \text{if } \ t \in [s_0, t_0] \end{cases}$$

for some $s_0 \in [0, 1/2]$ *and* $t_0 \in (1/2, 1]$. *Moreover, putting a function* $\varphi_{\max} = 2\psi - \varphi_0$ *we have* $\varphi_{\max} \in \Psi_2$ *and*

$$\psi = \frac{\varphi_0 + \varphi_{\max}}{2}.$$

For $0 \le \alpha \le \frac{1}{2} \le \beta \le 1$ and for the case $(\alpha, \beta) \neq (\frac{1}{2}, \frac{1}{2})$ we define

$$\psi_{\alpha,\beta}(t) = \begin{cases} 1 - t & \text{if } \ 0 \le t \le \alpha, \\ \frac{\alpha+\beta-1}{\beta-\alpha} t + \frac{\beta-2\alpha\beta}{\beta-\alpha} & \text{if } \ \alpha \le t \le \beta, \\ t & \text{if } \ \beta \le t \le 1. \end{cases}$$

For the case $(\alpha, \beta) = (\frac{1}{2}, \frac{1}{2})$ we put $\psi_{1/2,1/2} = \psi_\infty$. We clearly have $\psi_{\alpha,\beta} \in \Psi_2$ for all α, β with $0 \le \alpha \le \frac{1}{2} \le \beta \le 1$ and the corresponding norm is

$$\|(x_1, x_2)\|_{\psi_{\alpha,\beta}} = \begin{cases} |x_1| & \text{if } \ |x_2| \le \frac{\alpha}{1-\alpha}|x_1|, \\ \frac{\beta(1-2\alpha)}{\beta-\alpha}|x_1| + \frac{(2\beta-1)(1-\alpha)}{\beta-\alpha}|x_2| & \\ & \text{if } \ \frac{\alpha}{1-\alpha}|x_1| < |x_2|, \ \frac{1-\beta}{\beta}|x_2| < |x_1|, \\ |x_2| & \text{if } \ |x_1| \le \frac{1-\beta}{\beta}|x_2|. \end{cases}$$

Theorem 2 ([3], [4]) *Let* $\psi \in \Psi_2$. *Then the following are equivalent*:

(i) ψ *is an extreme point of* Ψ_2,
(ii) $\|\cdot\|_\psi$ *is an extreme point of* AN_2,
(iii) *There exist* α, β *with* $0 \le \alpha \le 1/2 \le \beta \le 1$ *such that* $\psi = \psi_{\alpha,\beta}$ *(resp.* $\|\cdot\|_\psi = \|\cdot\|_{\psi_{\alpha,\beta}}$*).*

Note that the equivalence of (ii) and (iii) in Theorem 2 is essentially the same as the next result given by R. Grząślewicz [2].

Corollary 3 ([2]) *Let* $\|\cdot\| \in AN_2$. *Then* $\|\cdot\|$ *is an extreme point of* AN_2 *if and only if all extreme points of the unit ball of* $(\mathbb{R}^2, \|\cdot\|)$ *are contained in the unit sphere of* $(\mathbb{R}^2, \|\cdot\|_\infty)$.

2. Proof of Theorem 1

Note that

$$\varphi(t) = \begin{cases} 2\psi(t) - 1 + t & \text{if } \ 0 \le t \le 1/2, \\ 2\psi(t) - t & \text{if } \ 1/2 \le t \le 1. \end{cases}$$

It is clear that φ is a convex function on $[0, 1/2]$ (resp. $[1/2, 1]$). For each $t \in (0, 1]$, we denote by $\varphi'_L(t)$ (resp. $\psi'_L(t)$) the left derivative of φ (resp. of ψ) at t. Similarly, for each $t \in [0, 1)$, we denote by $\varphi'_R(t)$ (resp. $\psi'_R(t)$) the right derivative of φ (resp. of ψ) at t. Since $\varphi(0) = 1$ and $\varphi(t) \ge \psi_\infty(t) = 1 - t$ for $t \in [0, 1/2]$, we have

$$\varphi'_R(0) = \lim_{t \to +0} \frac{\varphi(t) - \varphi(0)}{t} \ge \lim_{t \to +0} \frac{1 - t - 1}{t} = -1.$$

Since $\varphi(1) = 1$ and $\varphi(t) \ge \psi_\infty(t) \ge t$ for all $t \in [1/2, 1]$, we also have

$$\varphi'_L(1) = \lim_{t \to -0} \frac{\varphi(1 + t) - \varphi(1)}{t} \le \lim_{t \to -0} \frac{1 + t - 1}{t} = 1.$$

Hence, if $0 < s < t < 1/2$, then

$$-1 \le \varphi'_R(0) \le \varphi'_R(s) \le \varphi'_L(t) \le \varphi'_R(t) \le \varphi'_L(1/2) = 2\psi'_L(1/2) + 1,$$

and if $1/2 < s < t < 1$, then

$$2\psi'_R(1/2) - 1 = \varphi'_R(1/2) \le \varphi'_R(s) \le \varphi'_L(t) \le \varphi'_R(t) \le \varphi'_L(1) \le 1.$$

We define a mapping G from $[0, 1]$ into the subsets of $\mathbb{R}$ as

$$G(t) = \begin{cases} [-1, \varphi'_R(0)] & \text{if } \ t = 0, \\ [\varphi'_L(t), \varphi'_R(t)] & \text{if } \ 0 < t < 1/2, \ 1/2 < t < 1, \\ \{\varphi'_L(1/2)\} & \text{if } \ t = 1/2, \\ [\varphi'_L(1), 1] & \text{if } \ t = 1. \end{cases}$$

Let us give a necessary and sufficient condition of ψ that φ is convex on $[0, 1]$. Note that φ is convex on $[0, 1]$ if and only if $\varphi'_L(1/2) \le \varphi'_R(1/2)$.

Therefore we clearly have the following lemma.

Lemma 4 *φ is convex on $[0,1]$ if and only if $\psi'_R(1/2) \geq \psi'_L(1/2)+1$.*

We consider the case $\psi'_R(1/2) \geq \psi'_L(1/2)+1$. Since φ is convex on $[0,1]$ and $\varphi(0)=\varphi(1)=1$, we have $\varphi(t) \leq 1$ for all $t \in [0,1]$. Moreover, we have $\varphi = 2\psi - \psi_\infty \geq \psi_\infty$ by $\psi \geq \psi_\infty$. Thus $\varphi \in \Psi_2$.

We next suppose that $\psi'_R(1/2) < \psi'_L(1/2)+1$. By Lemma 4, φ is not convex on $[0,1]$ and hence $\varphi \notin \Psi_2$. From $\varphi'_L(1/2) > \varphi'_R(1/2)$, there exists a real number $a \in (1/2, 1]$ such that

$$\varphi'_L(1/2) > \frac{\varphi(a)-\varphi(1/2)}{a-1/2}. \tag{2.1}$$

The following lemma is easy and so the proof is omitted.

Lemma 5 (i) *Let $0 \leq t_1 < t_2 \leq 1/2$. If $\lambda_1 \in G(t_1)$ and $\lambda_2 \in G(t_2)$, then $\lambda_1 \leq \lambda_2$.*
(ii) *Let $0 < s_0 < 1/2$ and let $\lambda_n \in G(s_n)$ for each n. Then $\lambda_n \nearrow \varphi'_L(s_0)$ if $s_n \nearrow s_0$, and $\lambda_n \searrow \varphi'_R(s_0)$ if $s_n \searrow s_0$.*

Put $A = \{(s,\lambda) : 0 \leq s \leq 1/2, \lambda \in G(s)\}$. Then we have

Lemma 6 *The set A is a compact connected subset of $[0,1/2] \times [-1, \varphi'_L(1/2)]$.*

Proof. By Lemma 5(ii) A is closed, which implies that A is compact. To prove that A is connected, it is enough to show that A is homeomorphic to the closed interval $I = [-1, 1/2+\varphi'_L(1/2)]$. Define a mapping f from A into I as $f(s,\lambda) = s+\lambda$ for $(s,\lambda) \in A$. Then f is injective by Lemma 5(i). We show that f is surjective. For each $t \in I$, put

$$s_t = \sup\{s \in [0,1/2] : s+\lambda \leq t \text{ for some } \lambda \in G(s)\}.$$

Then it follows from Lemma 5(ii) that $\min G(s_t) \leq t - s_t \leq \max G(s_t)$, and so $t - s_t \in G(s_t)$. Putting $\lambda_t = t - s_t$, we have $t = s_t + \lambda_t$ and $(s_t, \lambda_t) \in A$. Hence f is surjective. Since f is one to one continuous and A is compact, f is a homeomorphism from A onto I. Thus A is homeomorphic to the closed interval I, which completes the proof. $\square$

The following is a key lemma to prove Theorem 1.

Lemma 7 *There exist* $s_0 \in [0, 1/2]$ *and* $t_0 \in [a, 1]$ *such that*

$$\lambda_0 = \frac{\varphi(t_0) - \varphi(s_0)}{t_0 - s_0} \in G(s_0) \cap G(t_0).$$

Proof. Let $\Omega = A \times [a, 1]$. Then Ω is compact and connected by Lemma 6. For any $(s, \lambda, t) \in \Omega$, we define $F(s, \lambda, t) = \varphi(t) - \varphi(s) - \lambda(t - s)$. For any $t \in [a, 1]$ we have $(0, -1, t) \in \Omega$ and $F(0, -1, t) > 0$. Also, $(1/2, \varphi'_L(1/2), a) \in \Omega$ and $F(1/2, \varphi'_L(1/2), a) < 0$ by (2.1). Since Ω is connected and F is continuous on Ω, there exists an element $(s, \lambda, t) \in \Omega$ with $F(s, \lambda, t) = 0$. Put $C = \{(s, \lambda, t) \in \Omega : F(s, \lambda, t) = 0\}$. Since C is compact, we put $s_0 = \min\{s : (s, \lambda, t) \in C\}$, $\lambda_0 = \min\{\lambda : (s_0, \lambda, t) \in C\}$ and $t_0 = \min\{t : (s_0, \lambda_0, t) \in C\}$, respectively. Note

$$\lambda_0 = \frac{\varphi(t_0) - \varphi(s_0)}{t_0 - s_0} \in G(s_0).$$

We show $F(s_0, \lambda_0, t) \geq 0$ for all $t \in [a, 1]$. Let $t_1 \in [a, 1]$. Let $s_0 > 0$ and $0 < s_1 < s_0$. Put $A_0 = \{(s, \lambda) : 0 \leq s \leq s_1, \lambda \in G(s)\}$ and $B_0 = \{(s, \lambda) \in A_0 : F(s, \lambda, t_1) > 0\}$. As in the proof of Lemma 6, A_0 is connected. By the definition of s_0, $F(s, \lambda, t_1) \neq 0$ for all $(s, \lambda) \in A_0$. Hence B_0 is an open and closed set of A_0. Also, $B_0 \neq \emptyset$ by $F(0, -1, t_1) > 0$. Hence B_0 coincides with A_0. Thus $F(s, \lambda, t_1) > 0$ for all $(s, \lambda) \in A_0$. Let $s_n \nearrow s_0$ and let $\lambda_n \in G(s_n)$. By Lemma 5(ii) we have $\lambda_n \nearrow \varphi'_L(s_0)$ and so $F(s_0, \varphi'_L(s_0), t_1) \geq 0$. If $\varphi'_L(s_0) = \lambda_0$, then $F(s_0, \lambda_0, t_1) \geq 0$. If $\varphi'_L(s_0) < \lambda_0$, then $F(s_0, \varphi'_L(s_0), t_1) > 0$ by the definition of λ_0. Assume $F(s_0, \lambda_0, t_1) < 0$. Then $F(s_0, \lambda, t_1) = 0$ for some λ with $\varphi'_L(s_0) < \lambda < \lambda_0$, which contradicts the definition of λ_0. Hence $F(s_0, \lambda_0, t_1) \geq 0$. Let $s_0 = 0$. Assume $F(0, \lambda_0, t_1) < 0$. By $F(0, -1, t_1) > 0$ there exists $\lambda'_0 \in (-1, \lambda_0)$ such that $F(0, \lambda'_0, t_1) = 0$, which is a contradiction. Hence $F(0, \lambda_0, t_1) \geq 0$. Thus $F(s_0, \lambda_0, t) \geq 0$ for all $t \in [a, 1]$.

Finally we show $\lambda_0 \in G(t_0)$. Let $t \in [a, 1]$. By $F(s_0, \lambda_0, t) \geq F(s_0, \lambda_0, t_0)$ we obtain $\varphi(t) - \varphi(t_0) - \lambda_0(t - t_0) \geq 0$. This implies $\varphi'_L(t_0) \leq \lambda_0 \leq \varphi'_R(t_0)$ and thus $\lambda_0 \in G(t_0)$, which completes the proof. □

For s_0 and t_0 given in Lemma 7, we define a function φ_0 on $[0, 1]$ as

$$\varphi_0(t) = \begin{cases} \varphi(t) & \text{if } \ t \in [0, s_0] \cup [t_0, 1], \\ \frac{\varphi(t_0)-\varphi(s_0)}{t_0 - s_0} t + \frac{\varphi(s_0)t_0 - \varphi(t_0)s_0}{t_0 - s_0} & \text{if } \ t \in [s_0, t_0]. \end{cases}$$

Let us show that φ_0 is convex on $[0, 1]$. Note that φ_0 is convex on $[0, 1]$ if and only if $(\varphi_0)'_L(t) \le (\varphi_0)'_R(t)$ for $t = s_0, t_0$. By Lemma 7,

$$(\varphi_0)'_L(s_0) = \varphi'_L(s_0) \le \lambda_0 = (\varphi_0)'_R(s_0)$$

and

$$(\varphi_0)'_L(t_0) = \lambda_0 \le \varphi'_R(t_0) = (\varphi_0)'_R(t_0).$$

Hence φ_0 is convex on $[0, 1]$ and so $\varphi_0 \in \Psi_2$. If $s_0 = 1/2$, then since φ is convex on $[1/2, 1]$ and from $\varphi'_L(t_0) \le \lambda_0$, we have $\varphi(t) = \varphi_0(t)$ for $t \in [1/2, t_0]$, which is a contradiction. Thus $s_0 \ne 1/2$. Putting $\varphi_{\max}(t) = 2\psi(t) - \varphi_0(t)$, we have

$$\varphi_{\max}(t) = \begin{cases} \psi_\infty(t) & \text{if } \ t \in [0, s_0] \cup [t_0, 1], \\ 2\psi(t) - \varphi_0(t) & \text{if } \ t \in [s_0, t_0]. \end{cases}$$

Note that $\varphi_{\max}$ is convex if and only if $(\varphi_{\max})'_L(t) \le (\varphi_{\max})'_R(t)$ for $t = s_0, t_0$. By Lemma 7,

$$\begin{aligned} (\varphi_{\max})'_L(s_0) = (\psi_\infty)'_L(s_0) &= -1 \le \varphi'_R(s_0) - \lambda_0 - 1 \\ &= 2\psi'_R(s_0) - \lambda_0 = (\varphi_{\max})'_R(s_0) \end{aligned}$$

and

$$\begin{aligned} (\varphi_{\max})'_L(t_0) = 2\psi'_L(t_0) - \lambda_0 &= \varphi'_L(t_0) + 1 - \lambda_0 \le 1 \\ &= (\psi_\infty)'_R(t_0) = (\varphi_{\max})'_R(t_0). \end{aligned}$$

Hence $\varphi_{\max}$ is convex on $[0, 1]$ and so $\varphi_{\max} \in \Psi_2$. Thus this completes the proof of Theorem 1.

Example 8 We consider the case $\psi = \psi_2$. It is clear that $(\psi_2)'_L(1/2) = (\psi_2)'_R(1/2)$. Hence it follows from Lemma 4 that $\varphi(= 2\psi_2 - \psi_\infty)$ is not convex. Put $s_0 = \frac{1}{2} - \frac{\sqrt{7}}{14}$ and $t_0(= 1 - s_0) = \frac{1}{2} + \frac{\sqrt{7}}{14}$. Note that since φ

is symmetric to $t = 1/2$, we have $\varphi(s_0) = \varphi(t_0)$. Easy calculation shows that φ has local minimums at $t = s_0, t_0$ and hence $\lambda_0 = \frac{\varphi(s_0)-\varphi(t_0)}{s_0-t_0} = 0 \in G(s_0) \cap G(t_0)$. For s_0, t_0 and λ_0, two functions φ_0 and $\varphi_{\max}$ are the following:

$$\varphi_0(t) = \begin{cases} 2((1-t)^2+t^2)^{1/2} - 1 + t & \text{if } \ 0 \le t \le \frac{1}{2} - \frac{\sqrt{7}}{14}, \\ -\frac{1}{2} + \frac{\sqrt{7}}{2} & \text{if } \ \frac{1}{2} - \frac{\sqrt{7}}{14} \le t \le \frac{1}{2} + \frac{\sqrt{7}}{14}, \\ 2((1-t)^2+t^2)^{1/2} - t & \text{if } \ \frac{1}{2} + \frac{\sqrt{7}}{14} \le t \le 1 \end{cases}$$

and

$$\varphi_{\max}(t) = \begin{cases} 1-t & \text{if } \ 0 \le t \le \frac{1}{2} - \frac{\sqrt{7}}{14}, \\ 2((1-t)^2+t^2)^{1/2} + \frac{1}{2} - \frac{\sqrt{7}}{2} & \text{if } \ \frac{1}{2} - \frac{\sqrt{7}}{14} \le t \le \frac{1}{2} + \frac{\sqrt{7}}{14}, \\ t & \text{if } \ \frac{1}{2} + \frac{\sqrt{7}}{14} \le t \le 1. \end{cases}$$

3. Proof of Theorem 2

The equivalence of (i) and (ii) is clear (see [4]). (i) $\Rightarrow$ (iii). Suppose that ψ is an extreme point of Ψ_2. If $\psi'_R(1/2) \ge \psi'_L(1/2) + 1$, then we have by Theorem 1, $\psi = \frac{\varphi+\psi_\infty}{2}$ and $\psi = \varphi = \psi_\infty = \psi_{1/2,1/2}$. If $\psi'_R(1/2) < \psi'_L(1/2) + 1$, then we have by Theorem 1, $\psi = (\varphi_0 + \varphi_{\max})/2$ and so $\psi = \varphi_{\max} = \varphi_0$. This implies that $\psi = \psi_{s_0,t_0}$. We may put $\alpha = s_0$ and $\beta = t_0$, respectively.

(iii) $\Rightarrow$ (i). Suppose that $\psi = \psi_{\alpha,\beta}$ for some α, β. If $\psi_{\alpha,\beta} = \frac{1}{2}(\varphi_1 + \varphi_2)$ where $\varphi_1, \varphi_2 \in \Psi_2$, then $\psi_{\alpha,\beta} = \varphi_1 = \varphi_2$ on $[0, \alpha] \cup [\beta, 1]$. For each $t \in [\alpha, \beta]$, $\psi_{\alpha,\beta}(t) = \frac{1}{2}(\varphi_1(t) + \varphi_2(t))$. Since φ_1 and φ_2 are convex, we have $\psi_{\alpha,\beta} = \varphi_1 = \varphi_2$. Thus $\psi_{\alpha,\beta}$ is an extreme point of Ψ_2. This completes the proof.

Acknowledgement. The authors would like to thank the referees for their helpful comments and careful reading in this paper.

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