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Anisotropic motion by mean curvature in the context of Finsler geometry

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Abstract. We study the anisotropic motion of a hypersurface in the context of the geometry of Finsler spaces. This amounts in considering the evolution in relative geometry, where all quantities are referred to the given Finsler metric ϕ representing the anisotropy, which we allow to be a function of space. Assuming that ϕ is strictly convex and smooth, we prove that the natural evolution law is of the form "velocity $= H_{\phi}$", where H_{ϕ} is the relative mean curvature vector of the hypersurface. We derive this evolution law using different approches, such as the variational method of Almgren-Taylor-Wang, the Hamilton-Jacobi equation, and the approximation by means of a reaction-diffusion equation.

Key words: Finsler spaces, mean curvature flow, fronts propagation, surface area.

1. Introduction

The concepts of surface energy, particularly that of anisotropic surface energy and of related quantities such as the anisotropic mean curvature, are becoming increasingly important in different contexts, as in the field of phase changes and phase separation in multiphase materials [1], [29]. The role played by anisotropy becomes crucial in the crystalline case [2], [3], [12], [42], [43], [44], where the principal curvatures in the sense of differential geometry cannot in general be defined pointwise everywhere [40]. However the study of anisotropic evolution problems in the smooth case is a first step for a better understanding of the role of anisotropy in the general case in which no differentiability properties are assumed.

Anisotropic surface energy falls quite naturally within the geometry of Finsler spaces [6], [8], [32], and many of the tools of convex geometry [39] prove useful for related variational problems [9]. In particular, the idea is to endow the space $\mathbf{R}^N$ with the distance obtained by integrating the Finsler

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metric, which we allow to be a function of space, and to work in relative geometry. This approach is much in the spirit of the quoted papers of Taylor and Almgren-Taylor-Wang and, for the two dimensional case, of Gage [22], Gage-Li [23]. Some numerical simulations of anisotropic motion by mean curvature based on this approach can be found in [36].

Let us denote by $\phi : \Omega_x \times \mathbf{R}^N_\xi \to [0, +\infty[$ the Finsler metric, which we shall always assume to be strictly convex in ξ and smooth. The natural law of motion of a smooth hypersurface $\Sigma = \partial E$ subjected to the anisotropy ϕ turns out to be

$$\text{velocity} = \kappa_\phi \quad \text{in the direction } n_\phi, \tag{1.1}$$

where n_ϕ is the relative normal vector and κ_ϕ is the relative scalar mean curvature of Σ. Representing Σ as the zero level set $\{u = 0\}$ of a smooth function $u : \Omega \to \mathbf{R}$ (u positive inside E), then n_ϕ and κ_ϕ are defined by

$$\begin{aligned} n_\phi(x) &= \phi^o_\xi(x, \nabla u(x)), \\ \kappa_\phi(x) &= -\operatorname{div} n_\phi(x) - n_\phi(x) \\ &\qquad \cdot \nabla(\log(\det_N \phi(x))) =: F(x, \nabla u, \nabla^2 u), \end{aligned} \tag{1.2}$$

where $\phi^o = \phi^o(x, \xi^\star)$ is the dual of ϕ, $\phi^o_\xi = \nabla_{\xi^\star}(\phi^o)$, and $\det_N \phi(x)$ is the inverse of the Lebesgue measure of the set $\{\xi \in \mathbf{R}^N : \phi(x, \xi) \leq 1\}$. In material science the vector n_ϕ is also known as the Cahn-Hoffmann vector [13], [14].

In the evolution law (1.1) no mobility factor appears, and consequently the material properties just involve one anisotropic function. This differs from the evolution considered in [18], [30], where a mobility factor is present. In this respect it has to be stressed that, in the two dimensional case, setting $\phi^o(x, \xi^\star) = \phi^o(\xi^\star) = \varrho\psi(\theta)$ (that is, independence of the position and representation of ϕ^o using polar coordinates in the $\xi^\star$-plane), the evolution law (1.1) reads as

$$\begin{aligned} \text{velocity} &= \kappa\psi(\psi + \psi'') \quad \text{in the euclidean direction,} \\ \text{velocity} &= \kappa(\psi + \psi'') \quad \text{in the relative } n_\phi \text{ direction,} \end{aligned} \tag{1.3}$$

where κ is the euclidean scalar curvature.

Let us describe in detail the content of this paper, which has been announced in [9]. In Section 2 we give some notation and we prove some elementary properties of the Finsler metric ϕ and of its dual ϕ^o. In Section

3 we introduce the relative derivative operators ∇_ϕ, div_ϕ, Δ_ϕ and, consequently, the concepts of normal vector n_ϕ, scalar mean curvature κ_ϕ, and mean curvature vector $H_\phi = \kappa_\phi \nu_\phi$ of a smooth hypersurface $\Sigma = \partial E$ with respect to ϕ. Here ν_ϕ is the inner normal vector to Σ in the euclidean sense, but normalized in such a way that $\phi^o(x, \nu_\phi(x)) = 1$. Such definitions are given in a global way, by viewing Σ as a zero level set of a smooth function with non vanishing gradient on Σ. It turns out immediately that κ_ϕ is not, in general, a function of the euclidean scalar mean curvature.

The definitions of κ_ϕ and H_ϕ do not depend on the choice of u (Proposition 3.1); this is basically consequence of the fact that the function F defined in (1.2) is *strongly geometric* in the sense of Giga-Goto [24] (see also [15], [25]), i.e.,

$$F(x, \lambda p, \lambda X + p \otimes q + q \otimes p) = \frac{\lambda}{|\lambda|} F(x, p, X),$$

for any $\lambda \neq 0$, $p \in \mathbf{R}^N \setminus \{0\}$, $q \in \mathbf{R}^N$, and any symmetric $N \times N$ matrix X. It turns out also that κ_ϕ can be characterized by using the properties of the signed distance function $\delta_\phi^{\partial E}$ (see (2.8) and Section 3.3) to the boundary ∂E (see (3.16)).

A number of examples are given in Section 4. In particular, we consider the general two dimensional case (see (4.3) and (4.1)) related to (1.3). The cases $\phi^o(x, \xi^\star) = a(x)|\xi^\star|$ and $\phi^o(x, \xi^\star) = (\sum_{k=1}^N |\xi_k^\star|^p)^{1/p}$ are discussed in examples 4.1 and 4.6, respectively. In example 4.5 we show that, if ϕ is independent of the position x, then the relative curvature of the indicatrix $B_\phi := \{x \in \mathbf{R}^N : \phi(\xi) = 1\}$ is identically $N - 1$, which is in accordance with the isoperimetric property of B_ϕ (see [9], [20], [21], [32], [39]).

The rest of the paper is devoted to justify law (1.1). In Section 5 we prove that the first variation along an arbitrary vector field g of the perimeter functional P_ϕ introduced in [4], [9] (see (2.6)) is given by

$$-\int_{\partial E} H_\phi \cdot g \, d\mathcal{P}_\phi^{N-1},$$

where $d\mathcal{P}_\phi^{N-1}$ is the natural surface measure associated to ϕ (see (2.7)).

The accordance of (1.1) with the general approach proposed in [3] is considered at the end of Section 5. In Section 6 we consider the Hamilton-

Jacobi equation

$$\frac{u_t}{|\nabla u|\phi^o(x,\nu)} = \mathrm{div}_\phi n_\phi,$$

which is naturally associated to the anisotropic evolution, and in Section 7 we prove that a formal asymptotic expansion of the reaction-diffusion equation

$$u_t = \Delta_\phi u - \frac{1}{\epsilon^2}u(1-u^2)$$

yields, in the limit as $\epsilon \to 0^+$, again law (1.1). We conclude the paper with Section 8, by defining an evolution law with respect to a Finsler metric and to a measure (Remark 8.2), and by extending the previous results to the anisotropic mean curvature evolution with a forcing term.

2. Notations and preliminaries

In the sequel Ω will be an open connected subset of $\mathbf{R}^N$, $N \geq 2$.

Let E be a subset of $\mathbf{R}^N$; we indicate by 1_E the characteristic function of E, i.e., $1_E(x) = 1$ if $x \in E$, and $1_E(x) = 0$ if $x \in \mathbf{R}^N \setminus E$. We shall write $E \in \mathcal{C}^2_b(\Omega)$ if E is a bounded open subset of Ω of class $\mathcal{C}^2$. If $E \in \mathcal{C}^2_b(\Omega)$, its boundary will be oriented by the inner unit normal vector field ν_E to ∂E. Whenever no confusion is possible, we shall write ν in place of ν_E.

We denote by ω_m the Lebesgue measure of the euclidean unit sphere in $\mathbf{R}^m$, and by $\mathcal{H}^m$ the m-dimensional Hausdorff measure, for $m \in \mathbf{N}$, $0 \leq m \leq N$.

We denote by $\cdot$, $|\cdot|$ and by $d(\cdot,\cdot)$ the euclidean scalar product, the norm and the distance in $\mathbf{R}^N$, respectively; the symbol $\otimes$ stands for the tensor product.

Suppose that $\phi : \Omega \times \mathbf{R}^N \to [0,+\infty[$ is a continuous function satisfying the properties

$$\phi(x,t\xi) = |t|\phi(x,\xi) \qquad x \in \Omega, \quad \xi \in \mathbf{R}^N, \quad t \in \mathbf{R}, \tag{2.1}$$

$$\lambda|\xi| \leq \phi(x,\xi) \leq \Lambda|\xi| \qquad x \in \Omega, \quad \xi \in \mathbf{R}^N, \tag{2.2}$$

for two suitable positive constants $0 < \lambda \leq \Lambda < +\infty$.

We say that ϕ is strictly convex if for any $x \in \Omega$ the map $\xi \to \phi^2(x,\xi)$ is strictly convex on $\mathbf{R}^N$. We say that ϕ is independent of the position if $\phi = \phi(\xi)$.

We shall indicate by $B_\phi(x) = \{\xi \in \mathbf{R}^N : \phi(x,\xi) \leq 1\}$ the unit sphere of ϕ at $x \in \Omega$. If ϕ is independent of the position we set $B_\phi = \{\xi \in \mathbf{R}^N : \phi(\xi) \leq 1\}$.

The dual function $\phi^o : \Omega \times \mathbf{R}^N \to [0,+\infty[$ of ϕ is defined by

$$\phi^o(x,\xi^\star) = \sup\{\xi^\star \cdot \xi \ : \ \xi \in B_\phi(x)\} \tag{2.3}$$

for any $(x,\xi^\star) \in \Omega \times \mathbf{R}^N$ (see, for instance, [37]). One can prove that ϕ^o is continuous, convex, satisfies properties (2.1) and (2.2), and that ϕ^{oo} coincides with the convex envelope of ϕ with respect to ξ. Moreover if $\nu \in \mathbf{S}^{N-1} := \{\xi \in \mathbf{R}^N : |\xi| = 1\}$, we have

$$\phi^o(x,\nu) = \inf d(0,\mathcal{P}_\nu), \tag{2.4}$$

where the infimum is taken among all affine hyperplanes $\mathcal{P}_\nu \subset \mathbf{R}^N$ which are orthogonal to ν and such that $\mathcal{P}_\nu \cap B_\phi(x) = \emptyset$.

We say that ϕ is a (strictly convex smooth) Finsler metric, and we shall write $\phi \in \mathcal{M}(\Omega)$ if, in addition to properties (2.1) and (2.2), ϕ and ϕ^o are strictly convex and of class $\mathcal{C}^2(\Omega \times (\mathbf{R}^N \setminus \{0\}))$. In particular $\phi^{oo} = \phi$. For references about Finsler metrics see [6], [8], [32]; for what concerns the geometric properties of convex sets we refer to [39], and references therein.

Given $\phi \in \mathcal{M}(\Omega)$ we define the continuously differentiable function $\det_N \phi : \Omega \to]0,+\infty[$ as

$$\det_N \phi(x) = \left(\mathcal{H}^N(B_\phi(x))\right)^{-1} \qquad x \in \Omega. \tag{2.5}$$

Accordingly, the volume element induced by $\det_N \phi$ is $d\mathcal{H}^N_\phi := \omega_N \det_N \phi d\mathcal{H}^N$.

Let $\phi \in \mathcal{M}(\Omega)$, and let $E \subseteq \mathbf{R}^N$ be a set of class $\mathcal{C}^1$; we define the perimeter $P_\phi(E,\Omega)$ of E in Ω (with respect to ϕ) as

$$P_\phi(E,\Omega) = \omega_N \int_{\Omega \cap \partial E} \phi^o(x,\nu_E) \det_N \phi \ d\mathcal{H}^{N-1}, \tag{2.6}$$

see [4], [9]. When $\Omega = \mathbf{R}^N$ the perimeter of E will be simply denoted by $P_\phi(E)$. When ϕ^o is independent of the position, the perimeter P_ϕ coincides with the surface energy integral considered in [3, Section 2.1.3].

We accordingly introduce the measure $d\mathcal{P}^{N-1}_\phi$ supported on $\Omega \cap \partial E$ defined by

$$d\mathcal{P}^{N-1}_\phi = \omega_N \phi^o(x,\nu_E) \det_N \phi \ d\mathcal{H}^{N-1}. \tag{2.7}$$

We denote by δ_ϕ the integrated distance associated to $\phi \in \mathcal{M}(\Omega)$, i.e., for any $x, y \in \Omega$ we set

$$\delta_\phi(x,y) = \inf\Big\{ \int_0^1 \phi(\gamma,\dot\gamma)\, dt : \gamma \in W^{1,1}([0,1];\Omega), \\ \gamma(0)=x, \gamma(1)=y \Big\}. \qquad (2.8)$$

In the sequel we shall use the following compact notation for $\phi \in \mathcal{M}(\Omega)$ and $i \in \{1,\ldots,N\}$:

$$\phi_x = \nabla_x \phi, \quad \phi_\xi = \nabla_\xi \phi, \quad \phi_{\xi^i} = \frac{\partial \phi}{\partial \xi^i},$$

$$\phi^o_x = \nabla_x(\phi^o), \quad \phi^o_\xi = \nabla_{\xi^\star}(\phi^o), \quad \phi^o_{\xi_i} = \frac{\partial \phi^o}{\partial \xi^\star_i},$$

$$\phi^o_{x^i\xi_j} = \frac{\partial^2(\phi^o)}{\partial x^i \partial \xi^\star_j}, \quad \phi^o_{\xi\xi} = \nabla^2_{\xi^\star}(\phi^o), \quad \phi^o_{\xi_i\xi_j} = \frac{\partial^2(\phi^o)}{\partial \xi^\star_i \partial \xi^\star_j}.$$

The symbols ∇u and $\nabla^2 u$ will denote the spatial gradient and the Hessian matrix, respectively, of any smooth function u.

We shall also adopt the Einstein convention of implicitly assuming summation over repeated indices from 1 to N.

2.1. Elementary properties of Finsler metrics

If $\phi \in \mathcal{M}(\Omega)$, $x \in \Omega$, $\xi, \xi^\star \in \mathbf{R}^N \setminus \{0\}$, $t \neq 0$, then (2.1) yields

$$\phi^o_\xi(x, t\xi^\star) = \frac{t}{|t|}\phi^o_\xi(x,\xi^\star), \quad \phi^o_{\xi\xi}(x,t\xi^\star) = \frac{1}{|t|}\phi^o_{\xi\xi}(x,\xi^\star), \qquad (2.9)$$

$$\phi(x,\xi) = \phi_\xi(x,\xi)\cdot\xi, \quad \phi^o(x,\xi^\star) = \phi^o_\xi(x,\xi^\star)\cdot\xi^\star. \qquad (2.10)$$

Lemma 2.1 *Let $\phi \in \mathcal{M}(\Omega)$. For any $x \in \Omega$ and $\xi, \xi^\star \in \mathbf{R}^N \setminus \{0\}$ we have*

$$\phi(x, \phi^o_\xi(x,\xi^\star)) = \phi^o(x, \phi_\xi(x,\xi)) = 1. \qquad (2.11)$$

Proof. We can ignore the dependence on x. Let us prove that $\phi^o(\phi_\xi(\xi)) = 1$, the other equality being similar. Let $\xi \in \mathbf{R}^N \setminus \{0\}$; in view of (2.9) we can assume that $\phi(\xi) = 1$. Clearly ϕ_ξ is orthogonal to ∂B_ϕ at ξ, and hence, using (2.4) and (2.10),

$$\phi^o\left(\frac{\phi_\xi(\xi)}{|\phi_\xi(\xi)|}\right) = \frac{\phi_\xi(\xi)\cdot\xi}{|\phi_\xi(\xi)|} = \frac{\phi(\xi)}{|\phi_\xi(\xi)|} = \frac{1}{|\phi_\xi(\xi)|},$$

which is the assertion. □

Thanks to (2.11) and (2.10) it follows that the vector $\phi^o_\xi(x,\xi^\star)$ realizes the maximum in definition (2.3) of $\phi^o(x,\xi^\star)$.

Lemma 2.2 *Let $\phi \in \mathcal{M}(\Omega)$. For any $x \in \Omega$ and $\xi, \xi^\star \in \mathbf{R}^N \setminus \{0\}$ we have*

$$\phi^o(x,\xi^\star)\phi_\xi(x,\phi^o_\xi(x,\xi^\star)) = \xi^\star, \quad \phi(x,\xi)\phi^o_\xi(x,\phi_\xi(x,\xi)) = \xi. \quad (2.12)$$

Proof. We can ignore the dependence on x. Let us prove the last equality in (2.12), the other one being similar. Let $\xi \in \mathbf{R}^N \setminus \{0\}$; in view of (2.9) and (2.1) we can assume that $\phi(\xi) = 1$. Define $\xi^\star := \phi_\xi(\xi)$; then by (2.10) we have $\xi^\star \cdot \xi = \phi(\xi) = 1$, and $\phi^o(\xi^\star) = 1$ by (2.11). Define $\xi^{\star\star} := \phi^o_\xi(\xi^\star)$; then by (2.10) we have $\xi^{\star\star} \cdot \xi^\star = \phi^o(\xi^\star) = 1$, and $\phi(\xi^{\star\star}) = 1$ by (2.11). Now both ξ and $\xi^{\star\star}$ realize the maximum in definition (2.3) of $\phi^o(\xi^\star)$; from the strict convexity of ϕ^o we deduce that $\xi = \xi^{\star\star}$. Therefore

$$\xi = \xi^{\star\star} = \phi^o_\xi(\xi^\star) = \phi^o_\xi(\phi_\xi(\xi)).$$

□

3. Definitions of ∇_ϕ, div_ϕ, Δ_ϕ, n_ϕ, κ_ϕ. Connections with δ_ϕ

Given $\phi \in \mathcal{M}(\Omega)$ and $x \in \Omega$, let $T(x,\cdot), T^o(x,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$ be the maps defined by

$$\begin{aligned} &T(x,\xi) = \phi(x,\xi)\phi_\xi(x,\xi), \quad T^o(x,\xi^\star) = \phi^o(x,\xi^\star)\phi^o_\xi(x,\xi^\star) \\ &\xi, \xi^\star \in \mathbf{R}^N \setminus \{0\}, \quad T(x,0) = T^o(x,0) = 0. \end{aligned}$$

For simplicity, whenever $x \in \Omega$ is fixed, we shall write $T(\xi)$, $T^o(\xi^\star)$ instead of $T(x,\xi)$, $T^o(x,\xi^\star)$.

Lemma 3.1 (Duality). *We have*

$$TT^o = T^oT = \mathrm{Id} \quad \text{on } \mathbf{R}^N.$$

Proof. Let $\xi \in \mathbf{R}^N$; using (2.11) and (2.12) we have

$$\begin{aligned} T^o(T(\xi)) &= \phi^o(x,T(\xi))\phi^o_\xi(x,T(\xi)) \\ &= \phi(x,\xi)\phi^o(x,\phi_\xi(x,\xi))\,\phi^o_\xi(x,\phi_\xi(x,\xi)) \\ &= \phi(x,\xi)\phi^o_\xi(x,\phi_\xi(x,\xi)) = \xi. \end{aligned}$$

The equality $TT^o = \mathrm{Id}$ can be proved in a similar way. □

If $u : \Omega \to \mathbf{R}$ is a smooth function with non vanishing gradient, we set

$$\nabla_\phi u = T^o(\nabla u) = \phi^o(x, \nabla u)\phi^o_\xi(x, \nabla u), \tag{3.1}$$

and, if $\eta(x) = (\eta^1(x), \ldots, \eta^N(x)) \in \mathbf{R}^N$ is a smooth vector field, we set

$$\begin{aligned} \mathrm{div}_\phi\, \eta &= \mathrm{div}\, \eta + \eta \cdot \nabla\left(\log(\det{}_N \phi)\right), \\ \Delta_\phi u &= \mathrm{div}_\phi \nabla_\phi u. \end{aligned} \tag{3.2}$$

Note that if $\phi(x, \xi) = (g_{ij}(x)\xi^i\xi^j)^{1/2}$ is a riemannian metric, then $(T(\xi))_i = g_{ij}\xi^j$, $(T^o(\xi^\star))^i = g^{ij}\xi^\star_j$ where (g^{ij}) is the inverse of (g_{ij}), and Δ_ϕ is the Laplace-Beltrami operator associated to (g_{ij}), since $\det_N \phi = \sqrt{\det(g_{ij})}$.

3.1. ϕ-normal vectors

Let $E \in \mathcal{C}^2_b(\Omega)$. Let $u : \Omega \to \mathbf{R}$ be a smooth function such that $\{u = 0\} = \partial E$, $\{u > 0\} = E$, and $\nabla u \neq 0$ on ∂E. We define the inner normal $n_\phi(x)$ to ∂E at $x \in \partial E$ with respect to $\phi \in \mathcal{M}(\Omega)$ as

$$n_\phi(x) := \phi^o_\xi(x, \nabla u) = \frac{\nabla_\phi u}{\phi^o(x, \nabla u)} \tag{3.3}$$

(compare [6, Ch. 1], [13], [14], [31], and [32, III.16]), where we simply write ∇u in place of $\nabla u(x)$. It is immediate to verify that n_ϕ depends only on $\{u = 0\}$ and not on u itself. The vector $n_\phi(x)$ can also be viewed as the inverse of the Gauss map of $\partial B_\phi(x)$ computed at the point $\nabla u(x)$ (see [39], p. 106). By (2.11) we have

$$\phi(x, n_\phi) = 1 \quad \text{on } \partial E. \tag{3.4}$$

If $\nu(x)$ is the unit inner normal (in the euclidean sense) vector to ∂E at $x \in \partial E$, we set

$$\nu_\phi(x) := \frac{\nu(x)}{\phi^o(x, \nu(x))} = \frac{\nabla u}{\phi^o(x, \nabla u)}, \quad \text{so that } \phi^o(x, \nu_\phi(x)) = 1. \tag{3.5}$$

Using (2.10), (3.4), and (2.12) we have

$$n_\phi \cdot \phi_\xi(x, n_\phi) = 1 = n_\phi \cdot \nu_\phi \quad \text{on } \partial E. \tag{3.6}$$

Finally, by (3.3), (3.5), and (2.12) we have

$$n_\phi = \phi^o_\xi(x, \nu_\phi) = T^o(\nu_\phi), \quad \nu_\phi = \phi_\xi(x, n_\phi) = T(n_\phi) \quad \text{on } \partial E. \tag{3.7}$$

Remark 3.1. If Ω is bounded of class $\mathcal{C}^1$, $u \in \mathcal{C}^1(\Omega)$ and $g \in \mathcal{C}^1(\Omega; \mathbf{R}^N)$, then the following Gauss-Green formula holds:

$$\int_\Omega u\mathrm{div}_\phi g \; d\mathcal{H}^N_\phi + \int_\Omega \nabla u \cdot g \; d\mathcal{H}^N_\phi = -\int_{\partial\Omega} ug \cdot \nu_\phi \; d\mathcal{P}^{N-1}_\phi \tag{3.8}$$

(recall that ν_Ω and ν_ϕ point inward Ω).

Proof. Definition (3.2) of div_ϕ yields

$$\begin{aligned} u\mathrm{div}_\phi g \; \det_N \phi &= u\mathrm{div}\, g \; \det_N \phi + ug \cdot \nabla(\log(\det_N \phi)) \; \det_N \phi \\ &= \mathrm{div}(ug \; \det_N \phi) - \nabla u \cdot g \; \det_N \phi. \end{aligned}$$

Hence, recalling definition (3.5), we get

$$\begin{aligned} \int_\Omega u\mathrm{div}_\phi g \; d\mathcal{H}^N_\phi &= -\omega_N \int_{\partial\Omega} ug \cdot \nu_\Omega \; \det_N \phi \; d\mathcal{H}^{N-1} - \int_\Omega \nabla u \cdot g \; d\mathcal{H}^N_\phi \\ &= -\int_{\partial\Omega} ug \cdot \nu_\phi \; d\mathcal{P}^{N-1}_\phi - \int_\Omega \nabla u \cdot g \; d\mathcal{H}^N_\phi, \end{aligned}$$

and this completes the proof. □

3.2. ϕ-mean curvature

Sticking with the notation of Section 3.1, we define the (scalar) mean curvature κ_ϕ of ∂E with respect to ϕ as

$$\kappa_\phi := -\mathrm{div}_\phi \; n_\phi = -\mathrm{div}\, n_\phi - n_\phi \cdot \nabla(\log(\det_N \phi)), \tag{3.9}$$

and the vector mean curvature H_ϕ to ∂E as $H_\phi := \kappa_\phi \nu_\phi$. When ϕ^o is independent of the position, the above definition of H_ϕ coincides with the relative curvature considered in [3, Section 2.2].

Denote by S^N the space of the real $N \times N$ symmetric matrices. From the definition of κ_ϕ we get

$$\kappa_\phi = -F(x, \nabla u, \nabla^2 u), \tag{3.10}$$

where $F : \Omega \times \mathbf{R}^N \setminus \{0\} \times S^N \to \mathbf{R}$ is the continuous function defined by

$$\begin{aligned} F(x,p,X) &:= \phi^o_{x^i\xi_i}(x,p) + \phi^o_{\xi_i\xi_j}(x,p)X_{ij} \\ &\quad + \phi^o_{\xi_i}(x,p)\frac{\partial}{\partial x^i}(\log(\det_N \phi)). \end{aligned} \tag{3.11}$$

Proposition 3.1 (Independence). *Suppose $u, v \in \mathcal{C}^2(\Omega)$ are such that $\Sigma := \{u = 0\} = \{v = 0\}$, $\nabla u \cdot \nabla v > 0$ on Σ, and Σ is a compact subset of*

Ω. *Then*

$$F(x, \nabla u, \nabla^2 u) = F(x, \nabla v, \nabla^2 v) \quad on\ \Sigma. \tag{3.12}$$

In particular κ_ϕ *depends just on the level set* $\{u = 0\}$ *and not on* u *itself.*

Proof. Fix $\bar{x} \in \Sigma$. Possibly multiplying v by a positive scalar factor, we can assume that $\nabla u(\bar{x}) = \nabla v(\bar{x})$. Set $w = u - v$. Then $\bar{x} \in \Sigma \subseteq \{w = 0\}$, and $\nabla w(\bar{x}) = 0$. A direct computation then shows that, if τ_1, τ_2 are two arbitrary tangent vectors to Σ at $\bar{x}$, then $\tau_1 \cdot \nabla^2 w(\bar{x}) \tau_2 = 0$. Now we observe that the matrix $\nabla^2 w(\bar{x})$ can be written as

$$\nabla^2 w(\bar{x}) = \nabla u(\bar{x}) \otimes q + q \otimes \nabla u(\bar{x}), \tag{3.13}$$

for a suitable $q \in \mathbf{R}^N$. This can be seen by writing $\nabla^2 w(\bar{x})$ in a orthogonal coordinate system formed by the normal vector p to Σ and by $N-1$ tangent vectors.

Observe now that the function F defined in (3.11) satisfies the following condition (see [24])

$$F(x, \lambda p, \lambda X + q \otimes p + p \otimes q) = \frac{\lambda}{|\lambda|} F(x, p, X) \tag{3.14}$$

for all $x \in \Omega$, $\lambda \neq 0$, $p \in \mathbf{R}^N \setminus \{0\}$, $q \in \mathbf{R}^N$, $X \in S^N$. Indeed, following [15, Examples 5.9, 5.10], by differentiating the last equality in (2.10) with respect to $\xi^\star$ we get $\phi^o_{\xi_i \xi_j}(x,p) p_j = 0$, hence by the symmetry of $\phi^o_{\xi\xi}$ we have

$$\operatorname{tr}(\phi^o_{\xi\xi}(x,p)\, [p \otimes q + q \otimes p]) = 0,$$

and (3.14) follows.

Then (3.12) follows from (3.13) and (3.14). □

Observe that the convexity of ϕ^o implies that F is *degenerate elliptic*, i.e.,

$$F(x, p, X + Y) \geq F(x, p, X) \qquad X, Y \in S^N, \quad Y \geq 0.$$

3.3. Connections between n_ϕ, κ_ϕ, and δ_ϕ

Denote by $\delta_\phi^{\partial E}$ the δ_ϕ signed distance function to ∂E positive inside E, i.e., $\delta_\phi^{\partial E}(x) = -\delta_\phi(x, E) + \delta_\phi(x, \mathbf{R}^N \setminus E)$; we shall assume that if E is smooth then $\delta_\phi^{\partial E}$ is smooth in a tubular neighbourhood of ∂E.

Lemma 3.2 *Let* $E \in \mathcal{C}^2_b(\Omega)$, $\phi \in \mathcal{M}(\Omega)$, *let* δ_ϕ *be the integrated distance* *associated to* ϕ, *and let* $\delta_\phi^{\partial E}$ *be the* δ_ϕ *signed distance function to* ∂E *positive inside* E. *Then*

$$\nabla \delta_\phi^{\partial E} = \nu_\phi \quad \text{on } \partial E, \tag{3.15}$$

and

$$\kappa_\phi = -\text{div}_\phi n_\phi = -\Delta_\phi \delta_\phi \quad \text{on } \partial E. \tag{3.16}$$

Proof. Let $x \in \partial E$. Since $\nabla \delta_\phi^{\partial E}$ is orthogonal to ∂E at x we have $\nabla \delta_\phi^{\partial E}(x) = \mu \nu(x)$ for a suitable $\mu \in \mathbf{R}$ to be determined. In [9] it is proved that

$$\phi^o(x, \nabla \delta_\phi^{\partial E}(x)) = 1 \tag{3.17}$$

on each point $x \in \Omega$ where $\delta_\phi^{\partial E}(x)$ is differentiable (and hence almost everywhere on Ω). Using (3.17) and the smoothness of ϕ^o and $\delta_\phi^{\partial E}$, we have $1 = \phi^o(x, \nabla \delta_\phi^{\partial E}(x)) = \mu \phi^o(x, \nu(x))$, so that $\mu = (\phi^o(x, \nu(x)))^{-1}$, i.e., (3.15).

Due to the independence of κ_ϕ with respect to the function u proved in Proposition 3.1, since $\phi^o(x, \nabla \delta_\phi^{\partial E}) = 1$ in a neighbourhood of ∂E, by (3.2) and (3.1) we have (3.16). □

4. Examples

Observe that κ_ϕ is not, in general, a function of the sum κ of the principal curvatures of ∂E: indeed, if ϕ, ϕ^o are independent of the position, by (3.10) and (3.11) it follows that, if $\nu = (\nu_1, \ldots, \nu_N)$ is the inner unit normal (in the euclidean sense) vector field to ∂E, then

$$\kappa_\phi = -\phi^o_{\xi_i \xi_j}(\nu) \frac{\partial \nu_j}{\partial x^i}$$

which obviously is not a function of $\kappa = -\text{div}\, \nu$. For instance, let $N = 3$, take $A, B, \alpha, \beta \in \,]0, +\infty[$, set $\xi^\star = (\xi_1^\star, \xi_2^\star, \xi_3^\star)$, $x = (x^{(1)}, x^{(2)}, x^{(3)})$, and $\phi^o(x, \xi^\star) := (A\xi_1^{\star 2} + B\xi_2^{\star 2} + \xi_3^{\star 2})^{1/2}$, $u(x) = x^{(3)} - \alpha(x^{(1)})^2 - \beta(x^{(2)})^2$, $\Sigma := \{u = 0\}$. Then, setting $\overline{\nu} := \nabla u(0) = (0, 0, 1)$, at the point $x = 0$ we have

$$\kappa_\phi = -\phi^o_{\xi_i \xi_j}(\nabla u) u_{x^i x^j} = 2\alpha \phi^o_{\xi_1 \xi_1}(\overline{\nu}) + 2\beta \phi^o_{\xi_2 \xi_2}(\overline{\nu}) = 2(\alpha A + \beta B),$$

which is not a function of $\kappa = 2(\alpha + \beta)$.

Observe also that if ∂E_1, ∂E_2 are two smooth hypersurfaces osculating each other at the point x then they have there the same κ_ϕ.

Let us now show some examples in which we calculate κ_ϕ for special choices of $\phi \in \mathcal{M}(\Omega)$.

Example 4.1. Assume that $\phi^o(x,\xi^\star) = a(x)|\xi^\star|$, for a suitable smooth function $a : \Omega \to \mathbf{R}$ with $0 < \lambda \le a \le \Lambda < +\infty$. Let $E \in \mathcal{C}^2_b(\Omega)$ having scalar mean curvature $\kappa = -\operatorname{div}\nu$ on ∂E. Then a direct computation yields $\det_N \phi = 1/(\omega_N a^N)$, $\nabla(\log(\det_N \phi)) = -N\nabla a/a$, and

$$n_\phi = a\nu, \qquad \operatorname{div} n_\phi = \nabla a \cdot \nu - a\kappa \quad \text{on } \partial E,$$

so that

$$\kappa_\phi = a\kappa + (N-1)\nabla a \cdot \nu \quad \text{on } \partial E.$$

Example 4.2. Let $N = 2$, $\phi \in \mathcal{M}(\Omega)$, $E \in \mathcal{C}^2_b(\Omega)$, and let $x \in \partial E$. Denote by $\tau(x)$ the unit tangent vector to ∂E at x. Then, if κ is the euclidean scalar curvature of ∂E we have

$$\begin{aligned}\kappa_\phi = {} & \kappa[\tau \cdot \phi^o_{\xi\xi}(x,\nu)\tau] - \phi^o_{x^i\xi_i}(x,\nu) \\ & - \phi^o_\xi(x,\nu)\cdot\nabla(\log(\det_N \phi)) \quad \text{on } \partial E,\end{aligned} \tag{4.1}$$

where τ is understood as a column vector. In particular, if ϕ does not depend on the position, we have

$$\kappa_\phi = \kappa[\tau \cdot \phi^o_{\xi\xi}(\nu)\tau] \quad \text{on } \partial E. \tag{4.2}$$

Indeed, let L and N be the tangent and the normal lines to ∂E at x generated by τ and ν, respectively. Let γ be the local parametrization of ∂E around x given by $\gamma(t) := x + t\tau + f(t)\nu$ for t in a neighbourhood of 0, where f is the signed distance function from any point of ∂E to L (f positive in the direction of ν). Hence $f(0) = f'(0) = 0$, and $f''(0) = \kappa(x)$. Set $u(y) = y^{(2)} - f(y^{(1)})$, where $y = (y^{(1)}, y^{(2)})$ are the local orthogonal coordinates associated to L and N. We then have $\partial E = \{u = 0\}$, and $\nabla u = \nabla y^{(2)} - f'(y^{(1)})\nabla y^{(1)}$, $\nabla^2 u = -f''(y^{(2)})\nabla y^{(1)} \otimes \nabla y^{(1)}$. It follows that $\nabla u(x) = \nu$, $\nabla^2 u(x) = -\kappa\tau\otimes\tau$. Then (4.1) follows from the definition of κ_ϕ (see (3.10)).

Example 4.3. Let $N = 2$, and assume that $\phi^o(x,\xi^\star) = \phi^o(\xi^\star) = \varrho\psi(\theta)$, where (ϱ,θ) are polar coordinates in the $\xi^\star$-plane, i.e., $\xi^\star_1 = \varrho\cos\theta$, $\xi^\star_2 =$

$\varrho \sin\theta$. Then the curvature κ_ϕ of a smooth curve $\Sigma = \partial E$ is

$$\kappa_\phi = \kappa(\psi + \psi'') \tag{4.3}$$

(compare [5], [28]). Indeed, if $\nu = (\cos\theta, \sin\theta)$, then $\tau = (\sin\theta, -\cos\theta)$, hence $\phi^o_{\xi_1\xi_1}(\xi^\star) = \varrho^{-1}(\psi + \psi'')\sin^2\theta$, $\phi^o_{\xi_1\xi_2}(\xi^\star) = -\varrho^{-1}(\psi + \psi'')\sin\theta\cos\theta$, $\phi^o_{\xi_2\xi_2}(\xi) = \varrho^{-1}(\psi + \psi'')\cos^2\theta$. Hence on Σ, using (4.2) we have

$$\kappa_\phi = \kappa(\psi + \psi'')(\sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta) = \kappa(\psi + \psi''),$$

which is (4.3).

Example 4.4. Let $N = 2$, $\phi \in \mathcal{M}(\Omega)$, and assume that $\phi^o(x, \xi^\star) = \varrho\psi(x, \theta)$ (see example 4.3). Let $E \in \mathcal{C}^2_b(\Omega)$, $\tau = (\sin\theta, -\cos\theta)$ and $\nu = (\cos\theta, \sin\theta)$. Then the relative curvature κ_ϕ of ∂E is

$$\kappa_\phi = \kappa(\psi + \psi_{\theta\theta}) - \nu \cdot \psi_x + \tau \cdot \psi_{x\theta} + (\psi_\theta\tau - \psi\nu) \cdot \nabla(\log(\det{}_2\,\phi)) \tag{4.4}$$

where

$$(\det{}_2\,\phi(x))^{-1} = \frac{1}{2}\int_0^{2\pi} \psi(x,\theta)\,(\psi(x,\theta) + \psi_{\theta\theta}(x,\theta))\ d\theta \tag{4.5}$$

(see [38]). Recalling that $\varrho = |\xi^\star|$, $\varrho_\xi = -\nu$, $\theta_\xi = |\xi^\star|^{-1}\tau$, one can check that

$$\phi^o_{x^i\xi_i}(x, \nu) = \phi^o_{x^i\xi_i}(x, \xi^\star) = \nu \cdot \psi_x - \tau \cdot \psi_{x\theta}. \tag{4.6}$$

In addition

$$n_\phi = \phi^o_\xi(x, \nu) = \psi\varrho_\xi + \varrho\theta_\xi\psi_\theta = \psi\nu - \psi_\theta\tau. \tag{4.7}$$

To prove (4.4), in view of (4.1), (4.3), (4.6), and (4.7), it remains to show (4.5). Set $\phi(x, \xi) = r\eta(x, \alpha)$, where (r, α) are polar coordinates in the ξ-plane. We have $\partial B_\phi(x) = \{(r, \alpha) : r = 1/\eta(x, \alpha)\}$, and hence

$$(\det{}_2\,\phi(x))^{-1} = \frac{1}{2}\int_0^{2\pi} \frac{1}{[\eta(x,\alpha)]^2} d\alpha. \tag{4.8}$$

Fix $x \in \Omega$. Parametrizing $\partial B_{\phi^o}(x)$ by $\gamma(\alpha^\star) = \nu(\alpha^\star)/\psi(x, \alpha^\star)$ with $\nu(\alpha^\star) = (\cos\alpha^\star, \sin\alpha^\star)$, setting $\tau(\alpha^\star) = (\sin\alpha^\star, -\cos\alpha^\star)$, we have

$$(\gamma')^\perp = \frac{\psi_\theta\tau(\alpha^\star)}{\psi^2} - \frac{\nu(\alpha^\star)}{\psi},$$

(where $\tau^\perp = \nu$, $\nu^\perp = -\tau$) and hence $|(\gamma')^\perp|^2 = |\gamma'|^2 = (\psi_\theta^2 + \psi^2)/\psi^4$.

Using (2.10) we have $\phi^o_\xi(x,\gamma)\cdot\gamma = \phi^o(x,\gamma) = 1$. Since we can characterize $(\gamma')^\perp = -\phi(x,-(\gamma')^\perp)\phi^o_\xi(x,\gamma)$, upon scalar product by γ we get

$$\phi(x,-(\gamma')^\perp) = -(\gamma')^\perp\cdot\gamma = \frac{1}{\psi^2(x,\alpha^\star)},$$

which implies, recalling that $\phi(x,-(\gamma')^\perp) = |\gamma'|\eta(x,\alpha)$,

$$\frac{1}{[\eta(x,\alpha)]^2} = \psi^4(x,\alpha^\star)|\gamma'|^2 = \psi^2 + \psi_\theta^2. \tag{4.9}$$

In addition one can show that

$$\frac{d\alpha}{d\alpha^\star} = \frac{\psi(\psi+\psi_{\theta\theta})}{\psi^2+\psi_\theta^2}. \tag{4.10}$$

Then (4.5) follows from (4.8), (4.9), and (4.10).

Example 4.5. Let $\phi \in \mathcal{M}(\Omega)$ be independent of the position. Then [9, Section 6] B_ϕ realizes the minimum perimeter P_ϕ among all sets with fixed Lebesgue measure (see also [20], [21], [39, p. 416]). Let us check that

$$\kappa_\phi = N-1 \quad \text{on } \partial B_\phi \tag{4.11}$$

(compare [32, Prop. 31.1]). Take $u(\zeta) = 1-\phi(\zeta)$ for $\zeta\in\mathbf{R}^N$; then $\nabla u(\zeta) = -\phi_\xi(\zeta)$. Hence from (2.12) we have $\phi^o_\xi(\nabla u(\zeta)) = -\zeta/\phi(\zeta)$ on $\mathbf{R}^N$. Consequently, using (3.3), $\kappa_\phi = -\mathrm{div}_\phi(\phi^o_\xi(\nabla u(\zeta))) = \mathrm{div}(\zeta/\phi(\zeta))$. Then, as $\zeta\cdot\phi_\xi(\zeta) = \phi(\zeta)$ by (2.10), we have

$$\mathrm{div}\left(\frac{\zeta}{\phi(\zeta)}\right) = \frac{\mathrm{div}\,\zeta}{\phi(\zeta)} - \frac{\zeta\cdot\phi_\xi(\zeta)}{\phi^2(\zeta)} = \frac{N}{\phi(\zeta)} - \frac{1}{\phi(\zeta)} = \frac{N-1}{\phi(\zeta)}.$$

Assertion (4.11) follows.

Example 4.6. Let $p\in\,]2,+\infty[$, and let us consider the L^p norm

$$\phi^o(x,\xi^\star) = \phi^o(\xi^\star) = \left(\sum_{k=1}^N |\xi_k^\star|^p\right)^{1/p}.$$

Then a direct computation yields (with no implicit summation)

$$\phi^o_{\xi_i\xi_j}(\xi^\star) = (p-1)\phi^o(\xi^\star)^{1-p}[|\xi_i^\star|^{p-2}\delta_{ij} - \phi^o(\xi^\star)^{-p}\xi_i^\star\xi_j^\star(|\xi_i^\star||\xi_j^\star|)^{p-2}],$$

where δ_{ij} is the Kronecker symbol. We use the previous formula to compute the κ_ϕ curvature of a paraboloid at the origin. Let $\alpha_i \in]0,+\infty[$ for $i = 1,\dots N-1$, $\Sigma = \{u=0\}$, where $u(x) = x^{(N)} - \sum_{j=1}^{N-1} \alpha_j (x^{(j)})^2$. Then

$$\kappa_\phi(0) = -\phi^o_{\xi_i\xi_j}(\nabla u(0)) u_{x^i x^j}(0) = 2(1-p)\sum_{i=1}^{N-1} \alpha_i(\delta_{iN} - \delta_{iN}) = 0.$$

This unexpected result, which shows that any smooth surface has zero relative κ_ϕ curvature at x whenever the normal at x is in the direction of one of the coordinate axes, can actually be explained by observing that the function $\phi(\xi) = (\sum_{i=1}^N |\xi_i|^{p'})^{1/p'}$, for $1/p + 1/p' = 1$, is not of class $\mathcal{C}^2(\mathbf{R}^N)$. Indeed, the indicatrix ∂B_ϕ, which has formally curvature $N-1$ (see (4.11)), is only of class $\mathcal{C}^{1,\alpha}$, $\alpha = p'-1 \in]0,1[$.

5. The first variation of the perimeter P_ϕ

In this section we are concerned with the first variation of the perimeter functional P_ϕ (compare [3, Section 2.2] and [43]). We need the following lemma.

Lemma 5.1 *Using the notation of Section* 3.1 *and setting* $\nu_\phi = ((\nu_\phi)_1, \dots, (\nu_\phi)_N)$, $n_\phi = (n^1_\phi, \dots, n^N_\phi)$, *the following relations hold*:

$$\phi^o_{x^i}(x,\nu_\phi) + n^j_\phi \frac{\partial(\nu_\phi)_j}{\partial x^i} = 0 \qquad i = 1,\dots,N;$$

$$\phi_{x^i}(x,n_\phi) + (\nu_\phi)_j \frac{\partial n^j_\phi}{\partial x^i} = 0 \qquad i = 1,\dots,N; \tag{5.1}$$

$$\phi_x(x,n_\phi) = -\phi^o_x(x,\nu_\phi). \tag{5.2}$$

Proof. Formulae (5.1) follow by differentiating (3.5) (respectively (3.4)) with respect to x^i and using (3.7).

Differentiating (3.6) with respect to x^i one gets $0 = n^j_\phi \frac{\partial(\nu_\phi)_j}{\partial x^i} + (\nu_\phi)_j \frac{\partial n^j_\phi}{\partial x^i}$. Adding the two equalities of (5.1), (5.2) follows. □

Concerning the variation of P_ϕ we shall distinguish two cases: in (5.3) we compute the variation along the n_ϕ direction, and in (5.4) we compute the variation along an arbitrary direction.

Theorem 5.1 *Let $E \in \mathcal{C}_b^2(\mathbf{R}^N)$, let $u \in \mathcal{C}^2(\mathbf{R}^N)$ be such that $E = \{u > 0\}$, $\partial E = \{u = 0\}$, and $\nabla u \neq 0$ on ∂E. Let U be a neighbourhood of ∂E and let $g \in \mathcal{C}_0^1(U; \mathbf{R}^N)$. Let $t \in \mathbf{R}$ be sufficiently small, set $\Phi_t : U \to \mathbf{R}^N$, $\Phi_t(x) = x + tg(x)$, extended as $\Phi_t(x) = x$ outside U, and let $E_t := \Phi_t(E)$. Then if $g = hn_\phi$ for $h \in \mathcal{C}_0^1(U)$ and n_ϕ defined in* (3.3), *we have*

$$\frac{d}{dt} P_\phi(E_t)_{|t=0} = -\int_{\partial E} \kappa_\phi h \; d\mathcal{P}_\phi^{N-1}, \tag{5.3}$$

where $d\mathcal{P}_\phi^{N-1}$ is defined in (2.7).

In the general case we have

$$\frac{d}{dt} P_\phi(E_t)_{|t=0} = -\int_{\partial E} H_\phi \cdot g \; d\mathcal{P}_\phi^{N-1}, \tag{5.4}$$

where $H_\phi = \kappa_\phi \nu_\phi$, and ν_ϕ is defined in (3.5).

Proof. Let $g = hn_\phi$; observe that $\phi(x, n_\phi) = 1$ on U by (3.3) and (2.11). Define $v : \mathbf{R}^N \to \mathbf{R}$ by $v(x + th(x)n_\phi(x)) = u(x)$. Then $\partial E_t = \{v = 0\}$ and $\nabla v \neq 0$ on ∂E_t for small t. In the sequel, for consistency with matrix notation, we shall understand ν, ∇u and ∇v as column vectors. Recall that, if $x \in \partial E$, then on ∂E_t

$$\begin{aligned} d\mathcal{H}^{N-1}(x + thn_\phi) &= d\mathcal{H}^{N-1}(x) + t\left[\operatorname{div}(hn_\phi) - \nu J \cdot \nu\right] \; d\mathcal{H}^{N-1}(x) \\ &\quad + o(t) d\mathcal{H}^{N-1}(x), \end{aligned} \tag{5.5}$$

where $J = \left[\dfrac{\partial}{\partial x^j}(hn_\phi^i)\right]_{ij}$ is the Jacobian of hn_ϕ, and $\operatorname{div}(hn_\phi) - \nu J \cdot \nu$ is the tangential divergence of hn_ϕ relative to ∂E [41]. We preliminarly show that

$$\frac{d}{dt}\left[\frac{\nabla v}{|\nabla v|}(x + thn_\phi)\right]_{|t=0} = -\nu J + (\nu J \cdot \nu)\nu, \quad x \in \partial E. \tag{5.6}$$

We have $\nabla v = \nabla u(\mathrm{Id} + tJ)^{-1}$, so that $\dfrac{d}{dt}\nabla v_{|t=0} = -\nabla u J$; moreover $\nabla v = \nabla u$ on ∂E (i.e., for $t = 0$). Hence

$$\begin{aligned} \frac{d}{dt}\left[\frac{\nabla v}{|\nabla v|}(x + thn_\phi)\right]_{|t=0} &= -\frac{\nabla u}{|\nabla v|} J + (\nabla u J \cdot \nabla v) \frac{\nabla v}{|\nabla v|^3} \\ &= -\frac{\nabla u}{|\nabla u|} J + \left(\frac{\nabla u}{|\nabla u|} J \cdot \frac{\nabla u}{|\nabla u|}\right) \frac{\nabla u}{|\nabla u|}, \end{aligned}$$

which is (5.6). Note also that

$$\begin{aligned} & P_\phi(E_t) \\ & = \omega_N \int_{\partial E} \phi^o \left[x + thn_\phi, \frac{\nabla v}{|\nabla v|}(x + thn_\phi)\right] \\ & \qquad \det_N \phi(x + thn_\phi) d\mathcal{H}^{N-1}(x + thn_\phi). \end{aligned} \tag{5.7}$$

Hence, using (3.3), (5.6), (5.5), by differentiating (5.7) with respect to t (and recalling the notation (2.7)) we have

$$\begin{aligned} \frac{d}{dt} P_\phi(E_t)_{|t=0} &= \omega_N \int_{\partial E} \phi^o_x(x,\nu) \cdot n_\phi \; h \; \det_N \phi \; d\mathcal{H}^{N-1} \\ & \quad + \omega_N \int_{\partial E} n_\phi \cdot (-\nu J + (\nu J \cdot \nu)\nu) \; \det_N \phi \; d\mathcal{H}^{N-1} \\ & \quad + \int_{\partial E} \nabla(\log(\det_N \phi)) \cdot n_\phi \; h \; d\mathcal{P}^{N-1}_\phi \\ & \quad + \int_{\partial E} (\operatorname{div}(hn_\phi) - \nu J \cdot \nu) \; d\mathcal{P}^{N-1}_\phi \\ & \quad =: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}. \end{aligned} \tag{5.8}$$

Recalling (3.5) and using the last equality of (3.6) we have

$$\mathrm{II} = \int_{\partial E} -\nu_\phi J \cdot n_\phi \; d\mathcal{P}^{N-1}_\phi + \int_{\partial E} \nu J \cdot \nu \; d\mathcal{P}^{N-1}_\phi,$$

so that

$$\mathrm{II} + \mathrm{IV} = \int_{\partial E} [-\nu_\phi J \cdot n_\phi + \operatorname{div}(hn_\phi)] \; d\mathcal{P}^{N-1}_\phi. \tag{5.9}$$

We observe now that, by (3.6), the last relation in (5.1), and (5.2) we have, for $i \in \{1, \dots, N\}$,

$$\begin{aligned} (\nu_\phi J)_i &= (\nu_\phi)_j \frac{\partial}{\partial x^i}\left(hn^j_\phi\right) = \frac{\partial}{\partial x^i} h + (\nu_\phi)_j h \frac{\partial n^j_\phi}{\partial x^i} \\ &= \frac{\partial h}{\partial x^i} - h\phi_{x^i}(x, n_\phi) = \frac{\partial h}{\partial x^i} + h\phi^o_{x^i}(x, \nu_\phi). \end{aligned} \tag{5.10}$$

Therefore, by (5.9) and (5.10) we have

$$\mathrm{I} = \int_{\partial E} (\nu_\phi J - \nabla h) \cdot n_\phi \; d\mathcal{P}^{N-1}_\phi,$$

so that

$$\begin{aligned} \mathrm{I}+\mathrm{II}+\mathrm{IV} &= \int_{\partial E} [-n_\phi \cdot \nabla h + \mathrm{div}(h n_\phi)]\, d\mathcal{P}_\phi^{N-1} \\ &= \int_{\partial E} \mathrm{div}\, n_\phi \, h \, d\mathcal{P}_\phi^{N-1}. \end{aligned}$$

Finally, by (5.8) and (3.9),

$$\begin{aligned} \frac{d}{dt} P_\phi(E_t)_{|t=0} &= \int_{\partial E} [\mathrm{div}\, n_\phi + n_\phi \cdot \nabla(\log(\det_N \phi))] h \, d\mathcal{P}_\phi^{N-1} \\ &= -\int_{\partial E} \kappa_\phi h \, d\mathcal{P}_\phi^{N-1}, \end{aligned} \tag{5.11}$$

and this proves formula (5.3).

Let now $g \in \mathcal{C}_0^1(U; \mathbf{R}^N)$ be an arbitrary vector field. Write $g = g^\tau + (g \cdot \nu_\phi) n_\phi$, where g^τ is a tangential vector field to ∂E. Repeating the computation in (5.8) with $h n_\phi$ replaced by g, one observes that the terms I, II, III, IV are linear in g, and that g^τ does not give any first order contribution to the variation. Using (5.11) we then get (5.4). □

5.1. Direction of maximal slope of P_ϕ

The following observation confirms the fact that, when dealing with evolution problems related to the functional P_ϕ, the natural direction (i.e., the direction of maximal slope of P_ϕ) of the displacement is $\kappa_\phi n_\phi$.

Let $\phi \in \mathcal{M}(\Omega)$, $\Sigma = \partial E$; then to each square-integrable vector field $g : \Sigma \to \mathbf{R}^N$ we can associate the norm $\|g\|_{\Sigma,\phi}^2 := \int_\Sigma (\phi(x,g))^2 \, d\mathcal{P}_\phi^{N-1}$. Let $L_\phi^2(\Sigma; \mathbf{R}^N)$ be the space $L^2(\Sigma; \mathbf{R}^N)$ endowed with this norm. Denote by dP_ϕ the variation of the perimeter functional P_ϕ as an element of the dual of the normed space $L_\phi^2(\Sigma; \mathbf{R}^N)$, and denote by $\langle \cdot, \cdot \rangle$ the duality. Then the following result holds.

Proposition 5.1 *A scalar multiple of the vector field $\kappa_\phi n_\phi$ is a solution of the problem*

$$\min\{\langle dP_\phi, g\rangle : g \in L_\phi^2(\Sigma; \mathbf{R}^N), \ \|g\|_{\Sigma,\phi} \le 1\}. \tag{5.12}$$

Proof. Let $\lambda \in \mathbf{R}$ and set $G(g) = \langle dP_\phi, g\rangle - \lambda(\|g\|_{\Sigma,\phi}^2 - 1)$ for $g \in$

$L^2_\phi(\Sigma;\mathbf{R}^N)$. Then, if $t \in \mathbf{R}$ and $f \in \mathcal{C}^1_0(\Sigma;\mathbf{R}^N)$, we have

$$\begin{aligned}\frac{d}{dt}G(g+tf)_{|t=0} &= \langle dP_\phi, f\rangle - \lambda\frac{d}{dt}\left(\int_\Sigma [\phi(x,g+tf)]^2 \ d\mathcal{P}^{N-1}_\phi\right)_{|t=0} \\ &= \langle dP_\phi, f\rangle - 2\lambda\int_\Sigma \phi(x,g)\phi_\xi(x,g)\cdot f \ d\mathcal{P}^{N-1}_\phi.\end{aligned}$$

If g is a solution of (5.12) we have $\frac{d}{dt}G(g+tf)_{|t=0} = 0$. Using the fact that dP_ϕ can be identified with $-\kappa_\phi\nu_\phi$ (see (5.4)), we then find

$$\int_\Sigma (\kappa_\phi\nu_\phi + 2\lambda\phi(x,g)\phi_\xi(x,g))\cdot f \ d\mathcal{P}^{N-1}_\phi = 0.$$

As f is arbitrary it follows that

$$\kappa_\phi\nu_\phi + 2\lambda\phi(x,g)\phi_\xi(x,g) = 0, \qquad \|g\|_{\Sigma,\phi} = 1.$$

By homogeneity and possibly rescaling g, we then get, using the notation of Section 3,

$$T(x,g) = \phi(x,g)\phi_\xi(x,g) = \kappa_\phi\nu_\phi.$$

Hence by Lemma 3.1 we get

$$g(x) = T^o(x,\kappa_\phi\nu_\phi) = \kappa_\phi\phi^o(x,\nu_\phi)\phi^o_\xi(x,\nu_\phi) = \kappa_\phi n_\phi,$$

which concludes the proof. □

5.2. The Almgren-Taylor-Wang approach

Motion by mean curvature can be approximated with the time discrete procedure introduced by Almgren-Taylor-Wang [3]. In the context of Finsler geometry such discrete process takes the following form.

Fix $E \in \mathcal{C}^2_b(\mathbf{R}^N)$, and let $\tau > 0$; the "evolved" set E_τ after the time step τ is defined as a minimum point for the energy functional

$$\mathcal{A}_\tau(B) = P_\phi(B) + \frac{1}{\tau}\int_{E\triangle B} \delta_\phi(x,\partial E)\ d\mathcal{H}^N_\phi,$$

where $E\triangle B = (E\setminus B)\cup(B\setminus E)$. Denoting by $\delta^{\partial E}_\phi$ the δ_ϕ signed distance function to ∂E, positive inside E, we have

$$\mathcal{A}_\tau(B) = P_\phi(B) - \frac{1}{\tau}\int_B \delta^{\partial E}_\phi \ d\mathcal{H}^N_\phi + C, \tag{5.13}$$

where $C = \frac{1}{\tau}\int_E \delta_\phi^{\partial E}\, d\mathcal{H}_\phi^N$ does not depend on B. Now, if E_τ minimizes $\mathcal{A}_\tau$, the first variation of $\mathcal{A}_\tau$ must vanish on E_τ. Let us compute the variation of the volume term of $\mathcal{A}_\tau$ in (5.13) in the n_ϕ direction. Using the notation of Section 5 we have $d\mathcal{H}^N(x+thn_\phi) = d\mathcal{H}^N(x) + t\mathrm{div}(hn_\phi)d\mathcal{H}^N(x) + o(t)d\mathcal{H}^N(x)$, so that

$$\begin{aligned}
&\frac{d}{dt}\int_{\Phi_t(B)} \delta_\phi^{\partial E}(y)\; d\mathcal{H}_\phi^N(y)_{|t=0} \\
&= \omega_N \int_B \nabla\delta_\phi^{\partial E}\cdot n_\phi\; h\; \det{}_N\,\phi d\mathcal{H}^N \\
&\quad + \omega_N \int_B \delta_\phi^{\partial E}\nabla(\log(\det{}_N\,\phi))\cdot n_\phi\; h\; \det{}_N\,\phi\; d\mathcal{H}^N \\
&\quad + \omega_N \int_B \delta_\phi^{\partial E}\mathrm{div}(hn_\phi)\; \det{}_N\,\phi\; d\mathcal{H}^N \\
&= \int_B \delta_\phi^{\partial E}\mathrm{div}_\phi(hn_\phi)\; d\mathcal{H}_\phi^N + \int_B \nabla\delta_\phi^{\partial E}\cdot n_\phi\; h\; d\mathcal{H}_\phi^N.
\end{aligned}$$

Using (3.8) with u replaced by $\delta_\phi^{\partial E}$ and g replaced by hn_ϕ, we then have, recalling definition (3.2) and (3.6),

$$\frac{d}{dt}\int_{\Phi_t(B)} \delta_\phi^{\partial E}(y)\; d\mathcal{H}_\phi^N(y)_{|t=0} = -\int_{\partial B} \delta_\phi^{\partial E} h\; d\mathcal{P}_\phi^{N-1}.$$

Using (5.3) we finally deduce that

$$0 = \frac{d}{dt}\mathcal{A}_\tau(\Phi_t(E_\tau))_{|t=0} = \int_{\partial E_\tau}\left(-\kappa_\phi + \frac{1}{\tau}\delta_\phi^{\partial E}\right) h\; d\mathcal{P}_\phi^{N-1}.$$

Hence each point $x \in \partial E_\tau$ has distance $\delta_\phi^{\partial E}(x) = \tau\kappa_\phi$, which is consistent with (1.1).

6. The Hamilton-Jacobi equation

In this section we consider the Hamilton-Jacobi equation for the mean curvature evolution (whose euclidean version is given by $u_t/|\nabla u| = \mathrm{div}(\frac{\nabla u}{|\nabla u|})$) with respect to the given Finsler metric $\phi \in \mathcal{M}(\mathbf{R}^N)$. In this section we shall assume that ϕ, ϕ^o satisfy the following further properties. Setting $\psi = \frac{1}{2}(\phi^o)^2$, there is a modulus σ and constants c, C with $0 < c \le C < +\infty$ such that

$$|\phi^o_{x^i\xi_i}(x,p) - \phi^o_{x^i\xi_i}(y,p)| \le \sigma(|x-y|),$$

$$|\phi^o_{\xi_i\xi_j}(x,p) - \phi^o_{\xi_i\xi_j}(y,p)| \leq C|x-y|,$$
$$|\phi^o_{\xi_i}(x,p)\frac{\partial}{\partial x^i}(\log(\det_N \phi(x))) - \phi^o_{\xi_i}(y,p)\frac{\partial}{\partial x^i}(\log(\det_N \phi(y)))|$$
$$\leq \sigma(|x-y|), \tag{6.1}$$

for any $i, j = 1, \ldots, N$, x, $y \in \mathbf{R}^N$, $p \in \mathbf{R}^N$ with $|p| = 1$, and

$$c|\xi^\star|^2 \leq \xi^\star \cdot \psi_{\xi\xi}(x,p)\xi^\star \leq C|\xi^\star|^2$$
$$x \in \mathbf{R}^N, \xi^\star \in \mathbf{R}^N, \quad p \in \mathbf{R}^N \setminus \{0\}. \tag{6.2}$$

Proposition 6.1 *The Hamilton-Jacobi equation for the mean curvature evolution with respect to $\phi \in \mathcal{M}(\mathbf{R}^N)$ reads as*

$$u_t = \phi^o(x, \nabla u)F(x, \nabla u, \nabla^2 u), \tag{6.3}$$

where F is defined in (3.11). *If $u_0 : \mathbf{R}^N \to \mathbf{R}$ is a continuous function which is constant outside a bounded subset of $\mathbf{R}^N$ and ϕ, ϕ^o satisfy assumptions* (6.1), (6.2), *equation* (6.3) *with initial condition $u(x,0) = u_0(x)$ admits a unique continuous viscosity solution. The level sets of the solution move with speed κ_ϕ in the direction n_ϕ, and with speed $\phi^o(x,\nu)\kappa_\phi$ in the euclidean normal direction ν.*

Proof. Since $|\xi|$ transforms into $\phi^o(x, \xi^\star)$, ∇u transforms into $\nabla_\phi u$, and div transforms into div_ϕ (see (3.1) and (3.2)), the equation $u_t/|\nabla u| = \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right)$ transforms into

$$\begin{aligned} u_t &= \phi^o(x,\nabla u)[\mathrm{div}(\phi^o_\xi(x,\nabla u)) + \phi^o_\xi(x,\nabla u)\cdot\nabla(\log(\det_N \phi))] \\ &= \Delta_\phi u - \phi^o_x(x,\nabla u)\cdot\phi^o_\xi(x,\nabla u), \end{aligned} \tag{6.4}$$

which can also be rewritten as

$$\begin{aligned} \frac{u_t}{|\nabla u|\phi^o(x,\nu)} &= \mathrm{div}_\phi(\phi^o_\xi(x,\nabla u)) = \mathrm{div}_\phi n_\phi = -\kappa_\phi \\ &= F(x,\nabla u,\nabla^2 u), \end{aligned} \tag{6.5}$$

where $\nu = \nabla u/|\nabla u|$.

In order to apply Theorem 4.9 of [25], which will imply the existence of a unique viscosity solution of (6.3) coupled with the initial condition $u(x,0) = u_0(x)$, we need to check properties (F1)–(F3), (F9), (F10) and the assumptions of Theorem 4.9 of [25]. We already know that the map

$$(x, p, X) \in \mathbf{R}^N \times (\mathbf{R}^N \setminus \{0\}) \times S^N \to -\phi^o(x,p)F(x,p,X)$$

is continuous and degenerate elliptic (see Section 3.2) which implies properties (F1), (F2) of [25], and is geometric. Properties (F3), (F9) and the last assumption of Theorem 4.9 in [25] follow from (6.1) and (6.2). It remains to check property (F10) of [25].

To this aim, thanks to assumptions (6.1), it suffices to focus attention to the second term in the right-hand side of (3.11), which (multiplied by $-\phi^o(x,p)$) reads as

$$-\phi^o(x,p)\mathrm{tr}(\phi^o_{\xi\xi}(x,p)X) = -\mathrm{tr}(A(x,p)X)$$

where $A(x,p) = \phi^o(x,p)\phi^o_{\xi\xi}(x,p)$. One can easily see that $A(x,p) = B(x,p) - \phi^o_\xi(x,p) \otimes \phi^o_\xi(x,p)$, where for simplicity we set $B(x,p) = \psi_{\xi\xi}$, i.e. A is a negative rank-one perturbation of the symmetric positive-definite matrix B, recall property (6.2). Using a well known interlacing property, eigenvalues $\lambda_2, \ldots, \lambda_N$ of A are interlaced with the eigevalues of B, and hence they belong to the interval $[c, C]$ (6.2). Moreover A is degenerate, so that $\lambda_1 = 0$ and the corresponding eigenvector is p (independent of x). With an orthogonal transformation $P(p)$, independent of x, matrix $P(p)^t A(x,p) P(p)$ is zero in its first row and column and the remaining $(N-1) \times (N-1)$ minor $\widetilde{A}$ is symmetric, positive definite, with all eigenvalues in $[c, C]$, therefore it can be factorized as $\widetilde{L}(x,p)\widetilde{L}^t(x,p)$ with a lower triangular matrix $\widetilde{L}(x,p)$ (Choleski factorization) [27]. From the upper bound on the eigenvalues, it follows that all elements of $\widetilde{L}(x,p)$ are bounded by some constant C independent of x and p. Moreover, since $\det \widetilde{L}(x,p)$ is bounded away from 0 (lower bound on the eigenvalues), we also have that each diagonal element in $\widetilde{L}(x,p)$ is bounded away from zero uniformly with respect to x and p. This is enough to recover Lipschitz continuity of $\widetilde{L}$ with respect to the elements of $\widetilde{A}$, and hence Lipschitz continuity in x. If now $L(x,p)$ is constructed by adding a zero column and row in front of $\widetilde{L}$, we can write $A(x,p) = \Sigma(x,p)\Sigma^t(x,p)$ with $\Sigma(x,p) = P(p)L(x,p)$. Reasoning as in [25, p. 463] we then get property (F10). Therefore equation (6.3) coupled with $u(x,0) = u_0(x)$ admits a unique viscosity solution u.

By (3.7), (3.6), and (3.5), we have $\nabla u \cdot n_\phi = |\nabla u| \nu \cdot n_\phi = |\nabla u| \phi^o(x, \nu)$. Hence the velocity of the front in the n_ϕ-direction, which is given by

$\dfrac{-u_t}{\nabla u \cdot n_\phi} = \dfrac{-u_t}{|\nabla u| \phi^o(x,\nu)}$ equals κ_ϕ by (6.5). We conclude that the solution of equation (6.4) (or (6.5)) is such that its level sets move with a speed κ_ϕ in the direction n_ϕ. In addition the speed of the fronts in the euclidean normal direction ν is $-u_t/|\nabla u| = \phi^o(x,\nu)\kappa_\phi$ (compare (1.3)). □

7. Asymptotic development of the reaction-diffusion equation

It is well known that the perimeter can be approximated, via De Giorgi's Γ-convergence [16], by a sequence of elliptic functionals [17], [33]. This result has been generalized by many authors (see, among others, [7], [10], [34], [35]). In particular, let $W : \mathbf{R} \to [0, +\infty[$ be defined as $W(s) = (1-s^2)^2$, set $w = W'/2$, and let $\phi \in \mathcal{M}(\Omega)$. If $\epsilon > 0$ let $\mathcal{F}_\epsilon : BV(\Omega) \to [0, +\infty]$ be defined as

$$\mathcal{F}_\epsilon(u) = \begin{cases} \displaystyle\int_\Omega \left[\epsilon\,(\phi^o(x,\nabla u))^2 + \epsilon^{-1} W(u)\right] \, d\mathcal{H}^N_\phi & \text{if } u \in H^1(\Omega), \\ +\infty & \text{elsewhere}, \end{cases} \tag{7.1}$$

where $BV(\Omega)$ is the space of the functions of bounded variation in Ω [19], [26]. Then one can prove that the Γ-limit of the sequence $\{\mathcal{F}_\epsilon\}_\epsilon$ on a set $E \in \mathcal{C}^2_b(\Omega)$ is $2c_0 P_\phi(E,\Omega)$, where $c_0 = \int_{-1}^{1} \sqrt{W(s)}\, ds$ (see [10], [34], [35]). In particular, there exists a sequence $\{u_\epsilon\}_\epsilon \subseteq H^1(\Omega)$ of functions converging to $1_E - 1_{\mathbf{R}^N \setminus E}$ in $L^1(\Omega)$ such that $\lim\limits_{\epsilon \to 0^+} \mathcal{F}_\epsilon(u_\epsilon) = 2c_0 P_\phi(E,\Omega)$. In [9] it is proved that such a minimizing sequence can be defined by means of the integrated distance δ_ϕ associated to ϕ defined in (2.8). This fact, together with (3.16), proves very useful for the asymptotic expansion of the Euler-Lagrange equation of $\mathcal{F}_\epsilon$. Precisely, let $\gamma : \mathbf{R} \to]-1,1[$ with $\gamma(0) = 0$ be the unique nondecreasing solution of the problem

$$\min\left\{ \int_{\mathbf{R}} [|\sigma'(t)|^2 + W(\sigma(t))] dt \ : \sigma \in H^1_{\mathrm{loc}}(\mathbf{R}), \ \lim_{t \to \pm\infty} \sigma(t) = \pm 1 \right\}$$

(with our choice of W one has $\gamma = \mathrm{tgh}$). Then the sequence $\{v_\epsilon\}_\epsilon$ of functions of $H^1(\Omega)$ defined by $v_\epsilon := \gamma\left(\dfrac{\delta_\phi}{\epsilon}\right)$ in a ϵ-tubular neighbourhood of ∂E (and suitably extended outside) converges to $1_E - 1_{\mathbf{R}^N \setminus E}$ in $L^1(\Omega)$ and is such that $\lim\limits_{\epsilon \to 0^+} \mathcal{F}_\epsilon(v_\epsilon) = 2c_0 P_\phi(E,\Omega)$ [9].

7.1. The reaction-diffusion equation associated to $\mathcal{F}_\epsilon$

Let us now calculate the first variation of $\mathcal{F}_\epsilon$. If $h \in \mathcal{C}_0^1(\Omega)$ we have, integrating by parts,

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_\epsilon(u+th)_{|t=0} &= \int_\Omega [2\epsilon\nabla_\phi u\cdot\nabla h + \epsilon^{-1}hW'(u)]\, d\mathcal{H}_\phi^N \\ &= \int_\Omega [-2\epsilon\Delta_\phi u + \epsilon^{-1}W'(u)]\, hd\mathcal{H}_\phi^N,\end{aligned}$$

where ∇_ϕ and Δ_ϕ are defined in (3.1) and (3.2). Therefore, by making the gradient flow of the functional $\mathcal{F}_\epsilon$ using the scalar product

$$(f,g) = \int_\Omega fgd\mathcal{H}_\phi^N, \tag{7.2}$$

we are led to the equation

$$u_t = \Delta_\phi u - \epsilon^{-2}w(u). \tag{7.3}$$

Let us briefly show how a formal inner asymptotic expansion suggests that equation (7.3) provides an approximation for an interface evolving by the law (1.1), see also [11]. Since the reasoning is formal, we shall assume that all quantities involved are sufficiently smooth.

Let u_ϵ be a solution of (7.3) on $\Omega\times(0,T)$ with suitable intial datum and boundary conditions, and for any $t\in(0,T)$ we set $\Sigma_\epsilon(t) := \{u_\epsilon(\cdot,t)=0\}$. Let $O_\epsilon(t)$ = outside of $\Sigma_\epsilon(t)$, $I_\epsilon(t)$ = inside of $\Sigma_\epsilon(t)$. The signed distance function $\delta_\phi^\epsilon : \mathbf{R}^N\times(0,T)\to\mathbf{R}$ from $\Sigma_\epsilon(t)$ is defined by

$$\delta_\phi^\epsilon(x,t) = \begin{cases} \delta_\phi(x,\Sigma_\epsilon(t)) & t\in(0,T),\ x\in I_\epsilon(t) \\ 0 & t\in(0,T),\ x\in\Sigma_\epsilon(t) \\ -\delta_\phi(x,\Sigma_\epsilon(t)) & t\in(0,T),\ x\in O_\epsilon(t), \end{cases}$$

where δ_ϕ is defined in (2.8). Let us denote for simplicity $\delta_\phi^\epsilon := \delta$. The projection $s(\cdot,t):\Omega\to\Sigma_\epsilon(t)$ is defined by $\delta_\phi(x,\Sigma_\epsilon(t)) = \delta_\phi(x,s(x,t))$. If we assume that $\Sigma_\epsilon(t)$ moves in the direction n_ϕ with velocity V_ϵ, we have $\partial_t\delta = -V_\epsilon$, since, for small σ, using (3.4),

$$\delta(x,t+\sigma) = \delta(x,t) - \phi(x,V_\epsilon\sigma n_\phi) = \delta(x,t) - V_\epsilon\sigma.$$

Here, in the first equality, we use the fact that the geodesic curve (with respect to δ_ϕ) connecting a point x close to $\Sigma_\epsilon(t)$ with its projection $s(x,t)$ on $\Sigma_\epsilon(t)$ has tangent vector at $s(x,t)$ in the direction of n_ϕ.

Let us introduce the stretched variable $y(x,t) = \delta(x,t)/\epsilon$, and for $s = s(x,t)$ define $U_\epsilon(y,s,t) := u_\epsilon(x,t)$. Suppose that both U_ϵ and V_ϵ can be expressed in terms of ϵ (inner expansion) as follows:

$$U_\epsilon(y,s,t) = \sum_{i=0}^{+\infty} \epsilon^i U_i(y,s,t), \qquad V_\epsilon(s,t) = \sum_{i=0}^{+\infty} \epsilon^i V_i(s,t).$$

Note that $U_i(0,s,t) = 0$ for all $i \geq 0$. With the notation $U_\epsilon^{(k)} = \partial_y^k U_\epsilon$, we have

$$\begin{aligned} \nabla U_\epsilon &= \frac{1}{\epsilon} U_\epsilon' \nabla \delta, \\ \frac{d}{dt} U_\epsilon &= -\frac{1}{\epsilon} U_\epsilon' V_\epsilon + \partial_s U_\epsilon \partial_t s + \partial_t U_\epsilon = -\frac{1}{\epsilon} U_0' V_0 + O(1). \end{aligned} \tag{7.4}$$

Using (2.1) for ϕ^o, (2.9), and (3.17), we find

$$\begin{aligned} \nabla_\phi U_\epsilon &= \phi^o\left(x, \frac{1}{\epsilon} U_\epsilon' \nabla \delta\right) \phi_\xi^o(x, \nabla\delta) \\ &= \frac{1}{\epsilon} U_\epsilon' \phi^o(x, \nabla\delta)\, \phi_\xi^o(x,\nabla\delta) = \frac{1}{\epsilon} U_\epsilon' \nabla_\phi \delta. \end{aligned} \tag{7.5}$$

Observe now that, by (3.17), (3.15), and the last equality in (2.10), we have

$$\nabla\delta \cdot \nabla_\phi \delta = \nabla\delta \cdot \phi_\xi^o(x, \nabla\delta) = 1.$$

Hence

$$\begin{aligned} \operatorname{div}(\nabla_\phi U_\epsilon) &= \frac{1}{\epsilon}\left[\frac{1}{\epsilon} U_\epsilon'' \nabla\delta \cdot \nabla_\phi \delta + U_\epsilon' \operatorname{div}(\nabla_\phi \delta)\right] \\ &= \frac{1}{\epsilon^2} U_\epsilon'' + \frac{1}{\epsilon} U_\epsilon' \operatorname{div}(\nabla_\phi \delta). \end{aligned} \tag{7.6}$$

Using (7.5) and definition (3.2) of div_ϕ, from (7.6) it follows that

$$\operatorname{div}_\phi(\nabla_\phi U_\epsilon) = \frac{1}{\epsilon^2} U_\epsilon'' + \frac{1}{\epsilon} U_\epsilon' \operatorname{div}(\nabla_\phi \delta) + \frac{1}{\epsilon} U_\epsilon' \nabla_\phi \delta \cdot \nabla(\log(\det_N \phi)).$$

Hence

$$\Delta_\phi U_\epsilon = \operatorname{div}_\phi(\nabla_\phi U_\epsilon) = \frac{1}{\epsilon^2} U_0'' + \frac{1}{\epsilon} U_0' \Delta_\phi \delta + \frac{1}{\epsilon} U_1'' + O(1). \tag{7.7}$$

Recall now that $\Delta_\phi \delta = -\kappa_\phi + O(\epsilon)$ by (3.16), where κ_ϕ is computed on Σ_ϵ, and that $w(U_\epsilon) = w(U_0) + \epsilon w'(U_0) \sum_{i=1}^{+\infty} \epsilon^{i-1} U_i + O(\epsilon^2)$. Let us insert (7.4) and (7.7) into (7.3), examine the resulting summands in increasing order and equate them to zero. The starting $\frac{1}{\epsilon^2}$-term yields

$$U_0''(y,t) - w(U_0(y,t)) = 0.$$

Using the boundary conditions provided by the outer expansion we obtain $U_0(y,t) = \gamma(y)$. The $\frac{1}{\epsilon}$-term yields

$$U_1'' - w'(\gamma)U_1 = \gamma'(\kappa_\phi - V_0).$$

For this problem to be solvable a compatibility condition between the right hand side and the kernel of $\mathcal{L}\eta := \eta'' - w'(\gamma)\eta$, subject to Dirichlet vanishing boundary conditions, must be enforced. Since $\operatorname{Ker}\mathcal{L} = \operatorname{span}(\gamma')$, we get $(\kappa_\phi - V_0)\int_{\mathbf{R}} |\gamma'|^2\, dy = 0$, which finally yields $V_0 = \kappa_\phi$ and $U_1 = 0$.

8. Final remarks

Remark 8.1. Most of the results of the present paper can be generalized to the case of a non symmetric ϕ, i.e., when ϕ satisfies the relation $\phi(x, t\xi) = t\phi(x,\xi)$ for any $(x,\xi) \in \Omega \times \mathbf{R}^N$, and $t > 0$, instead of (2.1). We can not however expect symmetric results with respect to changes of orientation of the surfaces. In the symmetric situation, for example, exchange of E into $\mathbf{R}^N \setminus E$ yields to a sign change in ν_ϕ, n_ϕ and κ_ϕ, while H_ϕ remains unchanged. As a consequence, the evolution law "velocity $= H_\phi$" remains unchanged. This is no longer true in the non symmetric case.

The following remark arises from the fact that, if the Finsler metric ϕ depends on the position $x \in \Omega$, then the measure $d\mathcal{H}^N_\phi$ defined in Section 2 is the Lebesgue measure multiplied by a suitable density function.

Remark 8.2. Let $m : \Omega \to]0, +\infty[$ be a function of class $\mathcal{C}^1(\Omega)$ such that $0 < c_1 \le m(x) \le c_2 < +\infty$ for any $x \in \Omega$, for two suitable positive constants c_1 and c_2. Define the non-negative Radon measure μ on Ω by $\mu = m\ dx$, where $dx = d\mathcal{H}^N$ stands for the Lebesgue measure. Then all results of the paper remain valid if one replaces $\omega_N \mathrm{det}_N\phi$, $d\mathcal{H}^N_\phi$, $d\mathcal{P}^{N-1}_\phi$, $\nabla(\log(\mathrm{det}_N\,\phi))$ by m, μ, $m\phi^o(x,\nu)d\mathcal{H}^{N-1}$, and $\nabla(\log m)$, respectively. As a consequence, the quantities div_ϕ, Δ_ϕ, κ_ϕ are modified by means of these substitutions, thus depending on the Finsler metric ϕ and on the measure μ, which are now independent. Given $\phi \in \mathcal{M}(\Omega)$, the choice $m = \omega_N \mathrm{det}_N\phi$ is quite natural, since the corresponding measure μ is the N-dimensional Hausdorff measure on Ω with respect to the distance δ_ϕ [9].

8.1. Anisotropic motion by mean curvature with a forcing term

All previuous results can be extended to the anisotropic mean curvature evolution with a forcing term $f \in \mathcal{C}^0(\Omega \times (0,T)) \cap L^\infty(\Omega \times (0,T))$ such that $|f(x,t) - f(y,t)| \leq C|x-y|$ for some constant $C > 0$ and any $x, y \in \Omega$, $t \in (0,T)$. Such evolution reads as

$$\text{velocity} = \kappa_\phi + f \quad \text{in the direction } n_\phi. \tag{8.1}$$

For instance, the functional P_ϕ of Section 5 must be replaced with the functional $P_\phi(E) + \int_E f \, d\mathcal{H}^N_\phi$, which, using the notation and the results of Section 5, is such that

$$\frac{d}{dt}\left(P_\phi(E_t) + \int_{E_t} f \, d\mathcal{H}^N_\phi\right)_{|t=0} = -\int_{\partial E} (\kappa_\phi + f) \; h \, d\mathcal{P}^{N-1}_\phi,$$

consistently with (8.1).

Similarly, the functional $\mathcal{A}_\tau$ in Section 5.2 must be modified into $\mathcal{A}_\tau(B) + \int_B f \, d\mathcal{H}^N_\phi$.

For what concerns Section 6 the Hamilton-Jacobi equation becomes

$$\frac{u_t}{\phi^o(x, \nabla u)} = -(\kappa_\phi + f). \tag{8.2}$$

Finally, it is enough to add to the functionals $\mathcal{F}_\epsilon$ in (7.1) the term $\int_\Omega u f \, d\mathcal{H}^N_\phi$. The reaction-diffusion equation associated to the Euler-Lagrange equation of this new functional then reads as follows:

$$u_t = \Delta_\phi u - \frac{1}{\epsilon^2} w(u) - \frac{1}{2c_0 \epsilon} f,$$

where $c_0 = \int_{-1}^{1} \sqrt{w(s)} \, ds$. One can check that, in this case, the term U_1 in general does not vanish.

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