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Models for coactions of finite groups on the AFD factor of type II_1

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Abstract. We proved in [Y1] that every coaction β of a finite group G on a II_1-factor gives rise to a normal subgroup N_β of G, called the inner part of β, and a "dual" 2-cocycle μ_β on $\ell^\infty(G/N_\beta)$. In this paper, we shall show that such a coaction β produces a function γ_β on $G\times G/N_\beta$ as well so that the triple $(N_\beta, \mu_\beta, \gamma_\beta)$ (modulo some equivalence) is a conjugacy invariant. It shall be shown too that, given an abstract triple (N, μ, γ) as above which satisfies suitable conditions, there exists a coaction $\beta=\beta_{(N,\mu,\gamma)}$, called a model coaction, of G on the AFD factor of type II_1 so that $(N_\beta, \mu_\beta, \gamma_\beta)$ realizes the given data (N, μ, γ).

Key words: coactions, Kac algebras, II_1-factors, inner parts, 2-cocycles, (twisted) crossed products.

0. Introduction

After Connes' breakthrough in classification of actions of cyclic groups and the integer group on the AFD (approximately finite-dimensional) factor of type II_1 [C1-2], his automorphism technique has been intensively developed and extended by several mathematicians [J], [O], [S-T1,2], [K-S-T] to the case of discrete amenable group actions on the AFD factors. When we consider a furhter possible extension of these results, it would be natural to ask ourselves what happens if we repalce groups by (discrete) quantized groups. As a result in connection with this question, we now know (see [L] and [Y]) that every finite-dimensional Kac algebra acts outerly on the AFD II_1 factor. Thus it is naturally expected that classification of actions of finite-dimensional Kac algebras should be possible as well along the line of Jones' work. If possible, this means that one needs to introduce a Kac algebra version of the characteristic invariant and the inner invariant. This program has been successfully completed by S. Popa and A. Wassermann in [P-W] in the case of cocommutative Kac algebras, i.e., in the case of coactions (of compact Lie groups), by classifying their dual actions. The results in §5 of [S-T1] also should be noted. Meanwhile, in connection with

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this program, we associated in [Y1] with every coaction β of a finite group G on a finite factor $\mathcal{A}$ a unique reduced Kac algebra of the group algebra of G, largest among the reduced Kac algebras the restriction (of β) to which is "inner." It was proven there too that there exists a "2-cocycle" μ_β on the Kac algebra dual C_β of the reduced algebra in such a way that the twisted algebra associated with μ_β, together with the restriction of the dual action $\hat{\beta}$ of β and "the derived coaction," corresponds to the characteristic invariant (of the dual action $\hat{\beta}$) in the sense of Popa and Wassermann. So, the next natural step one can think of would be to construct appropriate models for coactions on the AFD II_1 factor. The purpose of the present paper is to achieve this goal. It should be, however, remarked with emphasis that a construction of such models has been already given again by Popa and Wassermann in [P-W] (see also [S-T1]) by means of constructing models for dual actions (of compact Lie groups); they even exhibited models for a wider class of actions. Therefore, the author must frankly admit that the only excuse for presenting this note is that our (alternative) method employed in this paper has a close connection with "the inner parts of coactions" introduced in [Y1] and "dual 2-cocycles," and is thus more analogous to the argument set out in [J].

We now describe the organization of this paper. In Section 1, we review fundamental results obtained in [Y1], introducing the notation used in the following sections. In Section 2, we show that the derived coaction of a coaction on a finite factor is always inner. In Section 3, we deduce important properties of a function which completely determines the derived coaction. In Section 4, we study 2-cocycles on commutative Kac algebras and twisted crossed products by coactions of finite groups. Section 5 is devoted to constructing models for coactions of finite groups on the AFD factor of type II_1. In the last section, we give a definition of some conjugacy invariant for finite group coactions, which may be regarded as "the characteristic invariant" for coactions in our framework.

1. The inner part of coactions – Review

This section contains a review of the results concerning the inner part of finite group coactions on finite factors. In [Y1], it is shown that every coaction β of a finite group G on a finite factor $\mathcal{A}$ determines a unique normal subgroup N_β of G so that the "restriction" of β to the reduced algebra $R(G/N_\beta)$ of the group algebra $R(G)$ is "inner." It is proven there too that there exists a "2-cocycle" μ_β on $\ell^\infty(G/N_\beta)$ in such a way that the twisted product associated with the 2-cocycle is isomorphic to the relative commutant of $\mathcal{A}$ in the crossed product by the coaction. Both N_β and μ_β are conjugacy invariants for coactions of G on $\mathcal{A}$. In the following, we review the construction of these invariants in order to prepare for the discussion starting from Section 2. Readers are referred to [Y1] for more details.

Throughout this paper, G is a finite group. We use the standard notation $\ell^\infty(G)$ for the set $\mathbf{C}^G$ of all functions on G when each element of it is viewed as a multiplication operator on the Hilbert space $\ell^2(G)$. We always consider that $\ell^\infty(G)$ is equipped with the usual Kac (Hopf) algebraic structure: the coproduct Γ_G, the coinvolution j_G, the counit ε_G etc. are defined by

$$\Gamma_G(f)(s,t) = f(st), \qquad j_G(f)(s) = f(s^{-1}), \qquad \varepsilon_G(f) = f(e),$$

where $e \in G$ is the identity of G. When $\mathbf{C}^G$ is regarded as the (pre)dual of the C^*-algebra $\ell^\infty(G)$, it is denoted by $\ell^1(G)$, which is an involutive Banach algebra with product (convolution) $*$ and involution $\sharp$ defined by

$$(f * g)(t) = \sum_{s\in G} f(s)g(s^{-1}t), \quad f^\sharp(t) = \overline{f(t^{-1})}$$
$$(f, g \in \ell^1(G),\ t \in G).$$

We set $f^\vee(t) = f(t^{-1})$ and $\overline{f}(t) = \overline{f(t)}$. The symbol $\delta_s \in \ell^\infty(G)$, where $s \in G$, stands for the function on G given by $\delta_s(t) = \delta_{s,t}$. The left regular representation of G (or $\ell^1(G)$) is denoted by λ. The von Neumann algebra generated by $\lambda(G)$ on $\ell^2(G)$ is denoted by $R(G)$ (i.e., $R(G)$ is the group algebra of G). We also fix a coaction β of G on a finite factor $\mathcal{A}$. For the definition of a coaction, we refer readers to [N-T]. The coaction β determines a unique family $\{\Phi_s\}_{s\in G}$ of linear transformations on $\mathcal{A}$ characterized by the equation

$$\beta(a) = \sum_{s\in G} \Phi_s(a) \otimes \lambda(s) \qquad (a \in \mathcal{A}). \tag{1.1}$$

This family satisfies the following conditions:

$$\text{(CA0)} \quad \Phi_s(1) = \delta_{s,e}\cdot 1 \qquad \text{(CA1)} \quad \Phi_s \circ \Phi_t = \delta_{s,t}\Phi_s,$$
$$\text{(CA2)} \quad \Phi_s(a)^* = \Phi_{s^{-1}}(a^*), \qquad \text{(CA3)} \quad id_{\mathcal{A}} = \sum_{s\in G} \Phi_s,$$

$$\text{(CA4)} \quad \Phi_t(ab) = \sum_{s\in G} \Phi_s(a)\Phi_{s^{-1}t}(b) \qquad (a,b \in \mathcal{A}).$$

It is well-known that the coactions of G on $\mathcal{A}$ are in bijective correspondence with the families $\{\Phi_s\}_{s\in G}$ of linear maps of $\mathcal{A}$ satisfying (CA0)–(CA4). Every element X in the crossed product $\mathcal{A} \times_\beta R(G)$ by the coaction β can be written uniquely in the form $X = \sum_{s\in G} \beta(c(s))(1 \otimes \delta_s)$ for some $a = \{a(s)\}_{s\in G} \in \prod_{s\in G} \mathcal{A}$ (see Proposition 1.1 of [Y1]). We write X_a for X in this case. The relative commutant $\mathcal{A}^c$ of $\beta(\mathcal{A})$ in the crossed product is then the set of all elements X_c in which $\{c(s)\}$ satisfies

$$xc(t) = \sum_{s\in G} c(s)\Phi_{st^{-1}}(x) \tag{1.2}$$

for all $x \in \mathcal{A}$. We denote by $\mathrm{Rel}(\beta)$ the set of all families $\{c(s)\}_{s\in G}$ satisfying (1.2). The set $\mathrm{Rel}(\beta)$ has a $*$-algebraic structure coming from that of $\mathcal{A}^c$: $ac = \{a(s)c(s)\}_{s\in G} \in \mathrm{Rel}(\beta)$ and $c^* = \{c(s)^*\} \in \mathrm{Rel}(\beta)$ if $a = \{a(s)\}, c = \{c(s)\} \in \mathrm{Rel}(\beta)$. Moreover, if $c = \{c(s)\} \in \mathrm{Rel}(\beta)$ and $f \in \mathbf{C}^G$, then $c_f = \{\sum_{s\in G} b(s)f(s^{-1}t)\}_{t\in G}$ also belongs to $\mathrm{Rel}(\beta)$. Under this operation, $\mathrm{Rel}(\beta)$ is proven to be an $\ell^1(G)$-module. Namely, one has $(c_f)_g = c_{f*g}$ for any $f, g \in \ell^1(G)$. For any $a = \{a(s)\}, c = \{c(s)\} \in \mathrm{Rel}(\beta)$, $\sum_{s\in G} a(s)c(st)$ lies in the center of $\mathcal{A}$ for any $t \in G$. Since $\mathcal{A}$ is a factor, the equation

$$\sum_{s\in G} a(s)c(st)^* = f_{a,c}(t) \cdot 1$$

defines a function $f_{a,c}$ on G. With this notation, we have

$$f_{a,c}^\sharp = f_{c,a}, \qquad \overline{f_{a,c}} = f_{a^*,c^*},$$
$$g * f_{a,c} * h = f_{a_{g^\vee}, c_{\overline{h}}}$$

for any $g, h \in \mathbf{C}^G$. The linear span $\mathcal{I}_\beta$ of elements f in $\ell^1(G)$ of the form $f = f_{a,c}$ for some $a, c \in \mathrm{Rel}(\beta)$ then forms a two-sided ideal of $\ell^1(G)$. So there exists a unique central projection p_β in $\ell^1(G)$ such that $\mathcal{I}_\beta = \ell^1(G) * p_\beta$. We denote by $\mathrm{Int}(\beta)$ the set of all normalized irreducible characters χ of G such that $\chi * p_\beta \neq 0$, and call it the inner part of β. Here, by an irreducible character, we mean that it is the character of some irreducible representation of G. It can be shown that there exists an element $b = \{b(s)\} \in \mathrm{Rel}(\beta)$ so that $p_\beta = f_b (= f_{b,b})$ and $\sum_{s\in G} b(s) = 1$. With this $b = \{b(s)\}$, if we define an operator V_β in $\mathcal{A} \otimes R(G)$ by $V_\beta = \sum_{s\in G} b(s) \otimes \lambda(s)^*$, then it can be viewed as a unitary in the reduced algebra $\mathcal{A} \otimes R(G)_{\lambda(p_\beta)}$ (i.e.,

$V_\beta^* V_\beta = V_\beta V_\beta^* = 1 \otimes \lambda(p_\beta))$ and satisfies

$$\beta(a)(1 \otimes \lambda(p_\beta)) = V_\beta^*(a \otimes 1)V_\beta \qquad (\, a \in \mathcal{A}\,).$$

We describe this situation by saying that the restriction of β to $R(G)_{\lambda(p_\beta)}$ is inner. We showed in Theorem 3.6 of [Y1] that the reduced algebra $R(G)_{\lambda(p_\beta)}$ has a (reduced) Kac algebraic strucure (cf. [E-S]). Thus there exists a unique normal subgroup N_β of G such that $R(G)_{\lambda(p_\beta)}$ is isomorphic to $R(G/N_\beta)$ as Kac algebras. Hence the Kac algebra dual C_β of $R(G)_{\lambda(p_\beta)}$ is the $*$-subalgebra of $\ell^\infty(G)$ that is invariant under translation by elements of N_β, so that C_β can be identified with $\ell^\infty(G/N_\beta)$. With $b = \{b(s)\}$ as above, the equation

$$\mu_\beta(f,\, g) \cdot 1 = \sum_{t \in G} b_g(t) b_f(t) b(t)^* \qquad (\, f, g \in C_\beta\,)$$

defines a bilinear form μ_β on C_β. It turned out in Proposition 3.4 of [Y1] that μ_β is a cocycle on C_β in the sense of theory of Hopf algebras (see [M] for example). Thus, with the ordinary *sigma notation*, the equation

$$f \sharp g = \sum_{(f),(g)} \mu_\beta(f_{(1)},\, g_{(1)})\, f_{(2)} g_{(2)} \qquad (\, f, g \in C_\beta\,)$$

defines a new product, called the twisted product associated with μ_β, in C_β with 1 the identity with respect to the twisted product. Each vector f in C_β can be uniquely written in the form $f = f_{b^*, c^*}$ for some $c \in \mathrm{Rel}(\beta)$. From this, it follows that the equation

$$f_{b^*, c^*}^{\star} = f_{b^*, c}$$

defines a conjugate-linear map of C_β. It is in fact an involution in C_β with the twisted product. It is proven in Theorem 4.9 of [Y1] that the map $\Pi : \mathcal{A}^c \longrightarrow C_\beta$ defined by $\Pi(X_c) = f_{b^*, c^*}$ is a $*$-isomorphism from the realtive commutant $\mathcal{A}^c$ onto the twisted algebra C_β. For each $s \in G$, we define a linear map Ψ_s from $\mathrm{Rel}(\beta)$ into itself by

$$\Psi_s(c)(t) = \Phi_{tst^{-1}}(c(t)) \qquad (\, c \in \mathrm{Rel}(\beta)\,).$$

Then $\{\Psi_s\}_{s \in G}$ also satisfies conditions (CA0)–(CA4). Thus

$$\delta_\beta(f_{b^*, c^*}) = \sum_{s \in G} f_{b^*, \Psi_{s^{-1}}(c)^*} \otimes \lambda(s)$$

gives rise to a coaction δ_β of G on the twisted algebra C_β. We call it the derived coaction of β. The author was informed by Professor Y. Doi that this coaction is called the Miyashita(-Ulbrich) action in theory of Hopf algebras (see [D-T]). I am grateful to Prof. Doi for this information and sending me a reprint [D-T]. Under the isomorphism Π, the fixed-point algebra of the derived coaction coincides with the center of the crossed product.

2. The inner part of the derived coaction

Throughtout this section, we fix a coaction β of a finite group G on a finite factor $\mathcal{A}$ with τ the unique faithful normal tracial state. We shall freely use the notation introduced in Section 1.

Let $b = \{b(t)\}_{t\in G}$ be the element of $\mathrm{Rel}(\beta)$ that appeared in the preceding section. Suppose that $f \in C_\beta$. Then we have

$$f = f_{b^*,(b_f)^*} \tag{2.1}$$

Indeed, since $p_\beta = f_{b^*,b^*}$ and $p_\beta * f = f$ by definition, it follows from Lemma 1.6 of [Y1] that $f = f_{b^*,(b^*)_{\overline{f}}} = f_{b^*,(b_f)^*}$. Hence the derived coaction δ_β of G on the twisted algebra C_β can be described as follows.

$$\delta_\beta(f) = \sum_{s\in G} f_{b^*,\Psi_{s^{-1}}(b_f)^*} \otimes \lambda(s).$$

With this in mind, we define, for each $s \in G$, a linear map Ω_s from C_β into itself by

$$\Omega_s(f) = f_{b^*,\Psi_{s^{-1}}(b_f)^*} \qquad (f \in C_\beta) \tag{2.2}$$

Thus the family $\{\Omega_s\}_{s\in G}$ determines the derived coaction on C_β. By a simple calculation, we obtain

$$f_{b^*,\Psi_{s^{-1}}(b_f)^*}(t) = \sum_{r\in G} f_{\Psi_{rs^{-1}r^{-1}}(b),b}(rt^{-1})f(r).$$

So let us define a function γ_0 on $G \times G$ by

$$\gamma_0(s,t) = f_{\Psi_s(b),b}(t).$$

Then, with $f \in C_\beta$, we have

$$\Omega_s(f)(t) = \sum_{r\in G} \gamma_0(rs^{-1}r^{-1}, rt^{-1})f(r). \tag{2.3}$$

We put $\xi_0 = f_{b^*,b}$. From (2.1) and Lemma 1.6 of [Y1], we easily find that, for any $f \in C_\beta$,

$$\xi_0 * \overline{f} = f^\star \tag{2.4}$$

In the meantime, we assert that

$$\xi_0(t) = \sum_{s\in G} \overline{\mu_\beta(\delta_{t^{-1}s}, \delta_s)} = \sum_{s\in G} \overline{\mu_\beta(\delta_{t^{-1}s} * p_\beta, \delta_s * p_\beta)} \tag{2.5}$$

In fact, by the definition of the bilinear form μ_β, we have

$$\begin{aligned}\sum_{s\in G} \overline{\mu_\beta(\delta_{t^{-1}s}, \delta_s)} &= \sum_{r,s\in G} \overline{\tau(b_{\delta_s}(r) b_{\delta_{t^{-1}s}}(r) b(r)^*)} \\ &= \sum_{r,s\in G} \overline{\tau(b(rs^{-1}) b(rs^{-1}t) b(r)^*)} \\ &= \sum_{r,s\in G} \overline{\tau(b(s^{-1}) b(s^{-1}t) b(r)^*)} \\ &= \sum_{s\in G} \overline{\tau(b(s^{-1}) b(s^{-1}t))} \\ &= \sum_{s\in G} \tau(b(s^{-1})^* b(s^{-1}t)^*) = \xi_0(t).\end{aligned}$$

This proves (2.5). Since $(f^\star)^\star = f$ for any $f \in C_\beta$, it follows that $\xi_0 * \overline{\overline{\xi_0}} = p_\beta$. Moreover, from Lemma 1.6 of [Y1], we have $\xi_0^\vee = \xi_0$. To sum up, we obtain

$$\xi_0 * \overline{\overline{\xi_0}} = \xi_0 * \xi_0^\sharp = p_\beta. \tag{2.6}$$

From this, it results the equation

$$J_\mu f = f^\star \qquad (f \in C_\beta)$$

defines a conjugate-linear unitary involution J_μ on the Hilbert space $\{ f \in \ell^2(G) : f \in C_\beta \}$.

Now we turn attention to compute the inner part of the derived coaction. To do so, for each $t \in G$, we define a family $\{d(t)\}_{t\in G}$ of functions on G by $d(t) = \delta_{t^{-1}} * p_\beta$, which clearly belongs to C_β. Our immediate goal is to show that this family $d = \{d(t)\}_{t\in G}$ plays the same role as $b = \{b(s)\}_{s\in G}$ in the previous section does. First, we assert that $\{d(t)\}$ belongs to $\mathrm{Rel}(\delta_\beta)$. For this, let $f \in C_\beta$. From (2.2) and Lemma 4.5 of [Y1],

$$\sum_{r\in G} \{d(r) \sharp \Omega_{rt^{-1}}(f)\}(s)$$

$$
\begin{aligned}
&= \sum_{r\in G}\{f_{b^*,b^*_{\delta_{r^{-1}}}} \sharp f_{b^*,\Psi_{tr^{-1}}(b_f)^*}\}(s)\\
&= \sum_{r\in G}\{f_{b^*,b^*_{\delta_{r^{-1}}}\Psi_{tr^{-1}}(b_f)^*}\}(s)\\
&= \sum_{r,u\in G} \tau(b(u)^*\Psi_{tr^{-1}}(b_f)(us)b(usr))\\
&= \sum_{r,u\in G} \tau(b(u)^*\Phi_{ustr^{-1}s^{-1}u^{-1}}(b_f(us))b(usr))\\
&= \sum_{r,u\in G} \tau(b(u)^*\Phi_{ustr^{-1}}(b_f(us))b(r)).
\end{aligned}
$$

Since $\{b(s)\}$ is in $\mathrm{Rel}(\beta)$, we have $\Phi_g(x)b(s) = b(gs)\Phi_g(x)$ for any $x \in \mathcal{A}$ and $g, s \in G$. From this, together with (4.1) of [Y1], it follows that

$$
\begin{aligned}
&\sum_{r\in G}\{d(r)\sharp\Omega_{rt^{-1}}(f)\}(s)\\
&\qquad= \sum_{r,u\in G} \tau(b(u)^*b(ust)\Phi_{ustr^{-1}}(b_f(us)))\\
&\qquad= \sum_{u\in G} \tau(b(u)^*b(ust)b_f(us))\\
&\qquad= \sum_{u\in G} \tau(b(ust)b_f(us)b(u)^*) = \{f\sharp d(t)\}(s).
\end{aligned}
$$

Thus $\sum_{r\in G} d(r)\sharp\Omega_{rt^{-1}}(f) = f\sharp d(t)$, which shows that $d = \{d(t)\}_{t\in G}$ belongs to $\mathrm{Rel}(\delta_\beta)$. Moreover, $\{d(t)\}$ satisfies the following identities.

Lemma 2.7 *With $d(t) = \delta_{t^{-1}} * p_\beta$, we have*

(a) $f\sharp d(t) = \sum_{s\in G} d(s)\sharp\Omega_{st^{-1}}(f)$;

(b) $\sum_{s\in G} d(s)\sharp d(st)^\star = p_\beta(t)\cdot 1, \quad \sum_{s\in G} d(s)^\star\sharp d(t^{-1}s) = p_\beta(t)\cdot 1$;

(c) $\sum_{s\in G} d(s)\sharp f\sharp d(t^{-1}s)^\star = \sum_{s\in G} p_\beta(s^{-1}t)\Omega_s(f)$

for any $t\in G$ and $f\in C_\beta$.

Proof. We have already proven the first identity. For (b), note that

$d(s) = f_{b^*, b^*_{\delta_{s^{-1}}}}$. Thus, by Lemma 4.5 of [Y1], we have

$$\begin{aligned}
\sum_{s\in G}\{d(s)\,\sharp\, d(st)^\star\}(u) &= \sum_{s\in G}\{f_{b^*,b_{\delta_{s^{-1}}}}\,\sharp\, f^\star_{b^*,b^*_{\delta_{t^{-1}s^{-1}}}}\}(u) \\
&= \sum_{s\in G} f_{b^*,b^*_{\delta_{s^{-1}}}b_{\delta_{t^{-1}s^{-1}}}}(u) \\
&= \sum_{r,s\in G} \tau(b(r)^*b(rust)^*b(rus)) \\
&= \sum_{r,s\in G} \tau(b(r)^*b(st)^*b(s)) \\
&= \sum_{s\in G} \tau(b(st)^*b(s)) = f_b(t) = p_\beta(t).
\end{aligned}$$

This proves the first identity of (b). For the second identity, we use the last equality of (2.5) of [Y1]. From this,

$$\begin{aligned}
\sum_{s\in G}\{d(s)^\star\,\sharp\, d(t^{-1}s)\}(u) &= \sum_{s\in G} f_{b^*,b_{\delta_{s^{-1}}}b^*_{\delta_{s^{-1}t}}}(u) \\
&= \sum_{r,s\in G} \tau(b(r)^*b(rut^{-1}s)b(rus)^*) \\
&= \sum_{r,s\in G} \tau(b(rus)^*b(r)^*b(rut^{-1}s)) \\
&= \sum_{r,s\in G} \tau(b(s)^*b(r)^*b(rut^{-1}u^{-1}r^{-1}s)) \\
&= \sum_{r,s\in G} p_\beta(s^{-1}rutu^{-1}r^{-1})\tau(\Phi_s(b(r)^*)).
\end{aligned}$$

Recall that Φ_e is a conditional expectation from $\mathcal{A}$ onto the fixed-point algebra $\mathcal{A}^\beta$. In fact, it is the unique conditional expectation that preserves the trace τ, since $\Phi_e = (\iota\otimes\varphi_G)\circ\beta$, where φ_G denotes the Plancherel measure of G. Thus $\tau = \tau\circ\Phi_e$. From this, it follows that $\tau(\Phi_s(b(r)^*)) = \delta_{s,e}\tau(b(r)^*)$. Hence

$$\begin{aligned}
\sum_{s\in G}\{d(s)^\star\,\sharp\, d(t^{-1}s)\}(u) &= \sum_{r\in G} p_\beta(rutu^{-1}r^{-1})\tau(b(r)^*) \\
&= p_\beta(t).
\end{aligned}$$

This proves the second identity. For the last identity, let $f \in C_\beta$. we use the equality of (2.5) of [Y1] again in the following computation. Since

$f = f_{b^*,(b_f)^*}$, we have

$$\begin{aligned}
&\sum_{s\in G}\{d(s)\,\sharp\, f\,\sharp\, d(t^{-1}s)^*\}(u)\\
&\quad = \sum_{s\in G} f_{b^*, b^*_{\delta_{s^{-1}}} b_f^* b_{\delta_{s^{-1}t}}}(u)\\
&\quad = \sum_{r,s\in G} \tau(b(r)^* b(rut^{-1}s)^* b_f(ru) b(rus))\\
&\quad = \sum_{r,s\in G} \tau(b(r)^* b(s)^* b_f(ru) b(rutu^{-1}r^{-1}s))\\
&\quad = \sum_{r,s\in G} p_\beta(s^{-1}rut^{-1}u^{-1}r^{-1})\tau(b(r)^*\Phi_s(b_f(ru)))\\
&\quad = \sum_{r,s\in G} p_\beta(s^{-1}t^{-1})\tau(b(r)^*\Phi_{rusu^{-1}r^{-1}}(b_f(ru)))\\
&\quad = \sum_{r,s\in G} p_\beta(s^{-1}t^{-1})\tau(b(r)^*\Psi_s(b_f)(ru))\\
&\quad = \sum_{s\in G} p_\beta(s^{-1}t^{-1}) f_{b^*,\Psi_s(b_f)^*}(u)\\
&\quad = \sum_{s\in G} p_\beta(s^{-1}t)\Omega_s(f)(u).
\end{aligned}$$

This completes the proof. □

Proposition 2.8 *The restriction of the derived coaction on C_β to the reduced Kac algebra $R(G)_{\lambda(p_\beta)}$ is inner in the sense that there exists a unitary V_{δ_β} in $C_\beta \otimes R(G)_{\lambda(p_\beta)}$ such that*

$$\delta_\beta(f)(1\otimes\lambda(p_\beta)) = V_{\delta_\beta}^*(f\otimes 1)V_{\delta_\beta}$$

for all $f \in C_\beta$.

Proof. With $\{d(s)\}_{s\in G}$ in $\mathrm{Rel}(\delta_\beta)$ introduced above, we define an element V_{δ_β} in $C_\beta \otimes R(G)$ by

$$V_{\delta_\beta} = \sum_{s\in G} d(s)\otimes\lambda(s)^*.$$

From the first identity of Part (b) of Lemma 2.7, we find that $V_{\delta_\beta}V_{\delta_\beta}^* = 1\otimes\lambda(p_\beta)$. Remark that $1\otimes\lambda(p_\beta)$ is central in $C_\beta\otimes R(G)$. Thus the finite-dimensionality of $C_\beta\otimes R(G)$ implies that $V_{\delta_\beta}^*V_{\delta_\beta} = 1\otimes\lambda(p_\beta)$. Now, by identity (a) of Lemma 2.7, it can be verified that

$$V_{\delta_\beta}\delta_\beta(f) = (f \otimes 1)V_{\delta_\beta}$$

for all $f \in C_\beta$. Thus, we are done. □

3. The function γ

In the preceding section, we introduced the function γ_0 on $G \times G$ which completely describes the G-grading in C_β (recall (2.3)) determined by the derived coaction δ_β. The purpose of the present section is to closely examine some properties this function possesses. In what follows, we still fix a coaction β of G on $\mathcal{A}$ with τ the unique faithful normal tracial state, and keep all the notation established in Section 2, except that we write μ for μ_β for simplicity.

Lemma 3.1 *We have*

$$\overline{\gamma_0(s,t)} = \gamma_0(t^{-1}st, t^{-1}), \tag{3.2}$$

$$\sum_{r\in G} \gamma_0(s,r)p_\beta(r^{-1}t) = \gamma_0(s,t). \tag{3.3}$$

In particular, the function $t \in G \longmapsto \gamma_0(s,t)$ belongs to C_β for each $s \in G$.

Proof. First, by Lemma 4.6 of [Y1],

$$\begin{aligned}
\overline{\gamma_0(s,t)} &= \overline{f_{\Psi_s(b),b}(t)} \\
&= f_{\Psi_s(b)^*,b^*}(t) \\
&= \sum_{r\in G} \tau(\Psi_s(b)(r)^* b(rt)) \\
&= \sum_{r\in G} \tau(\Phi_{rsr^{-1}}(b(r))^* b(rt)) \\
&= \sum_{r\in G} \tau(\Phi_{rs^{-1}r^{-1}}(b(r)^*) b(rt)) \\
&= \sum_{r\in G} \tau(b(r)^* \Phi_{rsr^{-1}}(b(rt))) \\
&= \sum_{r\in G} \tau(\Psi_{t^{-1}st}(b)(rt) b(r)^*) \\
&= f_{\Psi_{t^{-1}st}(b),b}(t^{-1}) = \gamma_0(t^{-1}st, t^{-1}).
\end{aligned}$$

This proves (3.2). The second identity is clearly equivalent to $f_{\Psi_s(b),b} * p_\beta =$

$f_{\Psi_s(b),b}$. But this follows from the fact that $f_{\Psi_s(b),b} \in C_\beta$. □

In the subsections **A**–**F** that follow, we deduce identities that γ_0 needs to satisfy for $\{\Omega_s\}_{s\in G}$ to define the derived coaction.

A. By (2.3), we have

$$\Omega_s(1)(t) = \sum_{r\in G} \gamma_0(rs^{-1}r^{-1}, rt^{-1}).$$

Hence, the identity $\Omega_s(1) = \delta_{s,e}1$ is equivalent to

$$\sum_{r\in G} \gamma_0(rs^{-1}r^{-1}, rt^{-1}) = \delta_{s,e} \qquad (s, t \in G) \tag{3.4}$$

If we use (3.2), then identity (3.4) can be transformed into

$$\sum_{r\in G} \overline{\gamma_0(tst^{-1}, tr^{-1})} = \delta_{s,e}.$$

Upon replacing s by $t^{-1}st$ in the above equality, we obtain

$$\sum_{r\in G} \gamma_0(s, r) = \delta_{s,e} \qquad (s \in G) \tag*{(3.4)$'$}$$

B. Let $f \in C_\beta$. Then

$$\begin{aligned}\sum_{s\in G} \Omega_s(f)(t) &= \sum_{r,s\in G} \gamma_0(rs^{-1}r^{-1}, rt^{-1})f(r) \\ &= \sum_{r\in G}\Big(\sum_{s\in G} \gamma_0(s, rt^{-1})\Big)f(r).\end{aligned}$$

In the meantime,

$$f(t) = (p_\beta * f)(t) = \sum_{r\in G} p_\beta(r^{-1}t)f(r).$$

Thus the condition $\sum_{s\in G}\Omega_s(f) = f$ is equivalent to

$$\sum_{r\in G}\Big(\sum_{s\in G} \gamma_0(s, rt^{-1}) - p_\beta(r^{-1}t)\Big)f(r) = 0.$$

This is true for all $f \in C_\beta$. From this, together with (3.3), it follows that

$$\sum_{s\in G} \gamma_0(s, rt^{-1}) = p_\beta(r^{-1}t).$$

Therefore, we may conclude that

$$\sum_{s\in G} \gamma_0(s,t) = p_\beta(t). \tag{3.5}$$

for any $t \in G$.

C. Let f be in C_β. Then

$$\begin{aligned}
&\Omega_u(\Omega_v(f))(t)\\
&\quad = \sum_{r\in G} \gamma_0(ru^{-1}r^{-1}, rt^{-1})\Omega_v(f)(r)\\
&\quad = \sum_{s\in G}\Big(\sum_{r\in G} \gamma_0(ru^{-1}r^{-1}, rt^{-1})\gamma_0(sv^{-1}s^{-1}, sr^{-1})\Big)f(s).
\end{aligned}$$

In the meantime, we have

$$\delta_{u,v}\Omega_u(f)(t) = \sum_{s\in G} \delta_{u,v}\gamma_0(su^{-1}s^{-1}, st^{-1})f(s).$$

Hence the identity $\Omega_u(\Omega_v(f)) = \delta_{u,v}\Omega_u(f)$ for all $f \in C_\beta$ and $u, v \in G$ implies that

$$\sum_{r\in G} \gamma_0(ru^{-1}r^{-1}, rt^{-1})\gamma_0(sv^{-1}s^{-1}, sr^{-1}) = \delta_{u,v}\gamma_0(su^{-1}s^{-1}, st^{-1})$$

for all $s, t, u, v \in G$. It is easy to see that the above equality is the same as

$$\sum_{r\in G} \gamma_0(rur^{-1}, rt)\gamma_0(v^{-1}, r^{-1}) = \delta_{u,v}\gamma_0(u,t) \tag{3.6}$$

for all $t, u, v \in G$.

D. Let f, g be in C_β. Recall that $f \sharp g = \sum_{s,t\in G} \mu(\delta_s * p_\beta,\, \delta_t * p_\beta)(\delta_{s^{-1}} * f)(\delta_{t^{-1}} * g)$. Thus

$$\begin{aligned}
&\Omega_s(f \sharp g)(t)\\
&\quad = \sum_{r,u,v\in G} \gamma_0(rs^{-1}r^{-1}, rt^{-1})\mu(\delta_u * p_\beta,\, \delta_v * p_\beta)f(ur)g(vr)\\
&\quad = \sum_{r,u,v\in G} \gamma_0(rs^{-1}r^{-1}, rt^{-1})\mu(\delta_{ur^{-1}} * p_\beta,\, \delta_{vr^{-1}} * p_\beta)f(u)g(v).
\end{aligned}$$

Meanwhile, we have

$$\sum_{p\in G} \{\Omega_p(f) \sharp \Omega_{p^{-1}s}(g)\}(t)$$

$$
\begin{aligned}
&= \sum_{p,m,n\in G} \mu(\delta_m * p_\beta,\, \delta_n * p_\beta)\Omega_p(f)(mt)\Omega_{p^{-1}s}(nt) \\
&= \sum_{p,m,n,u,v\in G} \mu(\delta_m * p_\beta,\, \delta_n * p_\beta)\gamma_0(up^{-1}u^{-1}, ut^{-1}m^{-1}) \\
&\qquad\qquad \times\gamma_0(vs^{-1}pv^{-1}, vt^{-1}n^{-1})f(u)g(v).
\end{aligned}
$$

Hence the identity $\Omega_s(f \sharp g) = \sum_{p\in G} \Omega_p(f) \sharp \Omega_{p^{-1}s}(g)$ for all $f, g \in C_\beta$ and $s \in G$ implies that

$$
\begin{aligned}
&\sum_{r\in G} \gamma_0(rs^{-1}r^{-1}, rt^{-1})\mu(\delta_{ur^{-1}} * p_\beta,\, \delta_{vr^{-1}} * p_\beta) \\
&\quad = \sum_{p,m,n\in G} \mu(\delta_m * p_\beta,\, \delta_n * p_\beta)\gamma_0(up^{-1}u^{-1}, ut^{-1}m^{-1}) \\
&\qquad\qquad \times\gamma_0(vs^{-1}pv^{-1}, vt^{-1}n^{-1})
\end{aligned}
\tag{3.7}
$$

for all $s, t, u, v \in G$.

E. Let $f \in C_\beta$. From (2.4), we have

$$
\Omega_s(f)^\star(t) = \sum_{r,u\in G} \xi_0(r)\overline{\gamma_0(us^{-1}u^{-1}, ut^{-1}r)f(u)}.
$$

Meanwhile,

$$
\begin{aligned}
\Omega_{s^{-1}}(f^\star)(t) &= \sum_{r\in G} \gamma_0(rsr^{-1}, rt^{-1})f^\star(r) \\
&= \sum_{r,u\in G} \gamma_0(rsr^{-1}, rt^{-1})\xi_0(ru^{-1})\overline{f(u)}.
\end{aligned}
$$

Hence the identity $\Omega_s(f)^\star = \Omega_{s^{-1}}(f^\star)$ for all $f \in C_\beta$ and $s \in G$ yields

$$
\sum_{r\in G} \xi_0(r)\,\overline{\gamma_0(us^{-1}u^{-1}, ut^{-1}r)} = \sum_{r\in G} \gamma_0(rsr^{-1}, rt^{-1})\xi_0(ru^{-1})
$$

for all $s, t, u \in G$. This turns out to be equivalent to

$$
\sum_{r\in G} \xi_0(r)\,\overline{\gamma_0(s^{-1}, tr)} = \sum_{r\in G} \xi_0(r)\gamma_0(rsr^{-1}, rt)
\tag{3.8}
$$

for all $s, t \in G$.

F. Finally, we consider the property of the derived coaction that its restriction to the Kac algebra $R(G)_{\lambda(p_\beta)}$ is inner. In view of (the proof of)

Proposition 2.8, this property can be characterized as

$$f \sharp d(t) = \sum_{s \in G} d(s) \sharp \Omega_{st^{-1}}(f) \qquad (f \in C_\beta), \tag{3.9}$$

where $d(t) = \delta_{t^{-1}} * p_\beta \in C_\beta$. Recall again that $f \sharp g = \sum_{s,t \in G} \mu(\delta_s * p_\beta, \delta_t * p_\beta)(\delta_{s^{-1}} * f)(\delta_{t^{-1}} * g)$. We have

$$\begin{aligned} \{f \sharp d(t)\}(r) &= \sum_{u,v \in G} \mu_\beta(\delta_u * p_\beta, \delta_v * p_\beta) f(ur) p_\beta(tvr) \\ &= \sum_{u \in G} \mu_\beta(\delta_{ur^{-1}} * p_\beta, \delta_{(rt)^{-1}} * p_\beta) f(u). \end{aligned}$$

Meanwhile,

$$\begin{aligned} &\sum_{s \in G} \{d(s) \sharp \Omega_{st^{-1}}(f)\}(r) \\ &\quad = \sum_{s,m,n \in G} \mu_\beta(\delta_m * p_\beta, \delta_n * p_\beta) p_\beta(smr) \Omega_{st^{-1}}(f)(nr) \\ &\quad = \sum_{s,n,u \in G} \mu_\beta(\delta_{(rs)^{-1}} * p_\beta, \delta_n * p_\beta) \gamma_0(uts^{-1}u^{-1}, ur^{-1}n^{-1}) f(u). \end{aligned}$$

Thus identity (3.9) implies that

$$\begin{aligned} &\mu_\beta(\delta_{ur^{-1}} * p_\beta, \delta_{(rt)^{-1}} * p_\beta) \\ &\quad = \sum_{s,n,v \in G} \mu_\beta(\delta_{(rs)^{-1}} * p_\beta, \delta_n * p_\beta) \gamma_0(vts^{-1}v^{-1}, vr^{-1}n^{-1}) p_\beta(v^{-1}u) \\ &\quad = \sum_{s,n,v \in G} \mu_\beta(\delta_{(rs)^{-1}} * p_\beta, \delta_n * p_\beta) \\ &\qquad\qquad \times \gamma_0(vrts^{-1}r^{-1}v^{-1}, vn^{-1}) p_\beta(v^{-1}ur^{-1}). \end{aligned}$$

for any $r, t, u \in G$. Upon replacing u by ur and t by $r^{-1}t$ in the above equality, we obtain

$$\begin{aligned} &\mu_\beta(\delta_u * p_\beta, \delta_{t^{-1}} * p_\beta) \\ &= \sum_{s,n,v \in G} \mu_\beta(\delta_{(rs)^{-1}} * p_\beta, \delta_n * p_\beta) \gamma_0(vts^{-1}r^{-1}v^{-1}, vn^{-1}) p_\beta(v^{-1}u) \\ &= \sum_{s,n,v \in G} \mu_\beta(\delta_{s^{-1}} * p_\beta, \delta_n * p_\beta) \gamma_0(vts^{-1}v^{-1}, vn^{-1}) p_\beta(v^{-1}u). \end{aligned} \tag{3.10}$$

for any $u, t \in G$.

We summarize the discussion in the subsections **A**–**F** in the lemma that follows.

Lemma 3.11 *Define a function γ_0 on $G \times G$ by*

$$\gamma_0(s,t) = f_{\Psi_s(b),b}(t).$$

Then it satisfies the following identities.

$$(11.0)\quad \Omega_s(f)(t) = \sum_{r\in G} \gamma_0(rs^{-1}r^{-1}, rt^{-1})f(r);$$

$$(11.1)\quad \sum_{r\in G} \gamma_0(rs^{-1}r^{-1}, rt^{-1}) = \delta_{s,e};$$

$$(11.2)\quad \sum_{r\in G} \gamma_0(r,t) = p_\beta(t);$$

$$(11.3)\quad \sum_{r\in G} \gamma_0(rur^{-1}, rt)\gamma_0(v, r^{-1}) = \delta_{u,v}\gamma_0(u,t);$$

$$(11.4)\quad \sum_{r\in G} \gamma_0(rs^{-1}r^{-1}, rt^{-1})\mu(\delta_{ur^{-1}} * p_\beta,\ \delta_{vr^{-1}} * p_\beta)$$
$$= \sum_{p,m,n\in G} \mu(\delta_m * p_\beta,\ \delta_n * p_\beta)\gamma_0(up^{-1}u^{-1}, ut^{-1}m^{-1})$$
$$\times\, \gamma_0(vs^{-1}pv^{-1}, vt^{-1}n^{-1});$$

$$(11.5)\quad \sum_{r\in G} \xi_0(r)\overline{\gamma_0(s^{-1}, tr)} = \sum_{r\in G} \xi_0(r)\gamma_0(rsr^{-1}, rt);$$

$$(11.6)\quad \mu_\beta(\delta_u * p_\beta,\ \delta_{t^{-1}} * p_\beta)$$
$$= \sum_{r,p,q\in G} \mu_\beta(\delta_{r^{-1}} * p_\beta,\ \delta_p * p_\beta)\gamma_0(qtr^{-1}q^{-1}, qp^{-1})p_\beta(q^{-1}u)$$

for all $s,t,u,v \in G$ and $f \in C_\beta$.

From Theorem 3.6 of [Y1], $R(G)_{\lambda(p_\beta)}$ is isomorphic to the group algebra $R(G/N_\beta)$ of the quotient group G/N_β as Kac algebras for a unique normal subgroup N_β of G. Hence C_β is exactly the set of all functions on G that are invariant under translation by elements of N_β, and thus can be regarded as $\ell^\infty(G/N_\beta)$. In what follows, we do identify C_β with $\ell^\infty(G/N_\beta)$. The identification is thus done through the embedding $\pi_* : \ell^\infty(G/N_\beta) \longrightarrow \ell^\infty(G)$ defined by $\pi_*(f) = f\circ\pi$ $(f \in \ell^\infty(G/N_\beta))$, where π denotes the canonical surjection from G onto G/N_β. So μ shall be regarded as a bilinear form on $\ell^\infty(G/N_\beta)$ from now on. Note that, under this identification, the function $p_\beta = |N_\beta|^{-1}\chi_{N_\beta}$ is the same as $|N_\beta|^{-1}\delta_e$ (i.e., $\pi_*^{-1}(p_\beta) = |N_\beta|^{-1}\delta_e$), where δ_e in this case stands for the characteristic function of the identity e of G/N_β. In general, we have $\pi_*^{-1}(\delta_s * p_\beta) = |N_\beta|^{-1}\delta_{\pi(s)}$ for any $s \in G$, so that we identify $\delta_s * p_\beta$ with $|N_\beta|^{-1}\delta_{\pi(s)}$. Moreover, it is true that $\pi_*(g * h) = |N_\beta|\pi_*(g) * \pi_*(h)$ for all $g, h \in \ell^\infty(G/N_\beta)$. Our next purpose is to examine how the family $\{\Omega_s\}_{s\in G}$ and the function γ_0 can be described through this identification. For this end, we choose a section $\phi : G/N_\beta \longrightarrow G$ of π. By Lemma 3.1, the function $t \in G \longmapsto \gamma_0(s,t)$ belongs to C_β for each $s \in G$; hence it can be viewed as a function on $G \times G/N_\beta$. More precisely, the equation

$$\gamma(s,w) = \gamma_0(s,\phi(w)) \qquad (s \in G,\ w \in G/N_\beta)$$

defines a function γ on $G \times G/N_\beta$, and this definition is independent of the choice of the section ϕ. In fact, by definition, one has $\gamma(s,w) = f_{\Psi_s(b),b}(w)$. The function γ is characterized by the equality:

$$\gamma_0(s,t) = \gamma(s,\pi(t)) \qquad (s,t \in G).$$

In the later discussion, we shall find that this function γ is more important than γ_0. So, in the next lemma, we would like to restate equations (11.0)–(11.6) of Lemma 3.11 in terms of the function γ. We leave the verification to readers.

Lemma 3.12 *The function γ on $G \times G/N_\beta$ defined above satisfies the following equalities., where, in* (P5) *and* (P6), *η_0 is the function on G/N_β defined by $\eta_0(x) = \sum_{y\in G/N_\beta} \overline{\mu(\delta_{x^{-1}y}, \delta_y)}$ (i.e., $\eta_0 = |N_\beta|\pi_*^{-1}(\xi_0)$ with the previous notation).*

(P0) $\displaystyle \Omega_s(f)(w) = \sum_{r\in G} \gamma(rs^{-1}r^{-1}, \pi(r)w^{-1}) f(\pi(r));$

(P1) $\displaystyle \sum_{r\in G} \gamma(rs^{-1}r^{-1}, \pi(r)w^{-1}) = \delta_{s,e};$

(P2) $\displaystyle \sum_{r\in G} \gamma(r,w) = |N_\beta|^{-1}\delta_{w,e};$

(P3) $\displaystyle \sum_{r\in G} \gamma(rur^{-1}, \pi(r)w)\gamma(v, \pi(r)^{-1}) = \delta_{u,v}\gamma(u,w);$

(P4) $\displaystyle \frac{1}{|N_\beta|^2} \sum_{r\in G} \gamma(rs^{-1}r^{-1}, \pi(r)w^{-1})\mu(\delta_{\pi(ur)^{-1}}, \delta_{\pi(vr)^{-1}})$

$$= \sum_{p\in G}\sum_{x,y\in G/N_\beta} \mu(\delta_x,\delta_y)\gamma(up^{-1}u^{-1},\pi(u)w^{-1}x^{-1})$$
$$\times\,\gamma(vs^{-1}pv^{-1},\pi(v)w^{-1}y^{-1});$$

(P5) $$\sum_{x\in G/N_\beta}\eta_0(x)\overline{\gamma(s^{-1},wx)} = \frac{1}{|N_\beta|}\sum_{r\in G}\eta_0(\pi(r))\gamma(rsr^{-1},\pi(r)w);$$

(P6) $$\mu(\delta_w,\,\delta_{\pi(t)^{-1}}) = \sum_{r\in G}\sum_{x\in G/N}\sum_{\pi(q)=w}\mu(\delta_{\pi(r)},\,\delta_x)\gamma(qtrq^{-1},wx^{-1})$$

for all $s,u,v\in G$, $w\in G/N_\beta$ and $f\in C_\beta=\ell^\infty(G/N_\beta)$.

We now concentrate attention to the support of the function γ in the first variable. It will turn out that it has something to do with the spectrum of the derived coaction δ_β in the sense of Nakagami-Takesaki ([N-T]).

Let $c=\{c(t)\}_{t\in G}$ be in $\mathrm{Rel}(\beta)$. By the definition of p_β, we have $c=c_{p_\beta}$ with the notation in Section 1. Since $p_\beta=|N_\beta|^{-1}\chi_{N_\beta}$, we get

$$\begin{aligned} c(t) &= c_{p_\beta}(t)\\ &= \sum_{s\in G} c(ts)p_\beta(s^{-1})\\ &= \frac{1}{|N_\beta|}\sum_{s\in N_\beta} c(ts). \end{aligned}$$

It follows that $c(ts)=c(t)$ for any $t\in G$ and $s\in N_\beta$. Since N_β is normal, we may obtain the following lemma.

Lemma 3.13 *Let $c=\{c(t)\}_{t\in G}$ be in $\mathrm{Rel}(\beta)$. We have $c(rts)=c(t)$ for any $t\in G$ and $r,s\in N_\beta$.*

In the next lemma, recall that the grading $\{\Psi_s\}_{s\in G}$ in $\mathrm{Rel}(\beta)$ which determines the derived coaction is defined by $\Psi_s(c)=\{\Phi_{tst^{-1}}(c(t))\}_{t\in G}$ for any $c=\{c(t)\}\in\mathrm{Rel}(\beta)$.

Lemma 3.14 *Let G^{N_β} be the centralizer of N_β in G, i.e., G^{N_β} be the set of all elements $g\in G$ satisfying $sgs^{-1}=g$ for all $s\in N_\beta$. If $t\notin G^{N_\beta}$, we have $\Psi_t=0$ as a linear map.*

Proof. Let $g\in G$. In view of the preceding lemma, we have $\Psi_g(c)(ts)=\Psi_g(c)(t)$ for any $c\in\mathrm{Rel}(\beta)$, $t\in G$ and $s\in N_\beta$. But a straightforward calculation shows that the left-hand side of this identity equals $\Psi_{sgs^{-1}}(c)(t)$. This implies that $\Psi_{sgs^{-1}}=\Psi_g$ for all $s\in N_\beta$. Since $\Psi_u\circ\Psi_v=\delta_{u,v}\Psi_u$ for

$u, v \in G$, it follows that $\Psi_g = 0$ if $g \notin G^{N_\beta}$. □

By Lemma 3.14, we find that the spectrum of the derived coaction δ_β (see §1 of Chap. IV of [N-T] for the definition of the spectrum of a coaction) is contained in the normal subgroup G^{N_β}.

Corollary 3.15 *With the notation introduced above, we have $\gamma(t, w) = 0$ whenever $t \notin G^{N_\beta}$.*

Proof. This immediately follows from the definition of γ. □

Before we close this section, we briefly examine the dependence of γ on the choice of the element $b = \{b(s)\}_{s\in G}$ in $\mathrm{Rel}(\beta)$ with $f_b = p_\beta$ and $\sum_{s\in G} b(s) = 1$. So let $c = \{c(s)\}_{s\in G}$ be another such a choice. Recall (see Lemma 4.16 of [Y1]) that, if μ_b (resp. μ_c) is the 2-cocycle on C_β that arises from $b = \{b(s)\}$ (resp. $c = \{c(s)\}$), then they are related to each other in the following way:

$$\mu_c(f, g) = \sum_{t\in G} \mu_b(\overline{\eta} * f * \delta_{t^{-1}}, \overline{\eta} * g * \delta_{t^{-1}})\eta(t),$$

where $\eta' = f_{b,c} \in C_\beta$. Under the identification $C_\beta \cong \ell^\infty(G/N_\beta)$, this is equivalent to

$$\mu_c(f, g) = \sum_{x\in G/N_\beta} \mu_b(\overline{\eta_{c,b}} * f * \delta_{x^{-1}}, \overline{\eta_{c,b}} * g * \delta_{x^{-1}})\eta_{c,b}(x)$$
$$(f, g \in \ell^\infty(G/N_\beta)), \tag{3.16}$$

where $\eta_{c,b} = |N|\pi_*^{-1}(f_{b,c})$ with the previous notation. We would like to obtain a result similar to the one as above for the function γ.

Lemma 3.17 *In the situation described above, we denote by γ^b (resp. γ^c) the function on $G \times G/N_\beta$ defined by*

$$\gamma^b(s, w) = f_{\Psi_s(b),b}(w) \qquad (\textit{resp. } \gamma^c(s, w) = f_{\Psi_s(c),c}(w)).$$

Then they satisfy the identity

$$\gamma^c(s, w) = \frac{1}{|N|} \sum_{x\in G/N_\beta} \sum_{u\in G} \gamma^b(usu^{-1}, \pi(u)x^{-1})\overline{\eta_{c,b}}(\pi(u))\eta_{c,b}(xw)$$

for any $s \in G$ and $w \in G/N_\beta$.

Proof. By (4.15) of [Y1], we have $c = b_{\overline{f_{b,c}}}$. The assertion now follows from a direct computation. □

4. General theory on 2-cocycles and twisted crossed products

In this section, we shall give a new definition of a 2-cocycle on a commutative Kac algebra. Then we shall discuss crossed products by coactions twisted by 2-cocycles. The argument given here can be extended to the case of general finite-dimensional Kac algebras (cf. [Y2]). We shall deal with that case elsewhere in the future.

In the next definition, G is a finite group as usual. We always consider that the algebra $\ell^\infty(G)$ is equipped with the ordinary Kac algebraic structure.

Definition 4.1 A 2-cocycle on $\ell^\infty(G)$ is a bilinear form μ on $\ell^\infty(G) \otimes \ell^\infty(G)$ such that

(C1) in the involutive Banach algebra $\ell^1(G) \otimes \ell^1(G)$, we have

$$\mu^\sharp * \mu = \mu * \mu^\sharp = \varepsilon_G \otimes \varepsilon_G,$$

where ε_G, the counit of $\ell^\infty(G)$, is defined by $\varepsilon_G(f) = f(e)$;

(C2) μ satisfies the *cocycle condition*, i.e., with the conventional *sigma notation*,

$$\begin{aligned}\sum_{(f),(g)} &\mu(f_{(1)},\, g_{(1)})\mu(f_{(2)}g_{(2)},\, h)\\ &= \sum_{(g),(h)} \mu(g_{(1)},\, h_{(1)})\mu(f,\, g_{(2)}h_{(2)}) \quad (f, g, h \in \ell^\infty(G));\end{aligned}$$

(C3) μ is *normal* in the sense of [BCM], i.e.,

$$\mu(f,\, 1) = \mu(1,\, f) = \varepsilon_G(f) \qquad (f \in \ell^\infty(G));$$

(C4) for any $f, g \in \ell^\infty(G)$, we have

$$\sum_{s,t,u,v\in G} \overline{\mu(\delta_s,\, \delta_t)}\mu(\delta_{v^{-1}u},\, \delta_u) f(v^{-1}st^{-1})g(t^{-1}) = \mu(f,\, g);$$

(C5) for any $f \in \ell^\infty(G)$,

$$\sum_{s\in G} \mu(f * \delta_s,\, \delta_s) = \sum_{s\in G} \mu(f^\vee * \delta_s,\, \delta_s).$$

We denote by $Z^2(\ell^\infty(G))$ the set of all 2-cocycles on $\ell^\infty(G)$. It is easy to check that $\varepsilon_G \otimes \varepsilon_G$ also belongs to $Z^2(\ell^\infty(G))$. We call this 2-cocycle the trivial 2-cocycle.

Given a bilinear form μ on $\ell^\infty(G) \otimes \ell^\infty(G)$ satisfying (C1)–(C3) of Definition 4.1. Then, by [BCM], the equation

$$f \sharp_\mu g = \sum_{(f),(g)} \mu(f_{(1)}, g_{(1)}) f_{(2)} g_{(2)} \qquad (f, g \in \ell^\infty(G))$$

defines a new (associative) product, called the twisted product by μ, with 1 the identity with respect to this multiplication. We write $\ell^\infty_\mu(G)$ for $\ell^\infty(G)$ with the twisted product. We would like to introduce an involution on $\ell^\infty_\mu(G)$. First, for each $f \in \ell^\infty_\mu(G)$, we define an operator T_f on $\ell^2(G)$ by

$$T_f g = f \sharp_\mu g \qquad (g \in \ell^2(G)).$$

It is easily checked that

$$T_f = \sum_{s,t \in G} \mu(f * \delta_{s^{-1}}, \delta_t) \delta_s \lambda(t)^* \tag{4.2}$$

From this, it follows that $f \in \ell^\infty_\mu(G) \longmapsto T_f \in \mathcal{L}(\ell^2(G))$ is an injective homomorphism. It is obvious that this map is a homomorphism. To see that it is injective, suppose that $T_f = 0$ for some $f \in \ell^\infty_\mu(G)$. From (4.2) and the fact that $\{\delta_s \lambda(t)^*\}_{s,t \in G}$ forms a basis of $\mathcal{L}(\ell^2(G))$, we obtain $\mu(f * \delta_{s^{-1}}, \delta_t) = 0$ for all $s, t \in G$. Then, by (C3), we have $\varepsilon_G(f * \delta_{s^{-1}}) = 0$, which is equivalent to $f = 0$. For the moment, we denote by $\mathcal{D}_\mu$ the subalgebra $\{T_f : f \in \ell^\infty_\mu(G)\}$ of $\mathcal{L}(\ell^2(G))$. The following lemma is proven in [Y2] in more general setting. We, however, exhibit its proof for the sake of completeness.

Lemma 4.3 *Retain the notation described above. Then the following conditions are equivalent:*

(a) *the algebra $\mathcal{D}_\mu$ is self-adjoint;*

(b) *μ satisfies condition* (C4).

If one of the conditions (a) *and* (b) *occurs, then, for any $f \in \ell^\infty_\mu(G)$, we have*

$$T_f^* = T_{\eta_0 * \overline{f}},$$

where η_0 is a function on G defined by $\eta_0(t) = \sum_{s \in G} \overline{\mu(\delta_{t^{-1}s}, \delta_s)}$.

Proof. (a) $\Rightarrow$ (b): Let $f \in \ell^\infty_\mu(G)$. From identity (4.2),

$$\begin{aligned} T_f^* &= \sum_{s,t\in G} \overline{\mu(f * \delta_{s^{-1}}, \delta_t)}\, \lambda(t)\delta_s \\ &= \sum_{s,t\in G} \overline{\mu(f * \delta_{s^{-1}}, \delta_t)}\, \delta_{ts}\lambda(t) \\ &= \sum_{s,t\in G} \overline{\mu(f * \delta_{s^{-1}t^{-1}}, \delta_{t^{-1}})}\, \delta_s\lambda(t)^*. \end{aligned} \tag{4.4}$$

Hence, if there exists an element $h \in \ell^\infty_\mu(G)$ such that $T_f^* = T_h = \sum_{s,t\in G} \mu(h * \delta_{s^{-1}}, \delta_t)\delta_s\lambda(t)^*$, then we have

$$\overline{\mu(f * \delta_{s^{-1}t^{-1}}, \delta_{t^{-1}})} = \mu(h * \delta_{s^{-1}}, \delta_t) \qquad (s, t \in G).$$

By summing up both sides of the above identity with respect to t, we obtain, by (C3),

$$h(s) = \sum_{t\in G} \overline{\mu(f * \delta_{s^{-1}t}, \delta_t)}.$$

It is readily checked that $\sum_{t\in G} \overline{\mu(f * \delta_{s^{-1}t}, \delta_t)} = \eta_0 * \overline{f}$ with η_0 occurring in the assertion of this lemma. By substituting $\eta_0 * \overline{f}$ for f in (4.4), we have

$$T^*_{\eta_0 * \overline{f}} = \sum_{s,t\in G} \overline{\mu(\eta_0 * \overline{f} * \delta_{s^{-1}t^{-1}}, \delta_{t^{-1}})}\, \delta_s\lambda(t)^*.$$

This must be equal to $T_f = \sum_{s,t\in G} \mu(f * \delta_{s^{-1}}, \delta_t)\delta_s\lambda(t)^*$. From this, it follows that we have

$$\overline{\mu(\eta_0 * \overline{f} * \delta_{s^{-1}t^{-1}}, \delta_{t^{-1}})} = \mu(f * \delta_{s^{-1}}, \delta_t) \qquad (s, t \in G).$$

By letting $s = e$, we obtain

$$\overline{\mu(\eta_0 * \overline{f} * \delta_{t^{-1}}, \delta_{t^{-1}})} = \mu(f, \delta_t). \tag{4.5}$$

Suppose now that $g \in \ell^\infty_\mu(G)$. Then, from (4.5),

$$\begin{aligned} \mu(f, g) &= \sum_{t\in G} g(t)\overline{\mu(\eta_0 * \overline{f} * \delta_{t^{-1}}, \delta_{t^{-1}})} \\ &= \sum_{s,t\in G} g(t)\overline{(\eta_0 * \overline{f})(st)}\, \overline{\mu(\delta_s, \delta_{t^{-1}})} \\ &= \sum_{r,s,t\in G} g(t)\, \overline{\eta_0(r)} f(r^{-1}st)\, \overline{\mu(\delta_s, \delta_{t^{-1}})} \end{aligned}$$

$$= \sum_{r,s,t,u\in G} g(t)\,\mu(\delta_{r^{-1}u},\,\delta_u)f(r^{-1}st)\,\overline{\mu(\delta_s,\,\delta_{t^{-1}})}.$$

This shows that condition (C4) holds true.

(b) $\Rightarrow$ (a): The argument in the preceding paragraph shows that condition (C4) (or, equivalently, identity (4.5)) is equivalent to the identity $T^*_{\eta_0*\overline{f}} = T_f$. Therefore the algebra $\mathcal{D}_\mu$ is self-adjoint. □

Now we suppose that μ further satisfies condition (C4). Then, for each $f \in \ell^\infty_\mu(G)$, we set

$$f^\star = \eta_0 * \overline{f}$$

with η_0 the function defined in Lemma 4.3. In view of Lemma 4.3, this defines an involution on $\ell^\infty_\mu(G)$. Hence $\mathcal{D}_\mu$ is a $*$-subalgebra of $\mathcal{L}(\ell^2(G))$. It is clear that the vector $1 \in \ell^2(G)$ is a cyclic and separating vector for $\mathcal{D}_\mu$, so that the functional φ_0 on $\mathcal{D}_\mu$ defined by $\varphi_0(T_f) = (T_f1 \mid 1) = (\,f \mid 1\,)$ $(f \in \ell^\infty_\mu(G))$ is faithful. It is easily checked that $\varphi_0(T_g^*T_f) = (\,f \mid g\,)$. This implies that the Hilbert space $L^2(\varphi_0)$ obtained from φ_0 is isomorphic to $\ell^2(G)$, and that the pair $\{\mathcal{D}_\mu,\, \ell^2(G)\}$ is a standard representation.

Lemma 4.6 *With the notation as above, the following conditions are equivalent*:

(a) *the conjugate-linear map $J_\mu : f \in \ell^2(G) \longmapsto f^\star \in \ell^2(G)$ is a unitary involution, i.e., φ_0 is a trace*;

(b) *μ also satisfies condition* (C5);

(c) *the function η_0 defined in Lemma* 4.3 *satisfies $\eta_0^\vee = \eta_0$.*

If one of the above conditions is satisfied, then we have $\eta_0^\sharp * \eta_0 = \delta_e$.

Proof. The equivalence of (b) and (c) is obvious.

(a) $\Rightarrow$ (b): Let $f, g \in \ell^2(G)$. By assumption and the definition of the involution $\star$, we have

$$\begin{aligned}(f \mid g) &= (g^\star \mid f^\star\,)\\ &= (\eta_0^\sharp * \eta_0 * \overline{g} \mid \overline{f}\,).\end{aligned}$$

From this, it follows that $\eta_0^\sharp * \eta_0 = \delta_e$. In the meantime, the fact that $(\delta_e^\star)^\star = \delta_e$ yields $\eta_0 * \overline{\eta_0} = \delta_e$. Hence we conclude that $\eta_0^\sharp = \overline{\eta_0}$. Thus $\eta_0^\vee = \eta_0$.

(b) $\Rightarrow$ (a): By the preceding paragraph, we find that, if $\eta_0^\vee = \eta_0$, then we have $\eta_0^\sharp * \eta_0 = \delta_e$, which implies that $(f \mid g) = (g^\star \mid f^\star)$ for any $f, g \in \ell^2(G)$. $\square$

Corollary 4.7 *Let μ be a 2-cocycle on $\ell^\infty(G)$, i.e., $\mu \in Z^2(\ell^\infty(G))$. Then the equation*

$$\tilde{\mu}(f,\, g) = \mu(g,\, f) \qquad (\, f, g \in \ell^\infty(G)\,)$$

defines a 2-cocycle $\tilde{\mu}$ on $\ell^\infty(G)$. Moreover, we have $\mathcal{D}_{\tilde{\mu}} = \mathcal{D}'_\mu$.

Proof. It is easy to show that $\tilde{\mu}$ satisfies conditions (C1)–(C3).

For (C4), note first that $f \,\tilde{\sharp}\, g = g \,\sharp\, f$, where $\sharp$ and $\tilde{\sharp}$ are the twisted products associated with μ and $\tilde{\mu}$, respectively. Hence $\mathcal{D}_{\tilde{\mu}}$ is exactly the set of right multiplication operators by the elements of $f \in \ell^\infty_\mu(G)$, which is, by Lemma 4.6, the commutant $\mathcal{D}'_\mu$ of $\mathcal{D}_\mu$. In particular, $\mathcal{D}_{\tilde{\mu}}$ is self-adjoint. Thus, from Lemma 4.3, $\tilde{\mu}$ satisfies condition (C4).

For (C5), note that, from a direct computation, we have $\tilde{\eta_0} = \eta_0^\vee$, where $\tilde{\eta_0}$ is the function defined in Lemma 4.3 associated with $\tilde{\mu}$. In view of Lemma 4.6, we obtain $\tilde{\eta_0} = \eta_0$. So we have $\tilde{\eta_0}^\vee = \tilde{\eta_0}$. $\square$

For the moment, let us return to such bilinear forms $\mu = \mu_\beta$ as appeared in [Y1] (see Section 1), namely, bilinear forms that arise from coactions of finite groups on finite factors. So we consider the situation described in Section 1: β is a coaction of G on a finite factor $\mathcal{A}$ with τ the unique faithful normal tracial state; p_β is the central projection in $\ell^1(G)$ which determines the inner part of β; $\mu = \mu_\beta$ is the bilinear form on C_β, the commutative Kac algebra dual to $R(G)_{\lambda(p_\beta)}$ and so on. As usual, we set

$$N_\beta = \{ s \in G : \, \delta_s * p_\beta = p_\beta \}.$$

As in Section 3, we identify C_β with $\ell^\infty(G/N_\beta)$. So $p_\beta = |N_\beta|^{-1}\delta_e$, where e is the identity of G/N_β. Let $\pi : G \longrightarrow G/N_\beta$ denote the canonical surjection. From Proposition 3.4, Lemma 3.5 and Theorem 3.6 of [Y1], we find that μ_β satisfies conditions (C1)–(C3). We show below that condition (C4) is also satisfied, so that μ_β belongs to $Z^2(\ell^\infty(G/N_\beta))$.

Proposition 4.8 *The bilinear form μ_β belongs to $Z^2(\ell^\infty(G/N_\beta))$.*

Proof. As remarked, it suffices to prove that μ_β satisfies conditions (C4) and (C5). Condition (C5) is satisfied, due to (2.6) and Lemma 4.6. For (C4), let f, g be in $\ell^\infty(G/N_\beta)$. Recall that $|N_\beta|\,\xi_0 = \eta_0$ (see Lemma 3.12).

Since $\overline{f_{b,b^*}} = f_{b^*,b}$, we have

$$\begin{aligned}
&\sum_{x,y,z,w\in G/N_\beta} \overline{\mu_\beta(\delta_x,\,\delta_y)}\,\mu_\beta(\delta_{z^{-1}w},\,\delta_w)f(z^{-1}xy^{-1})g(y^{-1}) \\
&\quad = |N_\beta| \sum_{x,y,z\in G/N_\beta} \overline{\mu_\beta(\delta_x,\,\delta_y)}\, f_{b,b^*}(z)f(z^{-1}xy^{-1})g(y^{-1}) \\
&\quad = |N_\beta| \sum_{x,y\in G/N_\beta} \overline{\mu_\beta(\delta_x,\,\delta_y)}\,(f_{b^*,b} * f * \delta_y)(x)g(y^{-1}) \\
&\quad = |N_\beta| \sum_{y\in G/N_\beta} \overline{\mu_\beta(f_{b^*,b} * \overline{f} * \delta_y,\,\delta_y)}\, g(y^{-1}).
\end{aligned}$$

We take a section $\phi : G/N_\beta \longrightarrow G$ for π again. Then, by the definition of μ_β,

$$\begin{aligned}
\overline{\mu_\beta(f_{b^*,b} * \overline{f} * \delta_y,\,\delta_y)} &= \sum_{t\in G} \overline{\tau(b_{\delta_y\circ\pi}(t)b_{(f_{b^*,b}*\overline{f}*\delta_y)\circ\pi}(t)b(t)^*)} \\
&= \sum_{t\in G} \overline{\tau(b(t\phi(y)^{-1})\left(b_{f_{b^*,b}}\right)_{(\overline{f}*\delta_y)\circ\pi}(t)b(t)^*)} \\
&= \sum_{t\in G} \overline{\tau(b(t\phi(y)^{-1})\left(b^*\right)_{(\overline{f}*\delta_y)\circ\pi}(t)b(t)^*)} \\
&= \sum_{t\in G} \overline{\tau(b(t\phi(y)^{-1})\left(b_{f\circ\pi}\right)^*(t\phi(y)^{-1})b(t)^*)} \\
&= \sum_{t\in G} \tau(b(t)b_{f\circ\pi}(t\phi(y)^{-1})b(t\phi(y)^{-1})^*).
\end{aligned}$$

The third identity is due to the fact that $b_{f_{b^*,c^*}} = c$ for any $c \in \mathrm{Rel}(\beta)$ (see the proof of Lemma 4.5 of [Y1]). From the above calculation, it follows that

$$\begin{aligned}
&|N_\beta| \sum_{y\in G/N_\beta} \overline{\mu_\beta(f_{b^*,b} * \overline{f} * \delta_y,\,\delta_y)}\, g(y^{-1}) \\
&\quad = |N_\beta| \sum_{y\in G/N_\beta} \sum_{t\in G} \tau(b(t)b_{f\circ\pi}(t\phi(y)^{-1})b(t\phi(y)^{-1})^*)g(y^{-1}) \\
&\quad = \sum_{t\in G} \sum_{y\in G/N_\beta} \sum_{s\in N_\beta} \tau(b(t)b_{f\circ\pi}(t\phi(y)^{-1}s)\cdot b(t\phi(y)^{-1}s)^*) \\
&\qquad\qquad g(\pi(\phi(y)^{-1}s))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t,u\in G} \tau(b(t)b_{f\circ\pi}(tu)b(tu)^*)g(\pi(u)) \\
&= \sum_{t,u\in G} \tau(b(t)b_{f\circ\pi}(u)b(u)^*)g(\pi(t^{-1}u)) \\
&= \sum_{u\in G} \tau(b_{g\circ\pi}(u)b_{f\circ\pi}(u)b(u)^*) \\
&= \mu_\beta(f,\, g).
\end{aligned}
$$

Hence (C4) is satisfied. □

In the remainder of this section, we discuss twisted crossed products of von Neumann algebras by coactions. The main ingredients of twisted crossed products by coactions are (1) an action β of the commutant Kac algebra $R(G)'$ of the group algebra $R(G)$ of a finite group G on a von Neumann algebra $\mathcal{A}$ and (2) a 2-cocycle μ in $Z^2(\ell^\infty(G))$. One can think of β as a coaction of the opposite group of G on $\mathcal{A}$. Thus, if we wish to start with a coaction of G, then we need to take a 2-cocycle of $\ell^\infty(G^\sigma)$, where $G^\sigma = G$ has the opposite multiplication. Given a system $(\mathcal{A},\, G,\, \beta,\, \mu)$ as above, we define the twisted crossed product, denoted by $\mathcal{A}\times_{\beta,\mu}R(G)'$, to be the von Neumann algebra generated by $\beta(\mathcal{A})$ and $\mathbf{C}\otimes\{\, T_f \,:\, f\in\ell^\infty_\mu(G)\}$ with the notation introduced in the present section. In what follows, we shall deduce fundamental facts on twisted crossed products that will be needed in the next section. So let us fix such a system $(\mathcal{A},\, G,\, \beta,\, \mu)$. Let $a\in\mathcal{A}$. Then $\beta(a)$ has the form

$$
\beta(a) = \sum_{s\in G} \Phi_s(a)\otimes\rho(s),
$$

where ρ is the right regular representation of G. This defines a family $\{\Phi_s\}_{s\in G}$ of linear maps from $\mathcal{A}$ into itself, and determines the G-grading in $\mathcal{A}$ associated with β. In other words, the family satisfies identities (CA0)–(CA4) in Section 1.

Lemma 4.9 *Let $a\in\mathcal{A}$ and $f\in\ell^\infty_\mu(G)$. Then we have*

$$
(1\otimes T_f)\beta(a) = \sum_{t\in G} \beta(\Phi_t(a))(1\otimes T_{f*\delta_t}).
$$

Proof. This follows from a direct computation, since we already know an explicit form of T_f (see identity (4.2)). □

This lemma shows that the set of elements of the form $\sum_{s\in G}\beta(c(s))(1\otimes T_{\delta_s})$, where $\{c(s)\}_{s\in G}\subseteq\mathcal{A}$, forms a σ-strongly* dense $*$-subalgebra of the twisted crossed product $\mathcal{A}\times_{\beta,\mu}R(G)'$. Following the idea of the proof of Proposition 1.1 of [Y1], we shall prove that this set in fact coincides with the twisted crossed product. For this purpose, let ω_0 denote the vector state defined by the unit vector $|G|^{-1/2}\cdot 1\in\ell^2(G)$: $\omega_0(T)=|G|^{-1}(T\cdot 1\mid 1)$ $(T\in\mathcal{L}(\ell^2(G)))$. Then $E=id_{\mathcal{A}}\otimes\omega_0$ is a normal conditional expectation from $\mathcal{A}\otimes\mathcal{L}(\ell^2(G))$ onto $\mathcal{A}\cong\mathcal{A}\otimes\mathbf{C}$. Since $\mathcal{A}\times_{\beta,\mu}R(G)'$ is contained in $\mathcal{A}\otimes\mathcal{L}(\ell^2(G))$, it makes sense to restrict the map E to $\mathcal{A}\times_{\beta,\mu}R(G)'$. We still denote the restriction by E.

Lemma 4.10 *Let $X=\sum_{s\in G}\beta(c(s))(1\otimes T_{\delta_s})$ be in $\mathcal{A}\times_{\beta,\mu}R(G)'$, where $\{c(s)\}_{s\in G}\subseteq\mathcal{A}$. Then*

$$E(X)=\frac{1}{|G|}\sum_{s\in G}c(s).$$

Proof. Note that $X=\sum_{s,t\in G}\Phi_s(c(t))\otimes\rho(s)T_{\delta_t}$. It is easy to see that $\omega_0(\rho(s)T_{\delta_t})=|G|^{-1}$. Since $\sum_{s\in G}\Phi_s(c(t))=c(t)$ (see identity (CA3), we find that $E(X)=|G|^{-1}\sum_{t\in G}c(t)$. $\square$

Lemma 4.11 *Let X be as in the preceding lemma. Then we have*

$$c(t)=|G|\sum_{s\in G}\Phi_s\circ E((1\otimes T^*_{\delta_{ts^{-1}}})X)$$

for any $t\in G$.

Proof. Let η_0 be the function defined in Lemma 4.3. Since $\delta_u\,\sharp_\mu\,\delta_v=\sum_{r\in G}\mu(\delta_{ur^{-1}},\delta_{vr^{-1}})\delta_r$ for any $u,v\in G$, it follows that

$$\begin{aligned}(\delta_t^\star*\delta_u)\,\sharp_\mu\,\delta_s &= \sum_{g\in G}\eta_0(gu^{-1}t^{-1})(\delta_g\,\sharp_\mu\,\delta_s)\\ &= \sum_{g,h\in G}\eta_0(gu^{-1}t^{-1})\mu(\delta_{gh^{-1}},\delta_{sh^{-1}})\delta_h.\end{aligned}$$

Hence, by Lemma 4.9, we obtain

$$\begin{aligned}&(1\otimes T^*_{\delta_t})X\\ &\quad=\sum_{s\in G}(1\otimes T_{\delta_t^\sharp})\beta(c(s))(1\otimes T_{\delta_s})\end{aligned}$$

$$= \sum_{u,s\in G} \beta(\Phi_u(c(s)))(1 \otimes T_{(\delta_t^\sharp * \delta_u)\,\sharp_\mu\,\delta_s})$$
$$= \sum_{s,u,g,h\in G} \eta_0(gu^{-1}t^{-1})\mu(\delta_{gh^{-1}},\, \delta_{sh^{-1}})\beta(\Phi_u(c(s)))(1 \otimes T_{\delta_h}).$$

From this, together with Lemma 4.10, it follows that

$$\begin{aligned}
&E((1 \otimes T^*_{\delta_t})X) \\
&\quad = \frac{1}{|G|} \sum_{s,u,g,h\in G} \eta_0(gu^{-1}t^{-1})\mu(\delta_{gh^{-1}},\, \delta_{sh^{-1}})\beta(\Phi_u(c(s))) \\
&\quad = \frac{1}{|G|} \sum_{s,u,g\in G} \eta_0(gu^{-1}t^{-1})\overline{\eta_0}(sg^{-1})\beta(\Phi_u(c(s))) \\
&\quad = \frac{1}{|G|} \sum_{s,u\in G} (\overline{\eta_0} * \eta_0)(su^{-1}t^{-1})\, \Phi_u(c(s)) \\
&\quad = \frac{1}{|G|} \sum_{u\in G} \Phi_u(c(tu)).
\end{aligned}$$

The last identity is guranteed by the fact that $\eta_0 * \overline{\eta_0} = \delta_e$, i.e., $(\delta_e^\star)^* = \delta_e$. Applying the linear map Φ_s to both sides of the above equality, we conclude that

$$\Phi_s \circ E((1 \otimes T^*_{\delta_t})X) = \frac{1}{|G|}\Phi_s(c(ts)) \qquad (\, s, t \in G\,).$$

Thus we have $\Phi_s(c(t)) = |G|\, \Phi_s \circ E((1 \otimes T^*_{\delta_{ts^{-1}}})X)$. Since $c(t) = \sum_{s\in G} \Phi_s(c(t))$, we obtain the desired identity. □

Lemma 4.12 *We have*

$$\mathcal{A} \times_{\beta,\mu} R(G)' = \left\{ \sum_{s\in G} \beta(c(s))(1 \otimes T_{\delta_s}) : \{c(s)\}_{s\in G} \subseteq \mathcal{A} \right\}.$$

Proof. Let us denote by $\mathcal{S}$ the set on the right-hand side of the assertion, which is, as noted, σ-strongly* dense in the twisted crossed product. Suppose that $X \in \mathcal{A} \times_{\beta,\mu} R(G)'$. Then there is a net $\{X_i\}$ in $\mathcal{S}$ that converges σ-strongly* to X. Each X_i has the form $X_i = \sum_{s\in G} \beta(c(i,s))(1 \otimes T_{\delta_s})$. From Lemma 4.11, we find that

$$|G| \sum_{s\in G} \Phi_s \circ E((1 \otimes T^*_{\delta_{ts^{-1}}})X)$$

$$
\begin{aligned}
&= |G| \sum_{s\in G} \sigma\text{-}s^*\text{-}\lim_i \Phi_s \circ E((1 \otimes T^*_{\delta_{ts^{-1}}})X_i) \\
&= \sigma\text{-}s^*\text{-}\lim_i c(i,t).
\end{aligned}
$$

Hence, by setting $c(t) = |G| \sum_{s\in G} \Phi_s \circ E((1 \otimes T^*_{\delta_{ts^{-1}}})X) \in \mathcal{A}$, we get

$$
\begin{aligned}
X &= \sigma\text{-}s^*\text{-}\lim_i \sum_{s\in G} \beta(c(i,s))(1 \otimes T_{\delta_s}) \\
&= \sum_{s\in G} \beta(c(s))(1 \otimes T_{\delta_s}).
\end{aligned}
$$

This shows that X belongs to $\mathcal{S}$. □

Corollary 4.13 *Each element X of $\mathcal{A} \times_{\beta,\mu} R(G)'$ has a unique expression*

$$
X = \sum_{s\in G} \beta(c(s))(1 \otimes T_{\delta_s})
$$

for some $c = \{c(s)\}_{s\in G} \subseteq \mathcal{A}$. In this case, we write X_c for X.

Proof. The expression being unique immediately follows from Lemma 4.11. □

Proposition 4.14 *The linear map E is a faithful normal conditional expectation from the twisted crossed product $\mathcal{A} \times_{\beta,\mu} R(G)'$ onto $\mathcal{A}$, where $\mathcal{A}$ is identified with $\beta(\mathcal{A})$ via β.*

Proof. Let $a \in \mathcal{A}$. With the notation in Corollary 4.13, we have $\beta(a) = X_c$, where $c = \{c(s) \equiv a\}_{s\in G}$. Hence, by Lemma 4.10, $E(\beta(a)) = a$. This shows that E is a normal conditional expectation. To prove that it is faithful, we have only to note that, with $X = X_a$ in the twisted crossed product, where $a = \{a(s)\}_{s\in G} \subseteq \mathcal{A}$, we have $E(X_a X_a^*) = |G|^{-1} \sum_{s\in G} a(s)a(s)^*$. The verification of this identity is left to readers. □

The next objective is to examine the relative commutant of $\beta(\mathcal{A})$ in the twisted crossed product. For this end, we consider the relative commutant $\mathcal{A}^c$ of $\beta(\mathcal{A})$ in the *ordinary* crossed product $\mathcal{A} \times_\beta R(G)'$. As we noted before Lemma 4.9, $\mathcal{A} \times_\beta R(G)'$ can be viewed as the crossed product of $\mathcal{A}$ by the coaction β of the opposite group of G. Hence, by [Y1] (see §1), $\mathcal{A}^c$ is in bijective correspondence with the set $\mathrm{Rel}(\beta)$ of all elements $c = \{c(t)\}_{t\in G}$, where $c(t) \in \mathcal{A}$, satisfying

$$xc(t) = \sum_{s \in G} c(s)\Phi_{s^{-1}t}(x) \tag{4.15}$$

for all $x \in \mathcal{A}$, via the map $c = \{c(t)\} \in \mathrm{Rel}(\beta) \longmapsto \sum_{s\in G} \beta(c(s))(1 \otimes \delta_s) \in \mathcal{A}^c$. In the next lemma, we show that the relative commutant $\mathcal{A}^{c,\mu}$ of $\beta(\mathcal{A})$ in the twisted crossed product also can be identified with $\mathrm{Rel}(\beta)$ by the correspondence as above.

Lemma 4.16 *Let $X_c = \sum_{s\in G} \beta(c(s))(1 \otimes T_{\delta_s})$ be in the twisted crossed product. Then X_c lies in the relative commutant $\mathcal{A}^{c,\mu}$ if and only if $c = \{c(s)\}$ belongs to* $\mathrm{Rel}(\beta)$.

Proof. Let $x \in \mathcal{A}$. With the aid of Lemma 4.9, it can be easily checked that

$$X_c \cdot \beta(x) = \sum_{t\in G} \beta\Big(\sum_{s\in G} c(s)\Phi_{s^{-1}t}(x)\Big)(1 \otimes T_{\delta_t}).$$

Hence, by Corollary 4.13, the condition $[X_c, \beta(x)] = 0$ is equivalent to identity (4.15). So $c = \{c(t)\}$ belongs to $\mathrm{Rel}(\beta)$. □

The following corollary is an immediate consequence of Lemma 4.16.

Corollary 4.17 *If the action β is outer in the sense of [Y], then $\beta(\mathcal{A})$ has the trivial relative commutant in the twisted crossed product $\mathcal{A} \times_{\beta,\mu} R(G)'$.*

We close this section with the following remark concerning 2-cocycles on $\ell^\infty(G)$.

Remark 4.18. If the group G in question is commutative, then a 2-cocycle on $\ell^\infty(G)$ can be regarded, through the Fourier transform, as a 2-cocycle on the group $\hat{G}$, the dual of G, in the usual sense. Indeed, if $\hat{\mu}$ is the Fourier transform of a 2-cocycle μ on $\ell^\infty(G)$ (so $\hat{\mu}$ is a bilinear form on the group algebra of $\hat{G}$), then, with $\mu_0(p, q) = \hat{\mu}(\lambda(p), \lambda(q))$ $(p, q \in \hat{G})$, we have that (i) (C1) $\Longleftrightarrow |\mu_0(p, q)| = 1$; (ii) (C2) $\Longleftrightarrow \mu_0(p, q)\mu_0(pq, r) = \mu_0(p, qr)\mu_0(q, r)$; (iii) (C3) $\Longleftrightarrow \mu_0(e, p) = \mu_0(p, e) = 1$ with e the identity of $\hat{G}$; (iv) (C4) $\Longleftrightarrow \overline{\mu_0(p^{-1}, pq)}\mu_0(p, p^{-1}) = \mu_0(p, q)$; (C5) $\Longleftrightarrow \overline{\mu_0(p^{-1}, p)} = \overline{\mu_0(p, p^{-1})}$. Hence conditions (C4) and (C5) automatically follow from the other conditions in this case. The author does not know whether (C4) or (C5) is always redundant in the noncommutative case.

5. Models for finite group coactions

In [Y1] and Sections 3–4 of the present paper, we showed that every coaction β of a finite group G on a finite factor $\mathcal{A}$ gives rise to a triple (N_β, μ_β, γ_β): (i) a normal subgroup N_β of G which corresponds to the inner part of β; (ii) a 2-cocycle μ_β on $\ell^\infty(G/N_\beta)$ so that the associated twisted algebra $\ell^\infty_{\mu_\beta}(G/N_\beta)$ is isomorphic to the relative commutant of $\beta(\mathcal{A})$ in the crossed product $\mathcal{A}\times_\beta R(G)$; (iii) a function $\gamma=\gamma_\beta$ on $G\times G/N_\beta$ that determines the derived coaction δ_β of G on $\ell^\infty_{\mu_\beta}(G/N_\beta)$. The goal of this section is to prove the converse of this statement. More precisely, we shall show that, given an abstract triple (N, μ, γ) that satisfies suitable conditions, there exists a coaction $\beta=\beta_{(N,\mu,\gamma)}$ of G on the AFD factor of type II_1 so that (N_β, μ_β, γ_β) equals the given data (N, μ, γ). In this sense, coactions $\beta_{(N,\mu,\gamma)}$ may be considered as models for coactions of finite groups on the AFD II_1 factor.

Throughout this section, we fix a finite group G.

Definition 5.1 Let N be a normal subgroup of G. We define $\mathcal{E}(G,N)$ to be the set of all pairs $\mathbf{c}=(\mu,\gamma)$ in which

(1) μ is a 2-cocycle on $\ell^\infty(G/N)$, i.e., $\mu\in Z^2(\ell^\infty(G/N))$;
(2) γ is a function on $G\times G/N$ satisfying conditions (P1)–(P6) and;
(3) $\gamma(s,x)=0$ for all $x\in G/N$ and $s\notin G^N$, where G^N is the centralizer of N in G.

The results established in Sections 3 and 4 tell that, for any coaction β of G on a finite factor, the pair (μ_β, γ_β) belongs to $\mathcal{E}(G,N_\beta)$. We write $\mathbf{c}(\beta)=(\mu_\beta,\gamma_\beta)$ in this case.

Lemma 5.2 *Let N be a normal subgroup of G. If $\mathbf{c}=(\mu,\gamma)$ belongs to $\mathcal{E}(G,N)$, then so does the pair $\tilde{\mathbf{c}}=(\tilde\mu,\tilde\gamma)$ (see Corollary 4.7), where*

$$\begin{aligned}\tilde\mu(f,g) &= \mu(g,f) \qquad (f,g\in\ell^\infty(G/N));\\ \tilde\gamma(s,x) &= \gamma(s^{-1},x) \qquad (s\in G,\ x\in G/N).\end{aligned}$$

Clearly, we have $\mathbf{c}^{\tilde{}\tilde{}}=\mathbf{c}$.

Proof. We have already proven in Corollary 4.7 that $\tilde\mu$ is a 2-cocycle on $\ell^\infty(G/N)$. It is clear that $\tilde\gamma$ satisfies condition (3) of Definition 5.1.

It remains to prove that $\tilde\gamma$ satisfies (P1)–(P6). Let us denote by $\tilde\sharp$ the twisted product on $\ell^\infty(G/N)$ associated with $\tilde\mu$. As noted in the proof of Corollary 4.7, we have $f\,\tilde\sharp\,g=g\,\sharp_\mu f$ for any $f,g\in\ell^\infty(G/N)$. Fix an element

$s \in G$. For any $f \in \ell^\infty(G/N)$, we set

$$\Omega_s(f)(x) = \sum_{u\in G} \gamma(us^{-1}u^{-1}, \pi(u)x^{-1})f(\pi(u));$$
$$\tilde{\Omega}_s(f)(x) = \sum_{u\in G} \tilde{\gamma}(us^{-1}u^{-1}, \pi(u)x^{-1})f(\pi(u)).$$

Recall that, with this notation, γ satisfies (P1)–(P6) if and only if $\{\Omega_s\}$ determines a coaction of G on $\ell^\infty_\mu(G/N)$ which is inner, i.e., it satisfies (1) $\Omega_s(1)=\delta_{s,e}\cdot 1$; (2) $\sum_{s\in G}\Omega_s=\mathrm{id}$; (3) $\Omega_s \circ \Omega_t=\delta_{s,t}\Omega_s$; (4) $\Omega_s(f^\star)=\Omega_{s^{-1}}(f)^\star$; (5) $\Omega_t(f \sharp g)=\sum_{s\in G}\Omega_s(f)\sharp\Omega_{s^{-1}t}(g)$; (6) $f\sharp\delta_{\pi(t)^{-1}}=\sum_{s\in G}\delta_{\pi(s)^{-1}}\sharp\Omega_{st^{-1}}(f)$. Hence, in order to prove that $\tilde{\gamma}$ satisfies conditions (P1)–(P6), one only needs to show that $\{\tilde{\Omega}_s\}$ too satisfies the above identities (1)–(6) with respect to the product $\tilde{\sharp}$. But this can be done without difficulty, once we know that $\tilde{\Omega}_s = \Omega_{s^{-1}}$. The details are left to readers as an exercise. □

In what follows, we fix a normal subgroup N of G and an element $\mathbf{c} = (\mu, \gamma)$ of $\mathcal{E}(G, N)$. Set $\nu = \tilde{\mu}$ and $\phi = \tilde{\gamma}$. Thus $\tilde{\mathbf{c}} = (\nu, \phi)$. We take an outer action α of the commutant Kac algebra $R(G)'$ on the AFD factor $\mathcal{R}$ of type II_1. The existence of such an action is guaranteed, thanks to [L] or to [Y]. As we remarked in the preceding section, α can be regarded as an outer coaction of the opposite group of G on $\mathcal{R}$. As usual, with ρ the right regular representation of G, we write

$$\alpha(a) = \sum_{s\in G} \Phi^\alpha_s(a) \otimes \rho(s) \qquad (a \in \mathcal{R}).$$

Thus $\{\Phi^\alpha_s\}_{s\in G}$ satisfies identities (CA0)–(CA4). Next we take the unique central projection q_N in $R(G)'$ so that the reduced Kac algebra $R(G)'_{q_N}$ is (isomorphic to) $R(G/N)'$ (see [E-S] for the conditions q_N satisfies). So we have $q_N = \rho(p_N)$, where $p_N = |N|^{-1}\chi_N$. Then the equation

$$\theta(a) = \alpha(a)(1 \otimes q_N) \qquad (a \in \mathcal{R})$$

defines an injective $*$-homomorphism from $\mathcal{R}$ into $\mathcal{R} \otimes R(G/N)'$. In fact, it is easy to see that θ is an action of $R(G/N)'$ on $\mathcal{R}$. We write

$$\theta(a) = \sum_{x\in G/N} \Phi^\theta_x(a) \otimes \rho_{G/N}(x) \qquad (a \in \mathcal{R}),$$

where $\rho_{G/N}$ of course stands for the right regular representation of G/N.

With $\pi : G \longrightarrow G/N$ the canonical surjection, we have

$$\Phi_x^\theta = \sum_{\pi(s)=x} \Phi_s^\alpha \tag{5.3}$$

for any $x \in G/N$.

Lemma 5.4 *With the notation established so far, the action θ is outer.*

Proof. Suppose that X belongs to the relative commutant of $\theta(\mathcal{R})$ in the crossed product $\mathcal{R} \times_\theta R(G/N)'$. Since θ is a coaction of the opposite group of G, it follows from Proposition 1.1 of [Y1] that X has the form $X = \sum_{x\in G/N} \theta(c(x))(1 \otimes \delta_x)$, where $\{c(x)\}_{x\in G/N} \subseteq \mathcal{R}$. Moreover,the fact that X lies in the relative commutant implies that $\{c(x)\}$ satisfies $ac(x) = \sum_{y\in G/N} c(y)\Phi^\theta_{y^{-1}x}(a)$ for all $a \in \mathcal{R}$ (cf. (4.13)). Set $d(t) = c(\pi(t))$ $(t \in G)$. We assert that $\{d(t)\}_{t\in G}$ belongs to $\mathrm{Rel}(\alpha)$ with the notation in Section 4. Indeed, if $a \in \mathcal{R}$, then, by (5.3),

$$\begin{aligned}\sum_{s\in G} d(s)\Phi^\alpha_{s^{-1}t}(s) &= \sum_{y\in G/N} c(y) \sum_{\pi(s)=y} \Phi^\alpha_{s^{-1}t}(a) \\ &= \sum_{y\in G/N} c(y)\Phi^\theta_{y^{-1}\pi(t)}(a) \\ &= ac(\pi(t)) = ad(t).\end{aligned}$$

By assumption, $d(t) = c \cdot 1$ for some $c \in \mathbf{C}$. Therefore, $X = c \cdot 1$. □

Now we consider the twisted crossed product $\mathcal{Q} = \mathcal{R} \times_{\theta,\nu} R(G/N)'$ associated with θ and the 2-cocycle ν. In view of Corollary 4.15, $\mathcal{Q}$ is a factor. Moreover, since it is an infinite-dimensional subfactor of $\mathcal{R} \otimes \mathcal{L}(\ell^2(G/N))$, which is AFD, $\mathcal{Q}$ is the AFD factor of type II_1, and thus isomorphic to $\mathcal{R}$. Let τ be the faithful normal tracial state on $\mathcal{R}$. By Proposition 4.12, $\mathcal{Q}$ has a faithful normal conditional expectation E onto $\mathcal{R}$. Then it is not difficult to see from the construction of E that $\tilde{\tau} = \tau \circ E$ is the unique faithful normal tracial state on $\mathcal{Q}$.

Our objective is to construct a coaction of G on $\mathcal{Q}$. For this, it suffices to exhibit a family $\{\Phi_s\}_{s\in G}$ of linear maps from $\mathcal{Q}$ into itself satisfying (CA0)–(CA4). Let $X = \sum_{x\in G/N} \theta(c(x))(1 \otimes T_{\delta_x})$ be an arbitrary element of $\mathcal{Q}$. For each $s \in G$, we define a map Φ_s by

$$\Phi_s(X) = \sum_{x\in G/N} \sum_{t,u\in G} \phi(us^{-1}tu^{-1}, \pi(u)x^{-1})$$

$$\times\theta(\Phi_t^\alpha(c(\pi(u))))(1\otimes T_{\delta_x}). \tag{5.5}$$

This is clearly a linear transformation on $\mathcal{Q}$. If $c(x)\equiv 1$, i.e., $X=1$ in the above definition, then, by condition (P1), we have

$$\begin{aligned}
\Phi_s(1) &= \sum_{x\in G/N}\sum_{t,u\in G}\phi(us^{-1}tu^{-1},\pi(u)x^{-1})\delta_{t,e}(1\otimes T_{\delta_x})\\
&= \sum_{x\in G/N}\sum_{u\in G}\phi(us^{-1}u^{-1},\pi(u)x^{-1})(1\otimes T_{\delta_x})\\
&= \delta_{s,e}\sum_{x\in G/N}(1\otimes T_{\delta_x}) = \delta_{s,e}\cdot 1.
\end{aligned}$$

Thus $\{\Phi_s\}$ satisfies (CA1). With $X=\sum_{x\in G/N}\theta(c(x))(1\otimes T_{\delta_x})$ in $\mathcal{Q}$, from (P2), we obtain

$$\begin{aligned}
&\sum_{s\in G}\Phi_s(X)\\
&\quad= \sum_{x\in G/N}\sum_{s,t,u\in G}\phi(us^{-1}tu^{-1},\pi(u)x^{-1})\theta(\Phi_t^\alpha(c(\pi(u))))(1\otimes T_{\delta_x})\\
&\quad= \frac{1}{|N|}\sum_{x\in G/N}\sum_{t\in G}\sum_{\pi(u)=x}\theta(\Phi_t^\alpha(c(\pi(u))))(1\otimes T_{\delta_x})\\
&\quad= \sum_{x\in G/N}\sum_{t\in G}\theta(\Phi_t^\alpha(c(x)))(1\otimes T_{\delta_x})\\
&\quad= \sum_{x\in G/N}\theta(c(x))(1\otimes T_{\delta_x}) = X.
\end{aligned}$$

So $\{\Phi_s\}$ satisfies (CA3). If $u,v\in G$, then, with X as above, condition (P3) implies that

$$\begin{aligned}
&\Phi_u\circ\Phi_v(X)\\
&\quad= \sum_{s\in G/N}\sum_{s,t,g,h\in G}\phi(gu^{-1}hg^{-1},\pi(g)x^{-1})\phi(sv^{-1}ts^{-1},\pi(s)\pi(g)^{-1})\\
&\qquad\qquad\times\theta(\Phi_h^\alpha\circ\Phi_t^\alpha(c(\pi(s))))(1\otimes T_{\delta_x})\\
&\quad= \sum_{s\in G/N}\sum_{s,t,g\in G}\phi(gu^{-1}tg^{-1},\pi(g)x^{-1})\phi(sv^{-1}ts^{-1},\pi(s)\pi(g)^{-1})\\
&\qquad\qquad\times\theta(\Phi_t^\alpha(c(\pi(s))))(1\otimes T_{\delta_x}).
\end{aligned}$$

In the meantime, from (P3), we find that

$$\begin{aligned}
&\sum_{g\in G} \phi(gu^{-1}tg^{-1}, \pi(g)x^{-1})\phi(sv^{-1}ts^{-1}, \pi(s)\pi(g)^{-1}) \\
&\quad = \sum_{g\in G} \phi(gsu^{-1}ts^{-1}g^{-1}, \pi(g)\pi(s)x^{-1})\phi(sv^{-1}ts^{-1}, \pi(g)^{-1}) \\
&\quad = \delta_{u,v}\phi(su^{-1}ts^{-1}, \pi(s)x^{-1}).
\end{aligned}$$

From this, it follows that

$$\begin{aligned}
&\Phi_u \circ \Phi_v(X) \\
&\quad = \delta_{u,v} \sum_{s\in G/N} \sum_{s,t\in G} \phi(su^{-1}ts^{-1}, \pi(s)x^{-1})\theta(\Phi_t^\alpha(c(\pi(s))))(1\otimes T_{\delta_x}) \\
&\quad = \delta_{u,v}\Phi_u(X).
\end{aligned}$$

This shows that $\{\Phi_s\}$ satisfies (CA1). Let $X_c = \sum_{s\in G/N} \theta(c(x))(1\otimes T_{\delta_x})$ and $X_d = \sum_{s\in G/N} \theta(d(x))(1\otimes T_{\delta_x})$ be arbitrary two elements of $\mathcal{Q}$. Then, with the aid of Lemma 4.7, a simple calculation shows that

$$\begin{aligned}
&X_c X_d \\
&\quad = \sum_{x\in G/N} \theta\Big(\sum_{y,z,w\in G/N} \nu(\delta_{ywx^{-1}}, \delta_{zx^{-1}})c(y)\Phi_w^\theta(d(z))\Big)(1\otimes T_{\delta_x}).
\end{aligned} \tag{5.6}$$

Thus we have

$$\begin{aligned}
&\Phi_r(X_c X_d) \\
&\quad = \sum_{x\in G/N} \sum_{g,h\in G} \phi(gr^{-1}hg^{-1}, \pi(g)x^{-1}) \\
&\qquad \times\theta\Big(\sum_{y,z,w\in G/N} \nu(\delta_{yw\pi(g)^{-1}}, \delta_{z\pi(g)^{-1}})\Phi_h^\alpha(c(y)\Phi_w^\theta(d(z)))\Big)(1\otimes T_{\delta_x}) \\
&\quad = \sum_{x,y,z,w\in G/N} \sum_{s,g,h\in G} \phi(gr^{-1}hg^{-1}, \pi(g)x^{-1}) \\
&\qquad \times\nu(\delta_{yw\pi(g)^{-1}}, \delta_{z\pi(g)^{-1}})\theta(\Phi_s^\alpha(c(y))\Phi_{s^{-1}h}^\alpha \circ \Phi_w^\theta(d(z)))(1\otimes T_{\delta_x}) \\
&\quad = \sum_{x,y,z\in G/N} \sum_{s,g,h\in G} \phi(gr^{-1}hg^{-1}, \pi(g)x^{-1}) \\
&\qquad \times\nu(\delta_{y\pi(s^{-1}h)\pi(g)^{-1}}, \delta_{z\pi(g)^{-1}})\theta(\Phi_s^\alpha(c(y))\Phi_{s^{-1}h}^\alpha(d(z)))(1\otimes T_{\delta_x}) \\
&\quad = \sum_{x,y,z\in G/N} \sum_{s,g,h\in G} \phi(gr^{-1}shg^{-1}, \pi(g)x^{-1})
\end{aligned}$$

$$\times\nu(\delta_{y\pi(h)\pi(g)^{-1}},\, \delta_{z\pi(g)^{-1}})\theta(\Phi_s^\alpha(c(y))\Phi_h^\alpha(d(z)))(1\otimes T_{\delta_x}).$$

In the meantime, we want to compute $\sum_{p\in G}\Phi_p(X_c)\,\Phi_{p^{-1}r}(X_d)$. With

$$\begin{aligned} e(u,x) &= \sum_{s,t\in G}\phi(su^{-1}ts^{-1},\pi(s)x^{-1})\Phi_t^\alpha(c(\pi(s))) \\ &\qquad\qquad\qquad\qquad (u\in G,\, x\in G/N); \\ f(u,x) &= \sum_{g,h\in G}\phi(gu^{-1}hg^{-1},\pi(g)x^{-1})\Phi_h^\alpha(d(\pi(g))), \end{aligned}$$

equation (5.6) yields

$$\begin{aligned} &\sum_{p\in G}\Phi_p(X_c)\,\Phi_{p^{-1}r}(X_d) \\ &\quad=\sum_{p\in G}\sum_{x\in G/N}\theta\Big(\sum_{y,z,w\in G/N}\nu(\delta_{ywx^{-1}},\,\delta_{zx^{-1}})e(p,y)\Phi_w^\theta(f(p^{-1}r,z))\Big) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \times(1\otimes T_{\delta_x}). \end{aligned}$$

Since $\Phi_w^\theta\circ\Phi_h^\alpha=\delta_{w,\pi(h)}\Phi_h^\alpha$, we get

$$\begin{aligned} &\sum_{p\in G}\Phi_p(X_c)\,\Phi_{p^{-1}r}(X_d) \\ &\quad=\sum_{p\in G}\sum_{x\in G/N}\theta\Big(\sum_{y,z\in G/N}\nu(\delta_{y\pi(h)x^{-1}},\,\delta_{zx^{-1}}) \\ &\qquad\quad\times\sum_{s,t\in G}\phi(sp^{-1}ts^{-1},\pi(s)x^{-1})\Phi_t^\alpha(c(\pi(s))) \\ &\qquad\quad\times\sum_{g,h\in G}\phi(gr^{-1}phg^{-1},\pi(g)x^{-1})\Phi_h^\alpha(d(\pi(g)))\Big)(1\otimes T_{\delta_x}). \end{aligned}$$

From condition (P4), one deduces that

$$\begin{aligned} &\sum_{p\in G}\sum_{y,z\in G/N}\nu(\delta_{y\pi(h)x^{-1}},\,\delta_{zx^{-1}})\phi(sp^{-1}ts^{-1},\pi(s)x^{-1}) \\ &\qquad\qquad\qquad\times\phi(gr^{-1}phg^{-1},\pi(g)x^{-1}) \\ &\quad=\sum_{p\in G}\sum_{y,z\in G/N}\nu(\delta_y,\,\delta_z)\phi((sh)(h^{-1}p^{-1}th)(sh)^{-1},\pi(sh)x^{-1}y^{-1}) \\ &\qquad\qquad\qquad\times\phi(gr^{-1}phg^{-1},\pi(g)x^{-1}z^{-1}) \\ &\quad=\sum_{p\in G}\sum_{y,z\in G/N}\nu(\delta_y,\,\delta_z)\phi((sh)p^{-1}(sh)^{-1},\pi(sh)x^{-1}y^{-1}) \end{aligned}$$

$$\times\phi(g(r^{-1}th)pg^{-1},\pi(g)x^{-1}z^{-1})$$
$$=\frac{1}{|N|^2}\sum_{u\in G}\phi(ur^{-1}thu^{-1},\pi(u)x^{-1})\nu(\delta_{\pi(shu)^{-1}},\,\delta_{\pi(gu)^{-1}}).$$

From this, it follows that

$$\begin{aligned}
&\sum_{p\in G}\Phi_p(X_c)\,\Phi_{p^{-1}r}(X_d)\\
&\quad=\frac{1}{|N|^2}\sum_{x\in G/N}\sum_{g,h,s,t,u\in G}\phi(ur^{-1}thu^{-1},\pi(u)x^{-1})\\
&\qquad\times\nu(\delta_{\pi(shu)^{-1}},\,\delta_{\pi(gu)^{-1}})\theta(\Phi^\alpha_t(c(\pi(s)))\Phi^\alpha_h(d(\pi(g))))(1\otimes T_{\delta_x})\\
&\quad=\sum_{x,y,z\in G/N}\sum_{h,t,u\in G}\phi(ur^{-1}thu^{-1},\pi(u)x^{-1})\\
&\qquad\times\nu(\delta_{y\pi(hu)^{-1}},\,\delta_{z\pi(u)^{-1}})\theta(\Phi^\alpha_t(c(y))\Phi^\alpha_h(d(z)))(1\otimes T_{\delta_x}).
\end{aligned}$$

This shows that $\Phi_r(X_cX_d)=\sum_{p\in G}\Phi_p(X_c)\Phi_{p^{-1}r}(X_d)$, so that $\{\Phi_{s\in G}\}$ satisfies (CA4). Finally, with X_c as above, Lemma 4.7 implies that

$$X_c^*=\sum_{z\in G/N}\theta\Big(\sum_{x,y\in G/N}\eta_0(zy^{-1}x^{-1})\Phi^\theta_y(c(x)^*)\Big)(1\otimes T_{\delta_z}) \tag{5.7}$$

where $\eta_0(x)=\sum_{y\in G/N}\overline{\nu(\delta_{x^{-1}y},\,\delta_y)}$ (see Lemma 4.3). Hence

$$\begin{aligned}
\Phi_{s^{-1}}(X_c^*)\;=\;&\sum_{z\in G/N}\sum_{t,u\in G}\phi(ustu^{-1},\pi(u)z^{-1})\\
&\times\theta\Big(\Phi^\alpha_t\big(\sum_{x,y\in G/N}\eta_0(\pi(u)y^{-1}x^{-1})\Phi^\theta_y(c(x)^*)\big)\Big)(1\otimes T_{\delta_z}).
\end{aligned}$$

Since $\Phi^\alpha_t\circ\Phi^\theta_y=\delta_{\pi(t),y}\Phi^\alpha_t$, we obtain

$$\begin{aligned}
\Phi_{s^{-1}}(X_c^*)\;=\;&\sum_{x,z\in G/N}\sum_{t,u\in G}\phi(ustu^{-1},\pi(u)z^{-1})\eta_0(\pi(u)\pi(t)^{-1}x^{-1})\\
&\qquad\times\theta(\Phi^\alpha_t(c(x)^*))(1\otimes T_{\delta_z})\\
=\;&\frac{1}{|N|}\sum_{z\in G/N}\sum_{t,u,v\in G}\phi(ustu^{-1},\pi(u)z^{-1})\eta_0(\pi(ut^{-1}v^{-1}))\\
&\qquad\times\theta(\Phi^\alpha_t(c(\pi(v))^*))(1\otimes T_{\delta_z}).
\end{aligned}$$

From identity (P5), it results that

$$\begin{aligned}
&\frac{1}{|N|}\sum_{u\in G}\phi(ustu^{-1},\pi(u)z^{-1})\eta_0(\pi(ut^{-1}v^{-1}))\\
&\quad=\frac{1}{|N|}\sum_{u\in G}\phi(u(vtsv^{-1})u^{-1},\pi(u)(\pi(vt)z^{-1}))\eta_0(\pi(u))\\
&\quad=\sum_{x\in G/N}\eta_0(x)\overline{\phi(vs^{-1}t^{-1}v^{-1},\pi(vt)z^{-1}x)}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\Phi_{s^{-1}}(X_c^*) \;=\; &\sum_{x,z\in G/N}\sum_{t,v\in G}\eta_0(x)\,\overline{\phi(vs^{-1}t^{-1}v^{-1},\pi(vt)z^{-1}x)}\\
&\times\theta(\Phi_t^{\alpha}(c(\pi(v))^*))(1\otimes T_{\delta_z}).
\end{aligned}$$

Meanwhile, by applying (5.7) to $\Phi_s(X_c)$ in place of X_c, we get

$$\begin{aligned}
\Phi_s(X_c)^* \;=\; &\sum_{z\in G/N}\theta\Big(\sum_{x,y\in G/N}\eta_0(zy^{-1}x^{-1})\\
&\times\Phi_y^{\theta}(\{\sum_{u,t\in G}\phi(us^{-1}tu^{-1},\pi(u)x^{-1})\\
&\times\Phi_t^{\alpha}(c(\pi(u)))\}^*)\Big)(1\otimes T_{\delta_z})\\
=\;&\sum_{x,y,z\in G/N}\sum_{u,t\in G}\eta_0(zy^{-1}x^{-1})\,\overline{\phi(us^{-1}tu^{-1},\pi(u)x^{-1})}\\
&\times\theta(\Phi_y^{\theta}\circ\Phi_{t^{-1}}^{\alpha}(c(\pi(u))^*))(1\otimes T_{\delta_z}).
\end{aligned}$$

Since $\Phi_y^{\theta}\circ\Phi_{t^{-1}}^{\alpha}=\delta_{y,\pi(t)^{-1}}\Phi_{t^{-1}}^{\alpha}$, it follows that

$$\begin{aligned}
\Phi_s(X_c)^* \;=\; &\sum_{x,z\in G/N}\sum_{u,t\in G}\eta_0(z\pi(t)x^{-1})\,\overline{\phi(us^{-1}tu^{-1},\pi(u)x^{-1})}\\
&\times\theta(\Phi_{t^{-1}}^{\alpha}(c(\pi(u))^*))(1\otimes T_{\delta_z})\\
=\;&\sum_{x,z\in G/N}\sum_{u,t\in G}\eta_0(x^{-1})\,\overline{\phi(us^{-1}tu^{-1},\pi(ut^{-1})z^{-1}x^{-1})}\\
&\times\theta(\Phi_{t^{-1}}^{\alpha}(c(\pi(u))^*))(1\otimes T_{\delta_z})\\
=\;&\sum_{x,z\in G/N}\sum_{u,t\in G}\eta_0(x)\,\overline{\phi(us^{-1}t^{-1}u^{-1},\pi(ut)z^{-1}x)}\\
&\times\theta(\Phi_t^{\alpha}(c(\pi(u))^*))(1\otimes T_{\delta_z}),
\end{aligned}$$

which proves that $\{\Phi_s\}$ satisfies (CA2).

We summarize the discussion in the preceding paragraph in the theorem that follows.

Theorem 5.8 *Let $X = \sum_{x\in G/N} \theta(c(x))(1\otimes T_{\delta_x})$ be in the twisted crossed product $\mathcal{Q} = \mathcal{R} \times_{\theta,\nu} R(G/N)'$. Then the equation*

$$\Phi_s(X) = \sum_{x\in G/N}\sum_{t,u\in G} \phi(us^{-1}tu^{-1}, \pi(u)x^{-1}) \times\theta(\Phi_t^{\alpha}(c(\pi(u))))(1\otimes T_{\delta_x}) \quad (s\in G)$$

defines a linear map Φ_s from $\mathcal{Q}$ into itself so that the family $\{\Phi_s\}_{s\in G}$ satisfies conditions (CA0)–(CA4). *Therefore, the equation*

$$\beta(X) = \sum_{s\in G} \Phi_s(X)\otimes\lambda(s) \qquad (X\in\mathcal{Q})$$

in turn defines a coaction β of G on the AFD factor $\mathcal{Q}$ of type II_1.

Our next objective is to compute the inner part $\mathrm{Int}(\beta)$ of the coaction β constructed above. This is, by definition, the same as explicitly describing the central projection $p_\beta (= \sum_{\chi\in\mathrm{Int}(\beta)}\chi)$ of $\ell^1(G)$. Eventually, we shall prove that $\lambda(p_\beta) = q_N$. But we first show that $\lambda(p_\beta)\geq q_N$, by proving directly that the restriction of β to $R(G/N)$ is inner, i.e., implemented by a certain unitary V in $\mathcal{Q}\otimes R(G/N)$. To do so, we need some auxiliary results.

We define an operator V_N in $\mathcal{Q}\otimes R(G)$ by

$$V_N = \frac{1}{|N|}\sum_{s\in G} 1\otimes T_{\delta_{\pi(s)}}\otimes\lambda(s).$$

Then we have

$$V_N V_N^* = \frac{1}{|N|^2}\sum_{s,t\in G} 1\otimes T_{\delta_{\pi(s)}\,\sharp\,\delta^\star_{\pi(t^{-1}s)}}\otimes\lambda(t).$$

In the meantime,

$$\begin{aligned}\sum_{s\in G}\delta_{\pi(s)}\,\sharp\,\delta^\star_{\pi(t^{-1}s)} &= \sum_{s\in G}\sum_{w\in G/N}\eta_0(w\pi(s^{-1}t))(\delta_{\pi(s)}\,\sharp\,\delta_w) \\ &= \sum_{s\in G}\sum_{w\in G/N}\eta_0(w\pi(s^{-1}t))\nu(\delta_{\pi(s)z^{-1}},\,\delta_{wz^{-1}})\delta_z.\end{aligned}$$

From this, it follows that

$$\begin{aligned}
&V_N V_N^* \\
&= \frac{1}{|N|^2} \sum_{s,t\in G} \sum_{w,z\in G/N} \eta_0(w\pi(s^{-1}t))\nu(\delta_{\pi(s)z^{-1}}, \delta_{wz^{-1}})\delta_z \\
&= \frac{1}{|N|} \sum_{t\in G} \sum_{w,y,z\in G/N} \eta_0(wy^{-1}\pi(t))\nu(\delta_{yz^{-1}}, \delta_{wz^{-1}})(1 \otimes T_{\delta_z} \otimes \lambda(t)) \\
&= \frac{1}{|N|} \sum_{t\in G} \sum_{w,y,z\in G/N} \eta_0(w)\nu(\delta_{yz^{-1}}, \delta_{w\pi(t)^{-1}yz^{-1}})(1 \otimes T_{\delta_z} \otimes \lambda(t)) \\
&= \frac{1}{|N|} \sum_{t\in G} \sum_{w,y,z\in G/N} \eta_0(w)\nu(\delta_{\pi(t)w^{-1}y}, \delta_y)(1 \otimes T_{\delta_z} \otimes \lambda(t)) \\
&= \frac{1}{|N|} \sum_{t\in G} \sum_{w\in G/N} \eta_0(w)\overline{\eta_0}(w\pi(t)^{-1})(1 \otimes 1 \otimes \lambda(t)).
\end{aligned}$$

Since $\eta_0^\sharp * \eta_0 = \delta_e$ by Lemma 4.6, it results that

$$\begin{aligned}
V_N V_N^* &= \frac{1}{|N|} \sum_{t\in G} \delta_e(\pi(t)^{-1})(1 \otimes 1 \otimes \lambda(t)) \\
&= \frac{1}{|N|} \sum_{t\in N} (1 \otimes 1 \otimes \lambda(t)) \\
&= 1 \otimes 1 \otimes q_N.
\end{aligned}$$

Since $1 \otimes 1 \otimes q_N$ is a central projection and $\mathcal{Q} \otimes R(G)$ is finite, it follows that $V_N^* V_N = 1 \otimes 1 \otimes q_N$. Thus V_N can be considered as a unitary in the reduced algebra $\mathcal{Q} \otimes R(G)_{q_N}$. Moreover, we have

Proposition 5.9 *The unitary V_N constructed above satisfies*

$$\beta(X)(1 \otimes q_N) = V_N^*(X \otimes 1)V_N$$

for any $X \in \mathcal{Q}$. Therefore, the restriction of β to $R(G)_{q_N} \cong R(G/N)$ is inner.

Proof. By the discussion preceding this proposition, it suffices to show that

$$V_N \beta(X) = (X \otimes 1)V_N \qquad (X \in \mathcal{Q}).$$

With $c : G/N \longrightarrow \mathcal{R}$, from Lemma 4.9, we have

$$
\begin{aligned}
&V_N\beta(X_c)\\
&= \frac{1}{|N|}\sum_{x\in G/N}\sum_{r,s,t,u\in G}\phi(us^{-1}rtu^{-1},\pi(u)x^{-1})\\
&\qquad\qquad\times(1\otimes T_{\delta_{\pi(r)}}\theta(\Phi_t^\alpha(c(\pi(u))))(1\otimes T_{\delta_x})\otimes\lambda(s)\\
&= \frac{1}{|N|}\sum_{r,s,t,u\in G}\sum_{x,y\in G/N}\phi(us^{-1}rtu^{-1},\pi(u)x^{-1}))\\
&\qquad\qquad\times\theta(\Phi_y^\theta\circ\Phi_t^\alpha(c(\pi(u))))(1\otimes T_{(\delta_{\pi(r)}*\delta_y)\sharp\delta_x})\otimes\lambda(s)\\
&= \frac{1}{|N|}\sum_{r,s,t,u\in G}\sum_{x\in G/N}\phi(us^{-1}rtu^{-1},\pi(u)x^{-1}))\\
&\qquad\qquad\times\theta(\Phi_t^\alpha(c(\pi(u))))(1\otimes T_{(\delta_{\pi(rt)}\sharp\delta_x})\otimes\lambda(s)\\
&= \frac{1}{|N|}\sum_{r,s,t,u\in G}\sum_{x,y\in G/N}\nu(\delta_{\pi(rt)y^{-1}},\,\delta_{xy^{-1}})\phi(us^{-1}rtu^{-1},\pi(u)x^{-1}))\\
&\qquad\qquad\times\theta(\Phi_t^\alpha(c(\pi(u))))(1\otimes T_{\delta_y})\otimes\lambda(s)\\
&= \frac{1}{|N|}\sum_{r,s,t,u\in G}\sum_{x,y\in G/N}\nu(\delta_{\pi(rt)y^{-1}},\,\delta_x)\phi(us^{-1}rtu^{-1},\pi(u)y^{-1}x^{-1}))\\
&\qquad\qquad\times\theta(\Phi_t^\alpha(c(\pi(u))))(1\otimes T_{\delta_y})\otimes\lambda(s)\\
&= \frac{1}{|N|^2}\sum_{r,s,t,u,v\in G}\sum_{x\in G/N}\nu(\delta_{\pi(rtv^{-1})},\,\delta_x)\\
&\qquad\qquad\times\phi(us^{-1}rtu^{-1},\pi(uv^{-1})x^{-1}))\\
&\qquad\qquad\times\theta(\Phi_t^\alpha(c(\pi(u))))(1\otimes T_{\delta_{\pi(v)}})\otimes\lambda(s).
\end{aligned}
$$

Meanwhile, with the aid of (P6), we obtain

$$
\begin{aligned}
&\sum_{r,t,u\in G}\sum_{x\in G/N}\nu(\delta_{\pi(rtv^{-1})},\,\delta_x)\phi(us^{-1}rtu^{-1},\pi(uv^{-1})x^{-1}))\\
&\qquad\qquad\times\theta(\Phi_t^\alpha(c(\pi(u))))\\
&= \sum_{r,t,u\in G}\sum_{x\in G/N}\nu(\delta_{\pi(rtv^{-1}},\,\delta_x)\phi(uvs^{-1}rtv^{-1}u^{-1},\pi(u)x^{-1}))\\
&\qquad\qquad\times\theta(\Phi_t^\alpha(c(\pi(uv))))\\
&= \sum_{r,t,u\in G}\sum_{x\in G/N}\nu(\delta_{\pi(r)},\,\delta_x)\phi(uvs^{-1}ru^{-1},\pi(u)x^{-1}))
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad \times\theta(\Phi_t^\alpha(c(\pi(uv)))) \\
&= \sum_{r,u\in G}\sum_{x\in G/N} \nu(\delta_{\pi(r)},\, \delta_x) \\
&\qquad\qquad \times\phi(uvs^{-1}ru^{-1}, \pi(u)x^{-1}))\theta(c(\pi(uv))) \\
&= \sum_{y\in G/N}\Big(\sum_{r\in G}\sum_{x\in G/N}\sum_{\pi(u)=y} \nu(\delta_{\pi(r)},\, \delta_x) \\
&\qquad\qquad \times\phi(u(vs^{-1})ru^{-1}, yx^{-1}))\Big)\theta(c(y\pi(v))) \\
&= \sum_{y\in G/N} \nu(\delta_y,\, \delta_{\pi(sv^{-1})}\theta(c(y\pi(v))).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&V_N\beta(X_c) \\
&= \frac{1}{|N|^2}\sum_{s,v\in G}\sum_{y\in G/N} \nu(\delta_y,\, \delta_{\pi(sv^{-1})}\theta(c(y\pi(v))) \\
&\qquad\qquad\qquad\qquad \times(1\otimes T_{\delta_{\pi(v)}})\otimes\lambda(s) \\
&= \frac{1}{|N|}\sum_{s\in G}\sum_{x,y\in G/N} \nu(\delta_y,\, \delta_{\pi(s)x^{-1}}\theta(c(yx))(1\otimes T_{\delta_x})\otimes\lambda(s) \\
&= \frac{1}{|N|}\sum_{s\in G}\sum_{x,y\in G/N} \nu(\delta_{yx^{-1}},\, \delta_{\pi(s)x^{-1}}\theta(c(y))(1\otimes T_{\delta_x})\otimes\lambda(s) \\
&= \frac{1}{|N|}\sum_{s\in G}\sum_{x,y\in G/N} (\delta_y \sharp \delta_{\pi(s)})(x)\theta(c(y))(1\otimes T_{\delta_x})\otimes\lambda(s) \\
&= \frac{1}{|N|}\sum_{s\in G}\sum_{y\in G/N} \theta(c(y))(1\otimes T_{\delta_y})(1\otimes T_{\delta_{\pi(s)}})\otimes\lambda(s) \\
&= (X_c\otimes 1)V_N.
\end{aligned}
$$

This completes the proof. □

Corollary 5.10 *Let $\lambda(p_\beta)$ be the central projection in $R(G)$ corresponding to the inner part of the coaction β. Then we have $q_N \leq \lambda(p_\beta)$.*

Proof. This follows from Theorem 1.7 of [Y1]. □

Corollary 5.11 *The family $b = \{b(s)\}_{s\in G}$ of elements in $\mathcal{Q}$ defined by*

$$
b(s) = \frac{1}{|N|}(1\otimes T_{\delta_{\pi(s)^{-1}}})
$$

belongs to $\mathrm{Rel}(\beta)$*. Moreover, we have*

$$\sum_{s\in G} b(s)b(st)^* = p_N(t)\cdot 1$$

for any $t \in G$.

Proof. First note that $V_N = \sum_{s\in G} b(s)\otimes\lambda(s)^*$. Thus the last assertion is just a restatement of the identity $V_N V_N^* = 1\otimes 1\otimes q_N$. The first assertion results from the following general fact: □

Lemma 5.12 *Let β_0 be a coaction of G on a von Neumann algebra $\mathcal{A}$. Suppose that $\{c(s)\}$ is a family of elements in $\mathcal{A}$. Define an operator $T = \sum_{s\in G} c(s)\otimes\lambda(s)^*$ in $\mathcal{A}\otimes R(G)$. Then the family $\{c(s)\}$ belongs to* $\mathrm{Rel}(\beta_0)$ *if and only if T satisfies*

$$T\beta_0(x) = (x\otimes 1)T$$

for all $x \in \mathcal{A}$.

Proof. We write

$$\beta_0(x) = \sum_{s\in G} \Phi_s^{(0)}(x)\otimes\lambda(s).$$

With this notation, $\mathrm{Rel}(\beta_0)$ is, by definition, the set of all families $\{a(s)\}_{s\in G}$ in $\mathcal{A}$ satisfying $xa(t) = \sum_{s\in G} a(s)\Phi^{(0)}_{st^{-1}}(x)$ for all $x \in \mathcal{A}$. For $x \in \mathcal{A}$, it is easily verified that

$$\begin{aligned} T\beta_0(x) &= \sum_{t\in G}\Big(\sum_{s\in G} c(s)\Phi_{st}^{(0)}(x)\Big)\otimes\lambda(t); \\ (x\otimes 1)T &= \sum_{t\in G} xc(t^{-1})\otimes\lambda(t). \end{aligned}$$

From these identities, the assertion immediately follows. □

In order to prove that the central projection q_N really coincides with $\lambda(p_\beta)$, we need to examine the set $\mathrm{Rel}(\beta)$ in more detail.

As before, for any function $d : G/N \longrightarrow \mathcal{R}$, we write

$$X_d = \sum_{x\in G/N} \theta(d(x))(1\otimes T_{\delta_x}) \in \mathcal{Q}.$$

In addition to this notation, for any function $c : G \times G/N \longrightarrow \mathcal{R}$, we set

$$X_c(t) = \sum_{x \in G/N} \theta(c(t,x))(1 \otimes T_{\delta_x}) \in \mathcal{Q}.$$

Then, by the definition of $\mathrm{Rel}(\beta)$, the family $\{X_c(t)\}_{t \in G}$ belongs to $\mathrm{Rel}(\beta)$ if and only if it satisfies

$$X_d\, X_c(t) = \sum_{s \in G} X_c(s)\, \Phi_{st^{-1}}(X_d)$$

for all $d : G/N \longrightarrow \mathcal{R}$. By a direct calculation, the above identity is proven to be equivalent to

$$\begin{aligned} &\sum_{x,y,z \in G/N} \nu(\delta_{xzw^{-1}}, \delta_{yw^{-1}})\, d(x) \Phi_z^{\theta}(c(t,y)) \\ &= \sum_{s,u,v \in G} \sum_{x,y \in G/N} \phi(uts^{-1}vu^{-1}, \pi(u)y^{-1}) \nu(\delta_{x\pi(v)w^{-1}}, \delta_{yw^{-1}}) \\ &\qquad \times c(s,x) \Phi_v^{\alpha}(d(\pi(u))) \end{aligned} \tag{5.13}$$

for all $w \in G/N$ and $d : G/N \longrightarrow \mathcal{R}$. From this information, we shall find out an explicit form of the function $c(\cdot,\cdot)$. First, let us take d to be a constant function, say, $d(x) \equiv a$ for some $a \in \mathcal{R}$. Then, by (C3), the left-hand side (LHS) of (5.13) equals $a\, c(t,w)$. In the meantime, by (P1) and (C3), the right-hand side (RHS) of (5.13) is $\sum_{v \in G} c(vt, w\pi(v)^{-1}) \Phi_v^{\alpha}(a)$. Thus we get

$$a\, c(t,w) = \sum_{v \in G} c(vt, w\pi(v)^{-1}) \Phi_v^{\alpha}(a) \tag{5.14}$$

For each $w \in G/N$, set $c_w(t) = c(t^{-1}, w\pi(t))$. From (5.14),

$$\begin{aligned} ac_w(t) &= ac(t^{-1}, w\pi(t)) \\ &= \sum_{s \in G} c(st^{-1}, w\pi(t)\pi(s)^{-1}) \Phi_s^{\alpha}(a) \\ &= \sum_{s \in G} c(s^{-1}, w\pi(s)) \Phi_{s^{-1}t}^{\alpha}(a) \\ &= \sum_{s \in G} c_w(s) \Phi_{s^{-1}t}^{\alpha}(a). \end{aligned}$$

This shows that $\{c_w(t)\}_{t \in G}$ belongs to $\mathrm{Rel}(\alpha)$ for each $w \in G/N$. Since α is outer, there exists a function ξ_c on G/N, depending upon c, such that

$c_w(t) = \xi_c(w) \cdot 1$ for any $t \in G$. Thus we have

$$c(t,x) = \xi_c(x\pi(t)) \qquad (t \in G,\, x \in G/N). \tag{5.15}$$

Returning to (5.13) with this identity, we obtain

$$\begin{aligned}
&\sum_{x,y\in G/N} \nu(\delta_{xw^{-1}},\, \delta_{yw^{-1}})\, \xi_c(y\pi(t))\, d(x) \\
&\quad = \sum_{s,u,v\in G}\sum_{x,y\in G/N} \phi(uts^{-1}vu^{-1}, \pi(u)y^{-1})\nu(\delta_{x\pi(v)w^{-1}},\, \delta_{yw^{-1}}) \\
&\qquad\qquad\qquad \times \xi_c(x\pi(s))\, \Phi_v^\alpha(d(\pi(u))) \\
&\quad = \sum_{s,u,v\in G}\sum_{x,y\in G/N} \phi(uts^{-1}vu^{-1}, \pi(u)y^{-1})\nu(\delta_{xw^{-1}},\, \delta_{yw^{-1}}) \\
&\qquad\qquad\qquad \times \xi_c(x\pi(v^{-1}s))\, \Phi_v^\alpha(d(\pi(u))) \\
&\quad = \sum_{s,u,v\in G}\sum_{x,y\in G/N} \phi(uts^{-1}u^{-1}, \pi(u)y^{-1})\nu(\delta_{xw^{-1}},\, \delta_{yw^{-1}}) \\
&\qquad\qquad\qquad \times \xi_c(x\pi(s))\, \Phi_v^\alpha(d(\pi(u))) \\
&\quad = \sum_{s,u\in G}\sum_{x,y\in G/N} \phi(uts^{-1}u^{-1}, \pi(u)y^{-1})\nu(\delta_{xw^{-1}},\, \delta_{yw^{-1}}) \\
&\qquad\qquad\qquad \times \xi_c(x\pi(s))\, d(\pi(u))
\end{aligned} \tag{5.16}$$

for any $w \in G/N$ and $d : G/N \longrightarrow \mathcal{R}$. Let $a \in \mathcal{R}$. Applying the functional τ_a on $\mathcal{R}$ defined by $\tau_a(b) = \tau(ab)$ in equality (5.16), we get

$$\begin{aligned}
&\sum_{x,y\in G/N} \nu(\delta_{xw^{-1}},\, \delta_{yw^{-1}})\, \xi_c(y\pi(t))\, \tau(ad(x)) \\
&\quad = \sum_{s,u\in G}\sum_{x,y\in G/N} \phi(uts^{-1}u^{-1}, \pi(u)y^{-1})\nu(\delta_{xw^{-1}},\, \delta_{yw^{-1}}) \\
&\qquad\qquad\qquad \times \xi_c(x\pi(s))\tau(ad(\pi(u))).
\end{aligned} \tag{5.17}$$

Since the set of all functions g on G/N of the form $g(x) = \tau(ad(x))$ for some $a \in \mathcal{R}$ and $d : G/N \to \mathcal{R}$ coincides with $\ell_\nu^\infty(G/N)$, identity (5.17) is equivalent to

$$\begin{aligned}
&\sum_{x,y\in G/N} \nu(\delta_{xw^{-1}},\, \delta_{yw^{-1}})\, \xi_c(y\pi(t))\, f(x) \\
&\quad = \sum_{s,u\in G}\sum_{x,y\in G/N} \phi(uts^{-1}u^{-1}, \pi(u)y^{-1})\nu(\delta_{xw^{-1}},\, \delta_{yw^{-1}}) \\
&\qquad\qquad\qquad \times \xi_c(x\pi(s))f(\pi(u))
\end{aligned} \tag{5.18}$$

for all $f \in \ell^\infty_\nu(G/N)$. To clarify the meaning of (5.18), we define a map Ω_s $(s \in G)$ from $\ell^\infty_\nu(G/N)$ into itself by

$$\Omega_s(f)(x) = \sum_{u\in G} \phi(us^{-1}u^{-1}, \pi(u)x^{-1})f(\pi(u)) \qquad (x \in G/N).$$

Then $\{\Omega_s\}_{s\in G}$ determines a G-grading in $\ell^\infty_\nu(G/N)$ (recall how we deduced identities (P1)–(P5)). Hence the equation

$$\delta(f) = \sum_{s\in G} \Omega_s(f) \otimes \lambda(s) \qquad (f \in \ell^\infty_\nu(G/N))$$

defines a coaction δ of G on $\ell^\infty_\nu(G/N)$. By the definition of Φ_s, we easily see that $\Phi_s(1 \otimes T_f) = 1 \otimes T_{\Omega_s(f)}$ for any $f \in \ell^\infty_\nu(G/N)$. Hence, the coaction δ is, in some sense, the restriction of β to $\mathbf{C} \otimes \{\, T_f : f \in \ell^\infty_\nu(G/N)\,\}$. Having said this, we go back to (5.18). First,

$$\begin{aligned} \text{LHS of (5.18)} &= \sum_{x,y\in G/N} (\delta_x \sharp_\nu \delta_y)(w)\, (\xi_c * \delta_{\pi(t)^{-1}})(y)\, f(x) \\ &= \{f \sharp_\nu (\xi_c * \delta_{\pi(t)^{-1}})\}(w). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\text{RHS of (5.18)} \\ &\quad = \sum_{s,u\in G} \sum_{x,y\in G/N} \phi(uts^{-1}u^{-1}, \pi(u)y^{-1})(\delta_x \sharp_\nu \delta_y)(w) \\ &\qquad\qquad \times (\xi_c * \delta_{\pi(s)^{-1}})(x) f(\pi(u)) \\ &\quad = \sum_{s\in G} \sum_{x,y\in G/N} (\delta_x \sharp_\nu \delta_y)(w)\, (\xi_c * \delta_{\pi(s)^{-1}})(x)\Omega_{st^{-1}}(f)(y) \\ &\quad = \sum_{s\in G} \{(\xi_c * \delta_{\pi(s)^{-1}}) \sharp_\nu \Omega_{st^{-1}}(f)\}(w). \end{aligned}$$

This means that the family $\{\xi_c * \delta_{\pi(t)^{-1}}\}_{t\in G}$ belongs to $\mathrm{Rel}(\delta)$.

We summarize the results in the preceding paragraph in the proposition that follows.

Proposition 5.19 *Let $c : G \times G/N \longrightarrow \mathcal{R}$ be an arbitrary $\mathcal{R}$-valued function. Then the family $\{X_c(t)\}_{t\in G}$ of elements in $\mathcal{Q}$ belongs to* $\mathrm{Rel}(\beta)$ *if and only if there exists a function ξ_c on G/N, depending upon c, such that* (i) $c(t,x) = \xi_c(x\pi(t)) \cdot 1$; (ii) *the family $\{\xi_c * \delta_{\pi(t)^{-1}}\}_{t\in G}$ lies in* $\mathrm{Rel}(\delta)$, *i.e.*, *it satisfies*

$$f \sharp_\nu (\xi_c * \delta_{\pi(t)^{-1}}) = \sum_{s\in G} (\xi_c * \delta_{\pi(s)^{-1}}) \sharp_\nu \Omega_{st^{-1}}(f)$$

for all $f \in \ell^\infty_\nu(G/N)$.

We are now in a position to prove, with the help of Proposition 5.19, that the projection q_N equals $\lambda(p_\beta)$.

Theorem 5.20 *The central projection q_N coincides with $\lambda(p_\beta)$, the projection which determines the inner part of β. Therefore,* $\mathrm{Int}(\beta)$ *is the set of all irreducilbe characters χ of G such that $\delta_s * \chi = \chi$ for all $s \in N$.*

Proof. By the definition of the inner part, there exists an $\mathcal{R}$-valued function $c : G \times G/N \longrightarrow \mathcal{R}$ such that (i) $\{X_c(t)\}_{t\in G}$ belngs to $\mathrm{Rel}(\beta)$; (ii) $\sum_{s\in G} X_c(s)X_c(st)^* = p_\beta(t)\cdot 1$ for all $t \in G$. By Proposition 5.19, there is a function ξ_c on G/N so that $c(t,x) = \xi_c(x\pi(t))$. Hence we have $X_c(t) = 1 \otimes T_{\xi_c * \delta_{\pi(t)^{-1}}}$. A simple computation shows that equation (ii) is the same as

$$\sum_{s\in G} (\xi_c * \delta_{\pi(s)^{-1}}) \sharp_\nu (\xi_c * \delta_{\pi(st)^{-1}})^\star = p_\beta(t)\cdot 1. \tag{5.21}$$

In the meantme, recalling that $p_N = |N|^{-1}\chi_N$, we have

$$\begin{aligned}\sum_{t\in G} (\xi_c * \delta_{\pi(st)^{-1}})^\star p_N(t^{-1}u) &= \frac{1}{|N|}\sum_{t\in G} (\xi_c * \delta_{\pi(sut)^{-1}})^\star \chi_N(t^{-1}) \\ &= \frac{1}{|N|}\sum_{t\in N} (\xi_c * \delta_{\pi(sut)^{-1}})^\star \\ &= (\xi_c * \delta_{\pi(su)^{-1}})^\star.\end{aligned}$$

From this, together with (5.21), it follows that

$$\begin{aligned}(p_\beta * p_N)(u)\cdot 1 &= \sum_{t\in G} p_\beta(t)p_N(t^{-1}u)\cdot 1 \\ &= \sum_{s,t\in G} (\xi_c * \delta_{\pi(s)^{-1}}) \sharp_\nu (\xi_c * \delta_{\pi(st)^{-1}})^\star p_N(t^{-1}u) \\ &= \sum_{s\in G} (\xi_c * \delta_{\pi(s)^{-1}}) \sharp_\nu (\xi_c * \delta_{\pi(su)^{-1}})^\star = p_\beta(u)\cdot 1.\end{aligned}$$

Thus $p_\beta * p_N = p_\beta$. This shows that $\lambda(p_\beta) \le \lambda(p_N) = q_N$. In view of Corollary 5.10, we conclude that $\lambda(p_\beta) = q_N$. □

Our next goal is to describe the 2-cocycle μ_β and the function γ_β associated with β.

We take the family $b = \{b(s) = |N|^{-1}(1 \otimes T_{\delta_{\pi(s)^{-1}}})\}_{s\in G}$ of elements in $\mathcal{Q}$ defined in Corollary 5.11. By Corollary 5.11 and Theorem 5.20, we have $\sum_{s\in G} b(s)b(st)^* = p_\beta(t)\cdot 1$. We easily see that $\sum_{s\in G} b(s) = 1$. Hence this family gives rise to μ_β (see Section 1):

$$\mu_\beta(f, g)\cdot 1 = \sum_{t\in G} b_{g\circ\pi}(t) b_{f\circ\pi}(t) b(t)^* \qquad (f, g \in \ell^\infty(G/N)).$$

It is readily checked that $b_{h\circ\pi}(t) = 1 \otimes T_{h*\delta_{\pi(t)^{-1}}}$ for any $h \in \ell^\infty(G/N)$. Thus

$$\mu_\beta(f, g)\cdot 1 = \frac{1}{|N|}\sum_{t\in G}(1 \otimes T_{(g*\delta_{\pi(t)^{-1}})\sharp(f*\delta_{\pi(t)^{-1}})\sharp\delta^\star_{\pi(t)^{-1}}}).$$

Now we have

$$\begin{aligned}
&\frac{1}{|N|}\sum_{t\in G}(g*\delta_{\pi(t)^{-1}})\,\sharp\,(f*\delta_{\pi(t)^{-1}})\,\sharp\,\delta^\star_{\pi(t)^{-1}})\\
&\frac{1}{|N|}\sum_{t\in G}\sum_{x,y,z\in G/N} g(x\pi(t))f(y\pi(t))\eta_0(z\pi(t))\delta_x\,\sharp\,\delta_y\,\sharp\,\delta_z\\
&= \sum_{x,y,z,w,x_1,x_2\in G/N} g(xw)f(yw)\eta_0(zw)\nu(\delta_{xx_1^{-1}}, \delta_{yx_1^{-1}})\\
&\qquad\times\nu(\delta_{x_1x_2^{-1}}, \delta_{zx_2^{-1}})\delta_{x_2}\\
&= \sum_{x,y,z,w,x_1,x_2\in G/N} g(w)f(yx^{-1}w)\eta_0(zx^{-1}w)\nu(\delta_{xx_1^{-1}}, \delta_{yx_1^{-1}})\\
&\qquad\times\nu(\delta_{x_1x_2^{-1}}, \delta_{zx_2^{-1}})\delta_{x_2}\\
&= \sum_{x,y,z,w,x_1,x_2\in G/N} g(w)f(yx^{-1}w)\eta_0(zx^{-1}w)\nu(\delta_{x_1^{-1}}, \delta_{yx^{-1}x_1^{-1}})\\
&\qquad\times\nu(\delta_{x_1x_2^{-1}}, \delta_{zx^{-1}x_2^{-1}})\delta_{x_2x}\\
&= \sum_{x,y,z,w,x_1,x_2\in G/N} g(w)f(yw)\eta_0(zw)\nu(\delta_{x_1^{-1}}, \delta_{yx_1^{-1}})\\
&\qquad\times\nu(\delta_{x_1x_2^{-1}}, \delta_{zx_2^{-1}})\delta_{x_2x}\\
&= \sum_{y,z,w,x_1,x_2\in G/N} g(w)f(yw)\eta_0(zw)\nu(\delta_{x_1^{-1}}, \delta_{yx_1^{-1}})\\
&\qquad\times\nu(\delta_{x_1x_2^{-1}}, \delta_{zx_2^{-1}})\cdot 1
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{y,z,w,x_1\in G/N} g(w)f(yw)\eta_0(zw)\nu(\delta_{x_1^{-1}}, \delta_{yx_1^{-1}})\overline{\eta_0}(zx_1^{-1})\cdot 1\\
&= \sum_{y,w,x_1\in G/N} g(w)f(yw)(\eta_0 * \overline{\eta_0})(w^{-1}x_1^{-1})\nu(\delta_{x_1^{-1}}, \delta_{yx_1^{-1}})\cdot 1\\
&= \sum_{y,w\in G/N} g(w)f(yw)\nu(\delta_w, \delta_{yw})\cdot 1\\
&= \nu(g, f)\cdot 1 = \mu(f, g)\cdot 1.
\end{aligned}
$$

The second last equality is due to Lemma 4.6. The computation shows that we have μ_β coincides with the originally given 2-cocycle μ.

To compute the function γ_β, we first note that $\gamma_\beta(s,x) = 0$ whenever $s \notin G^N$ by Corollary 3.15. Moreover, by condition (3) of Definition 5.1, we have $\gamma(s,x) = 0$ if $s \notin G^N$. Thus $\gamma_\beta(s,x) = \gamma(s,x)$ if $s \notin G^N$ and $x \in G/N$. It remains to treat the case where $s \in G^N$. For this, we need to describe the element $\Psi_s(b)$ in detail. Let $s \in G^N$. By definition, we have $\Psi_s(b)(t) = \Phi_{tst^{-1}}(b(t))$. With the notation established in the discussion preceding Proposition 5.19, we obtain $\Psi_s(b)(t) = |N|^{-1}(1\otimes T_{\Omega_{tst^{-1}}(\delta_{\pi(t)^{-1}})})$. In the meantime, we have

$$
\begin{aligned}
\Omega_{tst^{-1}}(\delta_{\pi(t)^{-1}}) &= \sum_{u\in G} \phi(uts^{-1}t^{-1}u^{-1}, \pi(u)x^{-1})\delta_{\pi(t)^{-1}}(\pi(u))\\
&= \sum_{\pi(u)=\pi(t)^{-1}} \phi(uts^{-1}t^{-1}u^{-1}, \pi(u)x^{-1})\\
&= \sum_{h\in N} \phi(hs^{-1}h^{-1}, \pi(t)^{-1}x^{-1})\\
&= |N|\phi(s^{-1}, \pi(t)^{-1}x^{-1}).
\end{aligned}
$$

The second last equality is due to the fact that $\{\, ut : \pi(u) = \pi(t)^{-1} \,\} = N$ for each $t \in G$. Hence we obtain

$$
\Psi_s(b)(t) = \sum_{x\in G/N} \phi(s^{-1}, \pi(t)^{-1}x^{-1})(1\otimes T_{\delta_x}).
$$

Fom this, with $g \in G$, we have

$$
\begin{aligned}
&\gamma_\beta(s, \pi(g))\cdot 1\\
&\quad= f_{\Psi_s(b),b}(\pi(g))\cdot 1\\
&\quad= \sum_{r\in G} \Psi_s(b)(r)b(rg)^*
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{|N|} \sum_{r\in G} \sum_{x\in G/N} \phi(s^{-1}, \pi(r)^{-1}x^{-1})(1 \otimes T_{\delta_x \sharp \delta^\star_{\pi(rg)^{-1}}}) \\
&= \sum_{x,y\in G/N} \phi(s^{-1}, y^{-1}x^{-1})(1 \otimes T_{\delta_x \sharp \delta^\star_{\pi(g)^{-1}y^{-1}}}) \\
&= \sum_{x,y,z,w\in G/N} \phi(s^{-1}, y^{-1}x^{-1})\eta_0(zy\pi(g))\nu(\delta_{xw^{-1}}, \delta_{zw^{-1}})(1 \otimes T_{\delta_w}) \\
&= \sum_{x,y,z,w\in G/N} \phi(s^{-1}, y^{-1}x^{-1})\eta_0(zy\pi(g)) \\
&\qquad\qquad \times \nu(\delta_{w^{-1}}, \delta_{zx^{-1}w^{-1}})(1 \otimes T_{\delta_{wx}}) \\
&= \sum_{x,y,z,w\in G/N} \phi(s^{-1}, y^{-1}x^{-1})\eta_0(zxy\pi(g)) \\
&\qquad\qquad \times \nu(\delta_{w^{-1}}, \delta_{zw^{-1}})(1 \otimes T_{\delta_{wx}}) \\
&= \sum_{x,y,z,w\in G/N} \phi(s^{-1}, y^{-1})\eta_0(zy\pi(g))\nu(\delta_{w^{-1}}, \delta_{zw^{-1}})(1 \otimes T_{\delta_{wx}}) \\
&= \sum_{y,z,w\in G/N} \phi(s^{-1}, y^{-1})\eta_0(zy\pi(g))\nu(\delta_{w^{-1}}, \delta_{zw^{-1}}) \cdot 1 \\
&= \sum_{y,z\in G/N} \phi(s^{-1}, y^{-1})\eta_0(zy\pi(g))\overline{\eta_0}(z) \cdot 1 \\
&= \sum_{y\in G/N} \phi(s^{-1}, y^{-1})(\eta_0^\sharp * \eta_0)(y\pi(g)) \cdot 1 \\
&= \phi(s^{-1}, \pi(g)) = \gamma(s, \pi(g)).
\end{aligned}
$$

This proves that γ_β equals the function γ which we started with.

We now summarize the results obtained in the preceding discussion in the next theorem, which is our main theorem of this paper.

Theorem 5.22 *Let N be a normal subgroup of G. For any element $\mathbf{c} = (\mu, \gamma)$ in $\mathcal{E}(G,N)$, there exists a coaction $\beta = \beta_{(N,\mu,\gamma)}$ of G on the AFD factor of type II_1 such that* (i) $N = N_\beta$; (ii) $\mathbf{c} = \mathbf{c}(\beta)$.

6. Some equivalence relation on $\mathcal{E}(G,N)$

In the preceding section, we introduced the set $\mathcal{E}(G,N)$ for each normal subgroup N of G, which, in some sense, provides a "model" for the coactions of G on the AFD factor of type II_1. We saw that each coaction β of G on a finite factor $\mathcal{A}$ gives rise to an element $\mathbf{c}(\beta) = (\mu_\beta, \gamma_\beta)$ in $\mathcal{E}(G, N_\beta)$.

There is, however, some ambiguity in the notation $\mathbf{c}(\beta)$, since, as (3.16) and Lemma 3.17 suggest, both μ_β and γ_β heavily depend on the choice of the element $b = \{b(s)\}_{s\in G}$ of $\mathrm{Rel}(\beta)$ satisfying $f_b = p_\beta$ and $\sum_{s\in G} b(s) = 1$. In order to circumvent this situation, we shall introduce an equivalence relation on $\mathcal{E}(G,N)$ in general so that equivalence classes can make sense as an invariant. The equivalence relation we need is suggested in equation (3.16) and Lemma 3.17. As before, G is a finite group in what follows.

Definition 6.1 (1) Suppose that μ,ν are two 2-cocycles on $\ell^\infty(G)$, i.e., $\mu,\nu \in Z^2(\ell^\infty(G))$. We say that μ is cohomologous to ν, denoted by $\mu \sim \nu$, if there is a function η on G such that

(i) $\eta^\sharp * \eta = \eta * \eta^\sharp = \delta_e, \quad \sum_{s\in G} \eta(s) = 1$;

(ii) we have

$$\nu(f,\, g) = \sum_{s\in G} \mu(\overline{\eta} * f * \delta_{s^{-1}},\, \overline{\eta} * g * \delta_{s^{-1}})\eta(s)$$

for any $f,g \in \ell^\infty(G)$. The function η is called a connecting function (between μ and ν).

(2) Recall that the bilinear form $\varepsilon_G \otimes \varepsilon_G$, where ε_G is the counit of G, was called the trivial 2-cocycle. Any 2-cocycle that is cohomologous to the trivial one is said to be a coboundary.

It can be readily verified that the relation $\sim$ defined above is an equivalence relation on $Z^2(\ell^\infty(G))$. The definition is of course motivated by the ordinary group cohomology theory. The author does not know whether $Z^2(\ell^\infty(G))$ can be equipped with a suitable group structure so that the set of coboundaries forms a normal subgroup in such a way that it generates the equivalence relation $\sim$ just introduced.

Definition 6.2 Let N be a normal subgroup of G. We say that an element $\mathbf{c}_1 = (\mu_1,\gamma_1)$ of $\mathcal{E}(G,N)$ is equivalent to another element $\mathbf{c}_2 = (\mu_2,\gamma_2)$ if

(1) μ_1 is cohomologous to μ_2 with a function η on G/N a connecting function between them;

(2) the following identity holds true:

$$\gamma_2(s,w) = \frac{1}{|N|}\sum_{x\in G/N}\sum_{u\in G}\gamma_1(usu^{-1},\pi(u)x^{-1})\overline{\eta}(\pi(u))\eta(xw)$$

for any $f,g \in \ell^\infty(G/N)$, $s \in G$ and $w \in G/N$. In this case, we write

$\mathbf{c}_1 \sim \mathbf{c}_2$. The function η is still called a connecting function.

It is a routine to check that this relation $\sim$ is in fact an equivalence relation in $\mathcal{E}(G, N)$. The equivalence class of an element $\mathbf{c}$ of $\mathcal{E}(G, N)$ shall be denoted by $[\mathbf{c}]$. Then, by (3.16) and Lemma 3.17, the equivalence class $[\mathbf{c}(\beta)]$ does not depend on the choice of the element $b = \{b(s)\}$ for any coaction β. We denote the class by $\Lambda(\beta)$. It is then easy to see that $\Lambda(\beta)$ is a conjugacy invariant for coactions of G.

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