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About finite solvable groups with exactly four p-regular conjugacy classes

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Abstract. Let G be a finite solvable group and p a prime $\neq 2$. The purpose of this note is to give the structure of finite solvable groups with exactly four p-regular conjugacy classes.

Key words: finite solvable group, p-regular conjugacy class, p-length, p-nilpotent, structure of group.

1. Introduction

In [5], [6] and [7] Ninomiya describes the groups with exactly three p-regular classes. If F is a splitting field for G of characteristic p then, as well-known by Brauer, the number of non-isomorphic simple FG modules is equal to the number of p-regulare classes. Throughout this paper G denotes a finite group and p a prime $\neq 2$. The purpose of this note is to give the structure of finite solvable groups with exactly four p-regular classes. With $A \propto B$ we denote the semidirect product of a normal subgroup B with a subgroup A. All other notations are standard and can be found, for example, in [3].

Main Theorem *Let G be a solvable group with exactly four p-regular conjugacy classes, $O_p(G) = 1$ and $p \neq 2$. Then one of the following cases occurs :*

A) *If G is not p-nilpotent then $G \cong S_4$ and $p = 3$.*

B) *If G is p-nilpotent then G is one of the following types*

1. *The group G is one of the p'-groups*
   a) *G is the cyclic group Z_4 of order 4.*
   b) *$G \cong Z_2 \times Z_2$.*
   c) *G is the alternating group A_4 of degree 4, $p \neq 3$.*
   d) *G is the dihedral group of order 10, $p \neq 5$.*
2. *$O_{p'}(G)$ is elementary abelian of order 2^n and one of the following*

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statements for $O_{p'}(G)$ and a Sylow p-subgroup P of G holds.

a) $O_{p'}(G)$ *is a minimal normal subgroup,* $n = 4$ *and* $G = Z_5 \propto O_{p'}(G)$, $p = 5$.

b) $O_{p'}(G) \cong Z_2 \times N$, *where* N *is elementary abelian of order* $p + 1$, p *is a Mersenne prime,* P *is of order* p *and operates transitively on* $N \backslash 1$.

c) $G \cong (Z_p \propto N) \times (Z_p \propto N)$, *where* N *is elementary abelian of order* $p + 1$, p *is a Mersenne prime, and* Z_p *operates transitively on* $N \backslash 1$.

d) $O_{p'}(G)$ *is a minimal normal subgroup,* $n = 6$ *and* P *is a Sylow 3-group of GL(6,2), which acts naturally on* $O_{p'}(G)$.

3. $O_{p'}(G) \cong A(n, \theta)$ *is a Suzuki 2-group of order* 2^{2n} *and* θ *acts fixed point freely,* $G = P \propto A(n, \theta)$, *with* $|P| = p$.

4. $G = SL(2,3) \propto N$, *where* N *is an elementary abelian group of order 25 and* $SL(2,3)$ *operates transitively on* $N \backslash 1$, $p = 3$.

5. $G = (P \times Z_2) \propto N$, *where* P *is a cyclic* p*-group,* Z_2 *the cyclic group of order 2, and* $(P \times Z_2)$ *has two orbits on* $N \backslash 1$. *For the order* $\mid N \mid = q^a$, *and* $\mid P \mid = p^c$ *we have* $q^a - 1 = 4p^c$, *and if* $a > 1$ *then* $q = 5$ *and* a *is a prime.*

6. $G = Z_{15} \propto N$, $p = 5$, N *is elementary abelian of order 16 and* Z_{15} *operates as a Singer cycle on* N.

Conversly any of these groups has exactly four p-regular classes.

2. Preliminary results

Lemma 1 *If* G *is a solvable group with exactly four* p*-regular classes then the number of distinct prime divisors in the order of* G *is smaller than four.*

Proof. Let G be a counter example and p, r, s, t primes dividing the order of G. Let H be a $\{r, s, t\}$-Hall group. Then H contains only elements of prime order. This is a contradiction to [2] Theorem 3. □

Lemma 2 ([4] 2.7 Lemma p. 424). *Suppose that* $q^m - 1 = r^v$, *where* q, r *are primes and* m, v *are positive integers. Then either*

a) $q = 2$ *and* $v = 1$ *or*

b) $r = 2$ *and* $m = 1$ *or*

c) $q^m = 9$ *and* $r^v = 8$.

Lemma 3 *Let M be a set, on which G operates transitively and N a normal subgroup of G. Then all orbits of N on M have the same length.*

Proof. Let A_i, A_j be two orbits of N on M and $a_i \in A_i$, $a_j \in A_j$. Then there is $g \in G$ with $a_i g = a_j$. We show $x \mapsto xg$ is a bijection from A_i on A_j. Obviously there is $n \in N$ with $x = a_i n$. Therefore we have $xg = a_i ng = a_i gg^{-1}ng = a_j n^g = a_j n_1$ for $n_1 \in N \lhd G$ and so $xg \in A_j$. Clearly $x \rightarrow xg^{-1}$ is the converse morphism. □

Lemma 4 *Let R be a Sylow r-subgroup of a solvable group G. If all elements of $R\backslash 1$ are conjugate in G then R is abelian.*

Proof. Let G be a minimal counterexample and M be a minimal normal subgroup of G. If M is r-group then $M = R$. This is a contradiction. Hence M is a r'-group. Now RM/M is a Sylow r-subgroup of G/M. Obviously G/M is a counterexample of smaller order. □

Lemma 5 *Let G be a solvable group with exactly two p-regular conjugacy classes in $O_{p'}(G)$ and $O_p(G) = 1$. Then the Sylow p-subgroup P is cyclic and the p-length of G is 1.*

Proof. $O_{p'}(G)$ is a minimal normal subgroup and therefore abelian. Moreover $G/O_{p'}(G)$ operates transitively on $O_{p'}(G)\backslash 1$. By Lemma 3 the orbits of $O_{p'}(G)\backslash 1$ under $O_p(G/O_{p'}(G)) = O_{p'p}(G)/O_{p'}(G)$ have the same length. This length is obviously different from 1. By [4] 3.4. Lemma p. 268 $O_{p'p}(G)/O_{p'}(G)$ is cyclic. Hence the p-length of G is 1, which is seen by the constrained property. □

Lemma 6 *Let G be a solvable group with exactly four p-regulare conjugacy classes and $O_p(G) = 1$. Then the p-length of G is 1.*

Proof. Let G be a counter example. By Lemma 5 $O_{p'}(G)$ contains three conjugacy classes. Let $q(\neq p)$ be a prime divisor of the order of G and Q a Sylow q-subgroup of G. Now we have two cases. □

1. Let $O_{p'}(G)$ be a q-group.

If 3 distinct primes p, q, r divides the order of G then there is exactly one conjugacy class with elements of order r. Let R be a Sylow r-subgroup and $1 \neq x \in R$. Lemma 4 implies that R is abelian. Then by the Lemma of Burnside all elements of order r in R are conjugated under $N = N_G(R)$. The number of elements is obviously $\mid N : C_N(x) \mid= p^v$ for a suitable natural number v. Moreover RQ is a Frobenius group. Therefore the order of R is r. Hence we have $p^v + 1 = r$, which contradicts Lemma 2.

Therefore the order of G is divisible by the primes p, q only. We set $\mid Q \mid = q^b$ and $\mid O_{p'}(G) \mid = q^d$. Obviously $O_{p'}(G)$ contains $Z(Q)$ and $q^d = 1 + p^s + q^i p^t$ for suitable s, i and t. Moreover all p' -elements outside $O_{p'}(G)$ are conjugated. Hence $G/O_{p'p}(G)$ contains exactly two p' -classes. Consequently $q^{b-d} - 1$ is a p-power. By Lemma 2 $q^{b-d} - 1 = p$ and $q = 2$, i.e. G is a $\{2, p\}$-group and the order of $O_2(G)$ is 2^d.

1.1. First let $O_2(G)$ be a non-minimal normal subgroup and $1 \neq N < O_2(G)$ a G-normal subgroup. Let the order of N be 2^c. Then all elements of $N \backslash 1$ are conjugated. Hence $1 + p^s = 2^c$ or $2^c = 2^i p^t + 1$. Obviously the second case is impossible. By Lemma 2 we have $p = 2^c - 1$ and $s = 1$. The equation $2^d = 1 + p + 2^i p^t$ forces $i = c$, $t = 1$ and $d = 2c$. By [3] VI.6.5 Lemma S. 690 $C_G(O_2(G)/\Phi(O_2(G))) \leq O_2(G)$ holds. Therefore a Sylow p-subgroup P has a faithful representation on $O_2(G)/\Phi(O_2(G))$ and is consequently isomorphic to a subgroup of $GL(n, 2)$ with $1 \leq n \leq 2c$. Clearly $\mid GL(n, 2) \mid = (2^n - 1)(2^n - 2)...(2^n - 2^{n-1})$ and $c|m$ if $p = (2^c - 1)|(2^m - 1)$. Hence the order of P is smaller than p^3 and the p-length of G is one.

1.2. Now let $O_2(G)$ be now a minimal normal subgroup of G. Then it is the unique minimal normal subgroup of G. Therefore any homomorphic image of G has at most 2-length one, but G has 2-length two. By [3] VI 6.9 Hilfssatz S. 693 $O_2(G)$ has a complement $M \cong G/O_2(G)$ in G. Obviously M contains exactly two p-regular conjugacy classes, namely the 1-class and a class with elements of order two. Hence all 2-elements outside $O_2(G)$ have order 2 and so a Sylow 2-subgroup Q of G has exponent two. Then Q is abelian and therefore the 2-length of G is one. Since $O_p(G) = 1$, this case is impossible.

2. Let the order of $O_{p'}(G)$ be divisible by exactly two distinct primes. Then we have a minimal normal subgroup N and a subgroup U, such that $UN = O_{p'}(G)$ and $U \bigcap N = 1$. By the Frattini argument $N_G(U)O_{p'}(G) = G$. Since $O_{p'}(G)$ is a Frobenius group, we have $N_{O_{p'}(G)}(U) = U$. Hence $N_G(U)/U \cong G/O_{p'}(G)$ and $N_G(U)$ operates transitively on $U \backslash 1$. By Lemma 3 all orbits of $U \backslash 1$ under $O_p(N_G(U)/U) \cong O_{p'p}(G)/O_{p'}(G)$ have the same length. By [4] 3.4 Lemma p. 268 $O_{p'p}(G)/O_{p'}(G)$ is cyclic. It follows $l_p(G) = 1$.

Remark. In a similar way we can show, that $l_p(G) = 1$ if G has at most four p-regular conjugacy classes.

Lemma 7 *Let p, q be distinct primes and a, c be natural numbers such that $a > 1$, $p \neq 2$ and $q^a - 1 = 4p^c$. Then $q = 5$ and a is a prime.*

Proof. Obviously q is odd. We consider the equation modulo 8. Now it is easy to see that a is odd. Let $a = i \cdot j$ with a prime j and $i \geq 1$. We have $q^a - 1 = (q^i - 1)(q^{i(j-1)} + ... + q^i + 1)$ with $q^i - 1 = 4p^k$ and $p^l = q^{i(j-1)} + ... + q^i + 1$ for certain natural numbers k, l. Then $p^l = (4p^k + 1)^{j-1} + ... + (4p^k + 1) + 1 = A(p^k)^2 + j(j-1)2p^k + j$ for a natural number A. It is easy to see that only $k = 0$, $q = 5$ and $i = 1$ is possible. □

Remark. Some arguments of this proof are taken from the proof of Lemma 4.3 in [6]. We conjecture that furthermore holds $c = 1$.

3. Proof of the Main Theorem

In the proof we consider several cases. Let p, q and, if necessary, r be the primes dividing the order of G, and P, Q, R the Sylow subgroups, respectively.

1. Let $O_{p'}(G)$ only contain two classes.

Let the order of $O_{p'}(G)$ be a power of q, say q^d. First we will show that all p'-elements have prime power order. Suppose there is an element of order qr in G. Then we have outside of $O_{p'}(G)$ only p'-elements of order qr and r. In particular then $O_{p'}(G) = Q$. By Lemma 6 $H = R \propto P$ is a q'-Hall group of G. Hence $N_G(R) = C_G(R)$. Since all elements of $R \backslash 1$ are conjugate in G, the order of R is two and $r = 2$ by Burnside. Let $R =< a >$. Since G has elements of order $q2$, there is $u \in O_{p'}(G)\backslash 1$, which commutes with a. Therefore $C_G(u) \geq RQ$ and further $\mid G : C_G(u) \mid$ is a p-power. Since all nontrivial elements of $O_{p'}(G)$ are conjugate, we get $p^t = q^d - 1$ for a suitable t. Now by Lemma 2 we have the contradiction $p = 2$ or $q = 2$.

By Lemma 5 P is cyclic. Set $\overline{G} = G/O_{p'}(G)$. Then $C_{\overline{G}}(\overline{P}) \leq \overline{P}$, and so $\overline{G}/O_p(\overline{G})$ is isomorphically contained in $Aut(P)$. Hence $\overline{G}/O_p(\overline{G})$ is cyclic, and then it is a r-group or a q-group.

1.1. Let $G/O_{p'p}(G)$ be a r-group.

Let H be a p'-Hall group. Then H is a Frobenius group and R is cyclic. Since $R \cong G/O_{p'p}(G)$ contains exactly three conjugacy classes, $\mid R \mid = 3$. On the other hand R operates fixed point freely on P and $Q \cong O_{p'}(G)$.

Therefore R operates fixed point freely on $O_{p'p}(G)$. By the Theorem of Thompson $O_{p'p}(G)$ is nilpotent. This is a contradiction.

1.2. Let $G/O_{p'p}(G)$ be a q-group.

Let Q be a Sylow q-subgroup and $x \in Z(Q)$. By the Lemma of Hall and Higman [3] 6.5 Lemma S. 690 $x \in O_{p'}(G)$. If q^d is the order of $O_{p'}(G)$, we have $q^d - 1 = p$ and $q = 2$ in view of Lemma 2. By Lemma 5 P is cyclic of order p. Outside $O_{p'}(G)$ there are exactly two p'-classes and at most three consequently in $T := G/PO_{p'}(G)$. Hence $T/Z(T)$ contain at most two classes. Therefore $| T/Z(T) | \leq 2$ and $| Z(T) | \leq 2$. Consequently $| T | = 2$ and G is an abnilpotent group with index system $(2^d, p, 2)$. Moreover $p = 2^d - 1$ so that p is a Mersenne prime and d a prime. By [8] 4.2 Theorem, $2 \mid d$. Hence $d = 2$ and $G \cong S_4$. Obviously S_4 satisfy our assumptions.

2. Let $O_{p'}(G)$ only contain three classes.

Obviously there are exactly two p-regulare conjugacy classes in $G/O_{p'p}(G)$. By Lemma 6 $G/O_{p'p}(G)$ has order 2.

2.1. Let $O_{p'}(G)$ be a q-group and the order of G divisible by three distinct primes.

Then $| R | = 2$ and R operates non-trivially on P. Now let $< a > = R$ and az an involution with $z \in P$. By [1] 45.1 $D = < a, az >$ is a dihedral group and also a Frobenius group. By the Lemma of Hall and Higman D operates faithfully on $O_{p'}(G)$. But then there is an involution of D, which centralizes an element $1 \neq v \in O_{p'}(G)$. Hence we have an element of order $2q$. This is a contradiction.

2.2. Let $O_{p'}(G)$ be a q-group and the order of G divisible by two distinct primes.

We set $| G : N_G(Q) | = p^i$ and $| O_{p'}(G) | = 2^d$. Then there are $p^i | Q | - (p^i - 1) | O_{p'}(G) |$ p-regular elements in G. Hence in the unique class outside $O_{p'}(G)$ there are exactly $p^i(| Q | - | O_{p'}(G) |) = p^i 2^d$ elements. On the other hand $C_G(x) \supseteq < x, Z(Q) >$ for $x \in Q \backslash O_{p'}(G)$.Because $Z(Q) \cap O_{p'}(G) \neq 1$, we have $| C_G(x) | \geq 4$.Therefore the 2-power dividing $| G : C_G(x) |$ is at most 2^{d-1}. This is a contradiction to $| G : C_G(x) | = p^i 2^d$.

2.3. Let the order of $O_{p'}(G)$ be divisible by two distinct primes.

Then $O_{p'}(G)$ is a Frobenius group. Let Q_1 be the kernel and its order q^a, and let R_1 be the complement and its order r. If $r = 2 = | G/O_{p'p}(G) |$, the Sylow p-subgroup P operates trivially on R_1. Hence G/PQ_1 contains at most three conjugacy classes, but its order is four. This is a contradiction. Therefore $q = 2 = | G/O_{p'p}(G) |$. Now Q_1 is the unique minimal normal subgroup of G. Consequently any homomorph image of G has 2-length one, but the 2-length of G is two. By [3] VI 6.9 Hilfssatz S.693 Q_1 has a complement in G. Hence any non-trivial 2-element has order two. Consequently the Sylow 2-subgroup of G is abelian. This is a contradiction to $l_2(G) = 2$.

3. Let G be p-nilpotent.

3.1. Let the order of a certain element of $O_{p'}(G)$ be divisible by r and q.

Let N be a minimal normal subgroup of G. Then N is a Sylow subgroup. It may be assumed that $N = Q$. Therefore $O_{p'}(G) = RQ$ for an elementary abelian Sylow r-subgroup R. If $O_r(G) \neq 1$ then $O_r(G) = R$ and $O_{p'}(G)$ is abelian. Hence, if $O_{p'}(G)$ is not abelian, R operate faithfully on Q and PR operates transitvely on $Q \backslash 1$. If R operates irreducibly on Q, by the Lemma of Schur R is cyclic and thus the stabalizer of any element of $Q \backslash 1$ in R is 1. Assume R operates reducibly on Q. By Lemma 3 all orbits of $Q \backslash 1$ have the same length. By [4] Theorem 3.1 b p. 266 the stabalizer of any element of $Q \backslash 1$ in R is 1. Hence, if $O_{p'}(G)$ is not abelian, $O_{p'}(G)$ is a Frobenius group in contradiction to the assumption 3.1. Therefore $O_{p'}(G)$ is abelian. Obviously Q and R are normal subgroups of G. Let q^c and r^d denote their orders, respectively. Moreover we choose non-trivial elements $x \in Q$ and $y \in R$. Then $\mid G : C_G(x) \mid= p^a$, $\mid G : C_G(y) \mid= p^b$ and therefore $q^c - 1 = p^a$, $r^d - 1 = p^b$. This is a contradiction to $p \neq 2$.

3.2. Let the order of all elements of $O_{p'}(G)$ be a prime power and $O_{p'}(G)$ not a q-group.

By [2] $O_{p'}(G)$ is a Frobenius group or a 3-step group. In the second case we have a principal series $O_{p'}(G) > N_1 > N_2 > 1$, where $O_{p'}(G)/N_1$ and N_2 are 2-groups and N_1/N_2 is a q-group. Obviously N_2 is the unique minimal normal subgroup of G and $l_2(G) = 2$. Therefore any homomorphic image of G has at most 2-length one. By [3] VI 6.9 Hilfssatz S.693 N_2 has a complement in G. Hence all elements of $R \backslash 1$ are of order 2. Consequently R is abelian contradictly, $l_2(G) = 2$.

Now let $O_{p'}(G) = QR$ be a Frobenius group with complement Q and kernel R. Then Q is cyclic or a quaternion group. In the second case the order of Q is 8. Moreover G/R contains exactly three p-regular conjugacy classes. Let $xR \in G/R$ be an element of order four. Then $\mid G/R : C_{G/R}(xR) \mid= 2p^b = 6$ and hence $p = 3$. Obviously the representation σ of $P \propto Q \cong G/R$ on R is irreducible. Its degree is a, if $\mid R \mid= r^a$. By Clifford $\sigma \mid_Q$ decomposes into irreducible parts of the same degree. The faithful irreducible representation of a quaternion group has degree two or four (see [9] Hilfssatz 11). Therefore the degree a is even, say $a = 2c$. Since $P \propto Q$ operates transitively on $R\backslash 1$, $3^d 8 = r^a - 1 = (r^c - 1)(r^c + 1)$. It is easy to see that this equation has the unique solution $r = 5, c = 1, d = 1$. Since P operates faithfully on R and $GL(2,5)$ has the order 480, the order of P is 3. This is case B4) of the Main Theorem.

Now let Q be cyclic. Then G/R contains two or three p-regulare conjugacy classes. In the first case because of $p \neq 2$, $\mid Q \mid = 2$ and so Q inverts all elements of R. Hence R is abelian and PQ has two orbits in $R\backslash 1$. This implies $r^n - 1 = 2p^a + 2p^b$ for certain positive integers n, a, b. Obviously R is a minimal normal subgroup. Assume first $a \neq b$. Then P is not cyclic and irreducible on R. By [4] VIII 3.3 Lemma p. 268 we have a direct product $R = R_1 \otimes R_2 \otimes ... \otimes R_p$, where P permutes the R_i's transitivelly. Now the elements r_1, r_1^{-1}, $r_1 r_2$, $(r_1 r_2)^{-1}$, $r_1 r_2 r_3$, $(r_1 r_2 r_3)^{-1}$, $...r_i \in R_i$ belong to distinct conjugacy classes. This is a contradiction.

Now let $a = b$. By [4] 3.4 Lemma p. 268 P is cyclic of order p^a. These are case B5) and B1d) of the Main Theorem.

If G/R contains exactly three p-regulare conjugacy classes, we have $q - 1 = 2p^b$ for a natural number b. Then R is an elementary abelian 2-group. Moreover PQ permutes the set of non-identity elements of R transitively. Let F be the Fitting subgroup of PQ. Then each Sylow subgroup of F is normal in PQ and permutes the set of non-identity elements of R in orbits of equal length by Lemma 3. By [4] 3.4 Lemma p. 268 the Sylow subgroups and hence F are cyclic. Let $p^d q$ be the order of F and $\mid R \mid -1 = 2^n - 1 = p^c q$. In view of the proof of 3.5 Theorem p. 269 in [4] we have $2^n - 1 \mid p^d q n$. Therefore $n = p^{c-d}$ and if $c - d \geq 1$, $2^n - 1 = (2^p - 1)t = p^c q$. By Fermat's Theorem $q = 2^p - 1 = 2p^b + 1$ and thus $p^b = 2^{p-1} - 1$. By Lemma 2 we have $b = 1$ and therefore $p = 3$ and $q = 7$. Hence $3^c 7 = 2^n - 1$ and n is even. This is a contradiction. Consequently $c = d$. Then $F = PQ$ since PQ operate faithfully. Hence Q is a group of order three. Therefore n is even and as a consequence $P \cong Z_5$ or $P \cong 1$. These are case B6) and B1c) of the Main Theorem.

3.3.1. Let $O_{p'}(G)$ be a non-abelian q-group Q.

Either $Z(Q)$ or $Q/Z(Q)$ contains exactly two conjugacy classes. Therefore $q^d = 1 + p^e$ and $q = 2, e = 0$ or $e = 1$ by Lemma 2. Hence $Z(Q)$ contain exactly two conjugacy classes and $\mid Z(Q) \mid = 1 + p$ or 2, with a Mersenne prime p. If $Q_1 := Q/Z(Q)$ has three classes Q_1 is non-abelian because of $2^i \neq 1 + p^a + p^b$. Therefore $\mid Q_1 \mid = \mid Z(Q_1) \mid + 2^c p^d$ and $\mid Z(Q_1) \mid = 1 + p$ or 2. It is easy to see that also $\mid Q_1/Z(Q_1) \mid = 1 + p$ or 2. Let $UrZ(Q_1) := preimage(Z(Q_1))$ in G. We consider two cases.

a) Let $\mid Z(Q_1) \mid = 2$.

Hence $\mid Q_1/Z(Q_1) \mid = 1+p$, because Q_1 is not abelian. Moreover Q_1 has elements of order 4. Since Q_1 contains exactly three classes, all elements of $Q_1 \backslash Z(Q_1)$ are conjugated. Therefore Q_1 has exactly one involution. Hence Q_1 is the quaternion group of order 8 and $p = 3$. Moreover we have outside $UrZ(Q_1)$ exactly $\mid Q \mid - \mid UrZ(Q_1) \mid = 2p \mid Z(Q) \mid$ elements. Therefore $\mid C_Q(x) \mid = 1 + p = 4$ for $x \in Q \backslash UrZ(Q_1)$. On the other hand $Z(Q) \cap < x > = 1$ and x is of order 4. This is a contradiction.

b) Let $\mid Z(Q_1) \mid = 1 + p$.

If $\mid Q_1 : Z(Q_1) \mid = 2$, then we have outside $UrZ(Q_1)$ exactly $(1 + p) \mid Z(Q) \mid$ conjugate elements. On the other hand $\mid C_Q(x) \mid \geq 4$ for $x \in Q \backslash UrZ(Q_1)$. This is a contradiction. Therefore $\mid Q_1 : Z(Q_1) \mid = 1 + p$. Now we have in $G/Z(Q)$ outside $Z(Q_1)$ exactly $(1+p)p$ conjugate elements. But $\mid C_{Q_1}(x) \mid > 1 + p$ for $x \in Q_1 \backslash Z(Q_1)$ is a contradiction.

Therefore Q_1 contains exactly two classes and is elementary abelian. Hence $\Phi(Q) \leq Z(Q)$ and since $Z(Q)$ is minimal, $\Phi(Q) = Z(Q)$. Therefore $Z(Q) \backslash 1$ is the set of involutions of Q, which are all conjugate in G. By [3] III 3.19 Satz S.275 we see that $\mid A(Q) \mid$ divides $2^{2n}(2^n - 1)(2^n - 2)...(2^n - 2^{n-1})$ and the Sylow p-subgroup has order p. Let $\mid Q \mid = 2^{2n} = 1+p+2^{n-1}p+2^{n-1}p$ be the equation of the partition of Q into G-classes. Therefore Q is a Suzuki 2-group of type $A(n, \theta)$ (see: [4] VIII 7). In view of this equation and the centralizer of an element of order 4 in $A(n, \theta)$ it is easy to see that θ acts fixed point freely. Moreover it is clear that conversely groups of the type B3) of the Main Theorem have exactly four p-regulare classes.

3.3.2 Let $O_{p'}(G)$ be a abelian q-group Q.

Since $\mid Q \mid = q^n = 1+p^a+p^b+p^c$ for suitable a, b and c, it follows $q = 2$. If $\Phi(Q) > 1$ then $\Phi(Q)$ and $Q/\Phi(Q)$ have order 2 or $1 + p$ and contain exactly two conjugacy classes. By [3] III 3.19 Satz S.275 $\mid Q/\Phi(Q) \mid = 2$ and $P = 1$ or $\mid Q/\Phi(Q) \mid = 1 + p$ and $\mid P \mid = p$. If $P = 1$ we have case B1a). Now let$P$ be cyclic of order p. Hence the numbers a, b, c are 0 or 1. It is easy to check that $a = 0$, $b = c = 1$ and $\mid Q \mid = 2(1 + p)$. Because Q has more then one involution, we have exactly one conjugacy class of elements of order 4. On the other hand a cyclic group of order 4 has two elements of order 4. Therefore there is a even number of elements of order 4 in Q.

This is a contradiction. Hence $\Phi(Q) = 1$ and Q is elementary abelian. If Q is irreducible and P is not cyclic, by [4] VIII 3.3 Lemma p. 268 we have a direct product $Q = Q_1 \otimes Q_2 \otimes ... \otimes Q_p$, where P permutes the Q_i's. Now the elements q_1, q_1q_2, $q_1q_2q_3$, $...q_i \in Q_i$ belong to distinct conjugacy classes. Hence $p = 3$ and the number of elements in the class of q_1 is $3(\mid Q_1 \mid -1) = 3^a$. Therefore $\mid Q_1 \mid = 4$ and the order of Q is 64. According to the partition of Q into G-classes we have the equation $64 = 1+9+27+27$. Moreover P is a subgroup of $GL(6,2)$. The only 3-subgroups of $GL(6,2)$ with this property are the Sylow 3-subgroups. This is case B2.d) If Q is irreducible and P is cyclic, the stabilizer of any non-identity element of Q is 1. Hence $2^n - 1 = 3p^a$ and therefore n is even, $p^a = 5$, Q has order 16. This is case B2.a) of the Main Theorem.

If Q is reducible, one easily checks that the cases B1.b)-B1.d) occur.

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