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Besov spaces on symmetric manifolds

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Abstract. We investigate the spaces of Besov type on symmetric manifolds of the noncompact type. The paper is focused on finding the equivalent norms via the means typical for harmonic analysis on these manifolds.

Key words: Besov spaces, Helgason transform, symmetric spaces.

The whole scale of Besov spaces $B^s_{p,q}(X)$ on a Riemannian manifold with bounded geometry was defined by H. Triebel in 1986. The definition is of local nature. But in the case of symmetric manifolds of the non-compact type one can use the Fourier analysis similar as in the Euclidean case. In the paper we investigate the connection between the Besov spaces and the Helgason-Fourier transform on symmetric manifolds of the non-compact type for $p=2$. We focus on the problem of equivalent norms.

1. Preliminaries

1.1. Symmetric manifolds of the non-compact type

Let $X=G/K$ be a Riemannian symmetric manifolds of the noncompact type, i.e. G is a connected semi-simple Lie group with finite center and K is a maximal compact subgroup of G. We list briefly the customary notation associated with X and refer for example to [10] for more explicit definitions. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote Lie algebras of G and K respectively. Their complexifications will be denoted by the subscript $\mathbf{C}$. Let $\mathfrak{p}$ be an orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form $<,>$. Then

$$\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}\quad\text{(the Cartan decomposition).}\tag{1}$$

We assume that the Riemannian metric on X is generated by $<,>$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^\star$ its dual. An element $\lambda\in\mathfrak{a}^\star$ is called a restricted root of $\mathfrak{g}$ if $\lambda\neq 0$ and the corresponding root space $\mathfrak{g}_\lambda=\{X\in\mathfrak{g}:[H,X]=\lambda(H)X,\ \text{for all } H\in\mathfrak{a}\}$ is not trivial. The

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number $m_\lambda = \dim\mathfrak{g}_\lambda$ is called the multiplicity of λ. Let Σ_+ denote the set of positive roots on a fixed open Weyl chamber $\mathfrak{a}_+$. The direct sum $\mathfrak{n}$ of the corresponding roots subspaces is a nilpotent subalgebra of $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \quad \text{(the Iwasawa decomposition).} \tag{2}$$

The corresponding decompositions of the group G look as follows:

$$G = K\bar{A_+}K\ , \quad \text{where } A_+ = \exp \mathfrak{a}_+, \tag{3}$$

$$G = KAN, \quad \text{where } A = \exp a,\ N = \exp \mathfrak{n}. \tag{4}$$

If $g \in G$ then we will write its Cartan and Iwasawa decomposition in the following way:

$$g = k_1 \exp A(g) k_2,\ \ g = k(g) \exp H(g) n(g), \tag{5}$$

where $k(g) \in K$, $n(g) \in N$, $H(g) \in \mathfrak{a}$, and $A(g) \in \mathfrak{a}_+$ are uniquely determined.

Let M (resp. M') denote the centralizer (resp. the normalizer) of A in K. The factor group $W = M'/M$ is called the Weyl group of X. It is finite and acts on $\mathfrak{a}$ as a group of linear transformations by the operators $Ad_G(k)$, $k \in M'$. The group W acts also on the set of Weyl chambers and this action is free and transitive. We define an action of W on $\mathfrak{a}$ by $s\lambda(H) = \lambda(s^{-1}H)$, $s \in W$ and $H \in \mathfrak{a}$. The homogeneous manifold $B = K/M = G/MAN$ is called a boundary of X. We will denote the action of G on X by $g \cdot x$ and on B by $g(b)$. The point $o = eK$ is called the origin of X.

The Killing form induces the Euclidean measures on A, $\mathfrak{a}$ and $\mathfrak{a}^\star$. Multiplying these measures by $(2\pi)^{-l/2}$, ($l = \dim\mathfrak{a}$), we obtain invariant measures da, dH, $d\lambda$. The Haar measures dm on M and dk on K are normalized such that the total measure is 1. The Haar measures on G and N can be normalized in such a way that

$$\int_G f(g)dg = \int_{K\times A\times N} f(kan)e^{2\rho(\log a)}dkdadn, \tag{6}$$

$$\int_G f(g)dg = \int_{G/K}\left(\int_K f(gk)dk\right)dgK, \tag{7}$$

where $\rho = \frac{1}{2}\sum_{\lambda\in\Sigma_+} m_\lambda\lambda$ and log denotes the inverse of the map to $\exp : \mathfrak{a} \mapsto A$. Moreover the invariant measure $db = dkM$ on $B = K/M$ is normalized by $dkM(B) = 1$, cf. [11, I§5].

1.2. Helgason-Fourier transform and tempered distributions on X

Let $\pi : G \mapsto G/K$ be the natural projection and $o = \pi(K)$. Let δ_K denote the distribution $f \mapsto \int_K f(k)dk$ on G. For any $f \in C_o^\infty(G)$ we define the function $f^\natural \in C_o^\infty(G/K)$ by $f^\natural \circ \pi = f \star \delta_K$, where $\star$ denotes the convolution on G. Now, for any distribution $T \in D'(G/K)$ we can define a distribution $\tilde{T} \in D'(G)$ by $\tilde{T}(f) = T(f^\natural)$. If T is a function, then $\tilde{T} = T \circ \pi$. If T_1 and T_2 are distributions on G/K, one of compact support, then their "convolution" $T_1 \star T_2$ is the distribution defined by

$$(T_1 \star T_2)(f^\natural) = (\tilde{T_1} \star \tilde{T_2})(f), \tag{8}$$

cf. [11, II§5]. We have $(T_1 \star T_2)\tilde{} = (\tilde{T_1} \star \tilde{T_2})$, so the operation $\star$ satisfies the associative law and $T \star \delta_o = T$ where δ_o denotes the Dirac distribution at o.

We put $A(gK, kM) = -H(g^{-1}k)$ for $gK \in X$ and $kM \in B$. Then

$$A(g \cdot x, g(b)) = A(x, b) + A(g \cdot o, g(b)), \quad x \in X, \quad b \in B. \tag{9}$$

The functions $X \ni x \mapsto e^{\mu A(x,b)}$, $\mu \in \mathfrak{a}^\star_{\mathbf{C}}$, $b \in B$, are eigenfunctions of each invariant differential operator on X i.e. the differential operators invariant with respect to the action of G. In fact

$$D(e^{(i\lambda+\rho)A(x,b)}) = \gamma(D)(i\lambda) \cdot e^{(i\lambda+\rho)A(x,b)}, \quad \lambda \in \mathfrak{a}^\star, \tag{10}$$

and $D \mapsto \gamma(D)$ is an isomorphism of the algebra $D(X)$ of invariant operators onto the algebra $S(\mathfrak{a}^\star)_W$ of W-invariant polynomials with complex coefficients on $\mathfrak{a}^\star$. The algebra $S(\mathfrak{a}^\star)_W$ is generated by l algebraically independent homogeneous elements $p_1, \ldots, p_l$ and 1, $l = \dim \mathfrak{a}$. If d_j is the degree of p_j, $j = 1, \ldots, l$, then $\prod_{j=1}^{l} d_j = |W|$ (cf. [11, II.§4-5 and III.§3]).

If $f \in C_o(X)$ is a continuous function with compact support then its Helgason-Fourier transform $\mathcal{H}f$ is defined by

$$(\mathcal{H}f)(\lambda, b) = \int_X f(x) e^{(-i\lambda+\rho)A(x,b)} dx\,, \tag{11}$$

for all $(\lambda, b) \in \mathfrak{a}^\star \times B$ for which this integral converges absolutely. The Helgason-Fourier transform $\mathcal{H}f$ satisfies the identities

$$\int_B e^{(is\lambda+\rho)A(x,b)} \mathcal{H}f(s\lambda, b)db = \int_B e^{(i\lambda+\rho)A(x,b)} \mathcal{H}f(\lambda, b)db, \tag{12}$$

$s \in W$, and is inverted by

$$f(x) = |W|^{-1} \int_{\mathfrak{a}^\star \times B} \mathcal{H}f(\lambda, b) e^{(i\lambda+\rho)A(x,b)} |c(\lambda)|^{-2} d\lambda db, \tag{13}$$

where $c(\lambda)$ is Harish-Chandra's c-function and $|W|$ is the order of W. The map $f \mapsto \mathcal{H}f$ extends to an isometry of $L_2(X, dx)$ onto $L_2(\mathfrak{a}_+^\star \times B, |c(\lambda)|^{-2} d\lambda db)$ (the Plancherel theorem; cf. [13]).

Using the integration formulae (6) and (7) it is not hard to see that

$$\mathcal{H}f(\lambda, kM) = \mathcal{F}(\mathcal{R}(\cdot, kM))(\lambda), \tag{14}$$

where $\mathcal{R}$ is the Radon transform

$$\mathcal{R}f(H, kM) = e^{\rho(H)} \int_N f(k(\exp H)nK) dn \quad (H \in \mathfrak{a}, k \in K), \tag{15}$$

and $\mathcal{F}$ the classical Fourier transform on $\mathfrak{a}$

$$\mathcal{F}h(\lambda) = \int_{\mathfrak{a}} e^{-i\lambda(H)} h(H) dH \ (\lambda \in \mathfrak{a}^\star,\ H \in \mathfrak{a}). \tag{16}$$

The Radon transform maps $C_o^\infty(X)$ injectively into $C_o(\mathfrak{a} \times K/M)$. If the function f is K-invariant then its Helgason-Fourier transform coincides with the spherical Fourier transform and the Radon transform coincides with the Abel transform i.e.

$$\begin{aligned} (\mathcal{H}f)(\lambda) &= \int_X f(x) \varphi(-\lambda : x) dx, \\ A(f)(H) &= e^{\rho(H)} \int_N f((\exp H)nK) dn, \end{aligned} \tag{17}$$

where

$$\varphi(\lambda : x) = \int_K e^{(-i\lambda+\rho)A(x,kM)} dk, \quad \lambda \in \mathfrak{a}^\star, \quad H \in \mathfrak{a}. \tag{18}$$

Now we recall the definition of rapidly decreasing functions and tempered distributions on X, which is due to Harish-Chandra. We put $\sigma(g) = \|Y\|$ and $\Xi(g) = \int_K e^{-\rho H(gk)} dk$, $g \in G$, $g = k \cdot \exp Y$, $Y \in \mathfrak{p}$. The Schwartz space $\mathcal{C}(X)$ on X consists of C^∞ functions on X such that for any $m \in \mathbf{N}$ and any left(right)-invariant differential operator D (D') on G

$$\tau_{D,D',m}(f) = \sup |((D')Df)(g)| \Xi(g)^{-1} (1 + \sigma(g))^m < \infty. \tag{19}$$

The semi-norms $\tau_{D,D',m}$ convert $\mathcal{C}(X)$ into a Frechet space (cf. [6]). It is well known that $C_o^\infty(X) \subset \mathcal{C}(X) \subset L_p(X)$, $p \geq 2$ (topological embeddings) and $C_o^\infty(X)$ is dense in $\mathcal{C}(X)$. A distribution on X is said to be tempered if it can be extended to a continuous functional on $\mathcal{C}(X)$. Since $C_o^\infty(X)$ is continuously included and dense in $\mathcal{C}(X)$, the space of tempered distribution $\mathcal{C}'(X)$ can be regarded as the dual space to $\mathcal{C}(X)$. It should be clear that every distribution with compact support is tempered and that $L_2(X)$ is continuously included in $\mathcal{C}'(X)$.

The Helgason–Fourier transform maps $\mathcal{C}(X)$ onto a space $\mathcal{Z}(\mathfrak{a}^\star \times B)$ of all C^∞-functions ψ on $\mathfrak{a}^\star \times B$ such that:

i) ψ satisfies the condition (12),

ii) for each differential operator D with constant coefficients on $\mathfrak{a}^\star$, each invariant differential operator D' on B and each $m \in \mathbf{N}$

$$\eta_{m,D,D'}(\psi) = \sup_{(\lambda,kM)\in\mathfrak{a}^\star\times B} |(D(D'\psi))(\lambda, kM)|(1+\|\lambda\|)^m < \infty.$$

The space $\mathcal{Z}(\mathfrak{a}^\star \times B)$ equipped with the semi-norms $\eta_{m,D,D'}$ is a Frechet space. The transform $\mathcal{H}$ is a topological isomorphism of $\mathcal{C}(X)$ onto $\mathcal{Z}(\mathfrak{a}^\star \times B)$. If $f \in \mathcal{C}'(X)$ is a tempered distribution on X then we define its Helgason-Fourier transform by

$$(\mathcal{H}f)(\psi) = f(\mathcal{H}_o\psi), \tag{20}$$

where $\psi \in \mathcal{Z}(\mathfrak{a}^\star \times B)$, and $\mathcal{H}_o\psi(\lambda, kM) = H\psi(-\lambda, kM)$. The transform $\mathcal{H}$ is a topological isomorphism of $\mathcal{C}'(X)$ onto the space $\mathcal{Z}'(\mathfrak{a}^\star \times B)$ dual to $\mathcal{Z}(\mathfrak{a} \times B)$ (both equipped with their weak topologies), cf. [6]. It follows from (13) and the elementary estimates for $|c(\lambda)|$ that if f is a suitable function on X or a distribution with compact support then both definition of Helgason–Fourier transform coincides provided that we identify a function $h(\lambda, b)$ on $\mathfrak{a}^\star \times B$ with the following element of $\mathcal{Z}'(\mathfrak{a}^\star \times B)$:

$$\psi \mapsto |W|^{-1} \int_{\mathfrak{a}\times B} h(\lambda, b)\psi(\lambda, b)|c(\lambda)|^{-2} d\lambda db, \quad \psi \in \mathcal{Z}(\mathfrak{a}^\star \times B)$$

It is known that if $h \in C_o^\infty(X)$ is K-invariant and $f \in \mathcal{C}(X)$ then $f \star h \in \mathcal{C}(X)$ and $\mathcal{H}(f \star h) = \mathcal{H}f \cdot \mathcal{H}h$ (cf. [9]).

2. The spaces $B_q^s(X)$

First we describe resolutions of unity on $\mathfrak{a}^\star$ suitable to our purpose. Let $B(0, r)$ denote the ball in $\mathfrak{a}^\star$ with respect to the metric induce by the Killing form centered at 0 and with radius r.

Definition 1 Let Φ be the collection of all systems $\{\phi_j\} \subset \mathcal{Z}(\mathfrak{a}^\star \times B)$ with the following properties:

(i) $\phi_j(\lambda, b_1) = \phi_j(\lambda, b_2)$, for every $b_1, b_2 \in B$, $\lambda \in \mathfrak{a}$, $j = 0, 1, 2, \ldots$,

(ii) $\phi_j(s \cdot \lambda, b) = \phi_j(\lambda, b)$, for every $b \in B$, $\lambda \in \mathfrak{a}$, $s \in W$, $j = 0, 1, 2, \ldots$,

(iii) $\phi_j(\lambda, b) = \phi(2^{-j}\lambda, b)$, if $j = 1, 2, \ldots$, $\phi \in C_o^\infty(\mathfrak{a}^\star \times B)$,

(iv) $\operatorname{supp}\phi_0 \subset B(0,2) \times B$, $\operatorname{supp}\phi \subset \overline{(B(0,2) \setminus B(0,1/2))} \times B$,

(v) $\sum_{j=0}^\infty \phi_j(\lambda, b) = 1$ for every $(\lambda, b) \in \mathfrak{a} \times B$.

Remark 1. The family Φ is of course not empty. It is sufficient to take the smooth function on $\mathfrak{a}^\star$ radial with respect to the metric induced by the Killing form and supported in $\overline{(B(0,2) \setminus B(0,1/2))}$.

For the given system of functions and for every multi-index α there exist a positive number c_α such that

$$2^{|\alpha|}|D^\alpha \phi_j(\lambda)| \leq c_\alpha \quad \text{for all } \; j = 0, 1, 2\cdots, \text{ and all } \lambda \in \mathfrak{a}^\star.$$

Definition 2 Let $\{\phi_j\}_{j=0}^\infty \in \Phi$. Let $-\infty < s < \infty$ and $0 < q \leq \infty$, then

$$\begin{aligned} B_q^s(X) &= \{f \in \mathcal{C}'(X) : \|f \mid B_q^s\| \\ &= \left(\sum_{j=0}^\infty 2^{sjq} \|\mathcal{H}^{-1}\phi_j \mathcal{H} f \mid L_2(X)\|^q\right)^{1/q} < \infty\} \end{aligned}$$

(usual modification if $q = \infty$).

Remark 2. The above approach is not useful if we take the L_p-norm with $p \neq 2$ in the above definition. If $p < 2$ then $\mathcal{H}^{-1}\phi_j\mathcal{H}f \notin L_p(X)$ for any $f \in \mathcal{C}'(X)$ and every Fourier L_p-multiplier, $p \neq 2$, is an analytical function in certain tube neighborhood of $\mathfrak{a}^\star$ in $\mathfrak{a}^\star_{\mathbf{C}}$ (cf. [2]).

Lemma 1 *Let* $-\infty < s < \infty$ *and* $0 < q \leq \infty$.

(i) *The definition of* $B_q^s(X)$ *is independent of the chosen system* $\{\phi_j\} \in \Phi$.

(ii) *The operator* $I_{\sigma,t}(f) = \mathcal{H}^{-1}(t + |\rho|^2 + |\lambda|^2)^{\sigma/2}\mathcal{H}f$, $\sigma \in \mathbf{R}$, $t > -|\rho|^2$ *is a topological isomorphism of* $B_q^s(X)$ *onto* $B_q^{s-\sigma}(X)$.

Proof. The proof is standard therefore it is sketched only. Let $\{\phi_j\}$ and $\{\psi_j\}$ be the systems belonging to Φ. Then by the Plancherel formula

$$\|f \mid B_q^s(X)\|_\phi$$

$$= \Big(\sum_{j=0}^{\infty} 2^{sjq} \| \sum_{k=-1}^{1} \psi_{j+k}\phi_j \mathcal{H} f \mid L_2(\mathfrak{a}_+^\star \times B, |c(\lambda)|^{-2} d\lambda db) \|^q \Big)^{1/q}$$
$$\leq C \| f \mid B_q^s(X) \|_\psi .$$

Thus the point (i) is proved. The second point follows similarly from the estimate

$$C_1 2^{j\sigma} \leq (t + |\rho|^2 + |\lambda|^2)^{\sigma/2} \leq C_2 2^{j\sigma}, \quad \text{if } \lambda \in B(0, 2^{j+1}) \setminus B(0, 2^{j-1}).$$

□

Proposition 1 *Let* $-\infty < s < \infty$, $0 < q, q_1, q_2 \leq \infty$. *Let* $H^s(X) = \{ f \in D'(X) : \| f \mid H^s(X) \| = \| (I - \Delta)^{s/2} f \mid L_2(X) \| < \infty \}$ *be the Bessel potential space corresponding to the Laplace-Beltrami operator* Δ *on* X. *Then*

(i) $B_2^s(X) = H^s(X)$,

(ii) (*the real interpolation*) $(B_{q1}^{s_1}(X), B_{q2}^{s_2}(X))_{\theta,q} = B_q^s(X)$, $0 < \theta < 1$, $s = \theta s_1 + (1 - \theta) s_2$,

(iii) $\mathcal{C}(X) \subset B_q^s(X) \subset \mathcal{C}'(X)$ (*topological embeddings*). *If* $q < \infty$ *then* $\mathcal{C}(X)$ *is dense in* $B_q^s(X)$.

Proof. The point (i) follows for $s = 0$ directly from the Plancherel theorem. For $s \neq 0$ we can use Lemma 1. The above interpolation property can be prove in the similar way to the Euclidean case therefore the calculations are omitted (cf. [19, Th.2.4.2.]).

It is sufficient to prove the both embeddings with $q = \infty$ because $B_q^s(X) \subset B_\infty^s(X)$ and $B_\infty^s(X) \subset B_q^{s+t}(X)$, $t > 0$. There exist the positive integers m and M such that $|c(\lambda)|^{-2} < M(1 + \|\lambda\|)^m$ for all $\lambda \in \mathfrak{a}^\star$ (cf. [21, Prop. 9.1.7.3]). Moreover, there is the constant m_1 such that $2^{js}\phi_j(\lambda) \leq m_1(1 + \|\lambda\|)^{2js}$. Thus, for a sufficiently large integer k

$$\| f \mid B_\infty^s(X) \| \leq C \| (1 + \|\lambda\|)^k \mathcal{H} f \mid L_\infty(\mathfrak{a}_+^\star \times B) \|$$
$$\leq C \tau_{D,D',l}(f),$$

since $\mathcal{H}^{-1}$ is an isomorphism of $\mathcal{Z}(\mathfrak{a}^\star \times B)$ onto $\mathcal{C}(X)$. The space $C_o^\infty(X)$ is known to be dense in $H^s(X)$ therefore $\mathcal{C}(X)$ is dense in every $B_q^s(X)$, $q < \infty$, cf. [18]. It is also known that $(H^s(X))' = H^{-s}(X)$, (cf. [14]). By the duality theorem for the real method of interpolation $(B_1^s(X))' = B_\infty^{-s}(X)$, $s \in \mathbf{R}$. But $\mathcal{C}(X)$ is contained and dense in $B_1^s(X)$. Hence $B_\infty^s(X) \subset \mathcal{C}'(X)$ for every $s \in \mathbf{R}$. □

Remark 3. It follows from the above proposition that our spaces $B^s_q(X)$ coincide with the spaces $B^s_{2,q}(X)$ defined by Triebel in [18].

The following theorem will take the crucial part in the next section. It is a simpler version of the theorem proved in the Euclidean case for full scale $B^s_{p,q}(\mathbf{R}^n)$ by H. Triebel, cf. [15].

Theorem 1 *Let* $0 < q \leq \infty$ *and* $-\infty < s < \infty$. *Let* $s_0, s_1 \in \mathbf{R}$, $s_o < s < s_1$ *and* $s_1 > 0$. *Let* $\phi_0 \in C^\infty(\mathfrak{a}^\star)$, $\phi \in C^\infty(\mathfrak{a}^\star \setminus \{0\})$ *(both W-invariant) satisfy the following conditions*

$$|\phi_0(\lambda)| > 0 \quad \text{if } \lambda \in B(0,2),$$

$$|\phi(\lambda)| > 0 \quad \text{if } \lambda \in \overline{B(0,4) \setminus B\left(0, \frac{1}{4}\right)}, \tag{21}$$

$$\sup_{\lambda \in \mathfrak{a}^\star \setminus B(0,1/2)} \frac{|\phi_0(\lambda)|}{|\lambda|^{s_0}} < \infty \,, \qquad \sup_{\lambda \in \mathfrak{a}^\star \setminus B(0,1/2)} \frac{|\phi(\lambda)|}{|\lambda|^{s_0}} < \infty, \tag{22}$$

$$\sup_{\lambda \in B(0,4)} \frac{|\phi(\lambda)|}{|\lambda|^{s_1}} < \infty \,. \tag{23}$$

Then the expressions

$$\|\mathcal{H}^{-1}\phi_0\mathcal{H}f \mid L_2(X)\| + \left(\int_0^1 t^{-sq} \|\mathcal{H}^{-1}\phi(t\cdot)\mathcal{H}f \mid L_2(X)\|^q \frac{dt}{t}\right)^{1/q} \tag{24}$$

and

$$\left(\sum_{j=0}^{\infty} 2^{sjq} \|\mathcal{H}^{-1}\phi_j\mathcal{H}f \mid L_2(X)\|^q\right)^{1/q},$$

$$\text{where } \phi_j(\lambda) = \phi(2^{-j}\lambda), \tag{25}$$

(modification if $q = \infty$*) are equivalent quasi-norms in* $B^s_q(X)$.

Proof. The proof goes in the same way as Triebel's proof. Since we work with L_2-norms we can use the Plancherel theorem instead of the inequalities of Plancherel-Poly'a-Nikol'skij type which were used in Euclidean setting. We recall that the Plancherel measure is defined by $|c(\lambda)|^{-2}d\lambda db$ and that

Fourier-Helgason image of a distribution is invariant with respect to Weyl group, cf. Section 1.2.. So, we sketch the main steps only and refer to [15] for details. The norm of the space $L_2(\mathfrak{a}_+^\star \times B, |c(\lambda)|^{-2} d\lambda db)$ is denoted by $\|\cdot \mid L_2(\mathfrak{a}_+^\star \times B)\|$. This makes formulae shorter.

First we prove (25). Let $\{\psi_m\}_{m=0}^\infty \in \Phi$, and $\psi_m(\lambda) = \psi(2^{-m}\lambda)$, $m = 1, 2, \ldots$ It will be convenient to put $\psi_m = 0$ for $m = -1, -2, \ldots$. Then

$$\begin{aligned} 2^{sj} \|\mathcal{H}^{-1}\phi_j \mathcal{H} f \mid L_2(X)\| & \\ \leq 2^{sj} \| \sum_{l=-\infty}^{1} \psi_{j+l}\phi_j \mathcal{H} f \mid L_2(\mathfrak{a}_+^\star \times B)\| & \\ + 2^{sj} \| \sum_{l=2}^{\infty} \psi_{j+l}\phi_j \mathcal{H} f \mid L_2(\mathfrak{a}_+^\star \times B)\|. & \qquad (26) \end{aligned}$$

Estimating the sum consisted of the first terms in (26) we get

$$\begin{aligned} \Big(\sum_{j=0}^{\infty} 2^{sj} \| \sum_{l=-\infty}^{1} \psi_{j+l}\phi_j \mathcal{H} f \mid L_2(\mathfrak{a}_+^\star \times B)\|^q \Big)^{1/q} & \\ \leq C \|f \mid B_q^s(X)\|. & \qquad (27) \end{aligned}$$

In the similar way we can estimate the sum consisted of the second terms in (26). We get the inequality similar to (27) with $\sum_{l=2}^{\infty}$ instead of $\sum_{l=-\infty}^{1}$. Thus the expression (25) can be estimate from above by $C\|f \mid B_q^s(X)\|$. The opposite inequality is clear since $\operatorname{supp} \psi_j \subset \{\lambda : \phi_j(\lambda) \neq 0\}$.

It remains to prove the second equivalence. The above calculations with $\phi(\gamma\cdot)$, $1 \leq \gamma \leq 2$, instead of ϕ give

$$\begin{aligned} \Big(\sum_{j=1}^{\infty} 2^{sjq} \sup_{1 \leq \gamma \leq 2} \|\phi_j(\gamma\cdot) \mathcal{H} f \mid L_2(\mathfrak{a}_+^\star \times B)\|^q \Big)^{1/q} & \\ \leq C \|f \mid B_q^s(X)\|. & \qquad (28) \end{aligned}$$

In consequence

$$\begin{aligned} & \|\mathcal{H}^{-1}\phi_0 \mathcal{H} f \mid L_2(X)\| \\ & \quad + \Big(\int_0^1 t^{-sq} \|\mathcal{H}^{-1}\phi(t\cdot) \mathcal{H} f \mid L_2(X)\|^q \frac{dt}{t} \Big)^{1/q} \\ & \leq C \|f \mid B_q^s(X)\|. \end{aligned}$$

On the other hand

$$\|\psi_j \mathcal{H} f \mid L_2(\mathfrak{a}_+^\star \times B)\| \le C \inf_{1\le\gamma\le 2} \|\phi_j \mathcal{H} f \mid L_2(\mathfrak{a}_+^\star \times B)\|$$

because $\phi(\gamma\lambda) \neq 0$ if $\lambda \in \overline{B(0,2) \setminus B(0,\frac{1}{2})}$ and $1 \le \gamma \le 2$. So

$$\int_0^1 t^{-sq} \|\mathcal{H}^{-1}\phi(t\cdot)\mathcal{H} f \mid L_2(X)\|^q \frac{dt}{t}$$
$$\ge C \sum_{j=0}^{\infty} 2^{sjq} \inf_{1\le\gamma\le 2} \|\mathcal{H}^{-1}\phi_j(\gamma\cdot)\mathcal{H} f \mid L_2(X)\|^q.$$

This finishes the proof. □

We end this section by proving simple mapping properties of invariant differential operators.

Proposition 2 *Let $-\infty < s < \infty$ and $0 < q < \infty$. Let D be the invariant differential operator on X.*

(i) *If the order of D is equal to m then $\|Df \mid B_q^{s-m}(X)\| \le C\|f \mid B_q^s(X)\|$.*

(ii) *Let the degree of $\gamma(D) = p$ equal $2m$, $m \in \mathbf{N}$. Let p_{2m} denote the sum of monomials of maximal degree of p. Assume that there is a constant C such that the inequality $p_{2m}(\lambda) \ge C\|\lambda\|$ holds for every $\lambda \in \mathfrak{a}^\star$. Let $h \in B_q^s(X)$. If there is $f \in \bigcup_{t\in\mathbf{R}} B_q^t(X)$ such that $D(f) = h$ then $f \in B_q^{s+2m}(X)$.*

Proof. The immediate proof of (i) is omitted. We prove the point (ii). The polynomial p_{2m} belongs to $S(\mathfrak{a}^\star)_W$ because the generators of the algebra are homogeneous polynomials. Let D_1 and D_2 be differential operators such that $\gamma(D_1) = p_{2m}$ and $D = D_1 + D_2$. Then the operators D_1, D_2 are invariant and $(I + D_1)f = h + (I - D_2)f$. The order of $I - D_2$ is less or equal to $2m - 1$. There is $t \in \mathbf{R}$ such that $f \in B_q^t(X)$. We assume that $t < s + 2m$, otherwise there is nothing to prove. We have

$$\|f \mid B_q^{t+1}(X)\| \sim \|(I-\Delta)^m f \mid B_q^{t+1-2m}(X)\|$$
$$\le C\Big(\sum_{j=0}^{\infty} 2^{(t-2m+1)jq} \|\phi^j \mathcal{H} h \mid L_2(\mathfrak{a}^\star \times B)\|^q\Big)^{1/q}$$
$$+ C\Big(\sum_{j=0}^{\infty} 2^{(t-2m+1)jq} \|\phi^j \mathcal{H}(I - D_2) f \mid L_2(\mathfrak{a}^\star \times B)\|^q\Big)^{1/q}$$

$$\leq C(\|h \mid B^s_q(X)\| + \|f \mid B^t_q(X)\|.$$

The argument can be repeated as far as $t < s + 2m - 1$. This proves the point (ii). □

3. Equivalent quasi-norms

In this section we introduce equivalent quasi-norm via the means characteristic for harmonic analysis on symmetric spaces like Laplace-Beltrami operator, spherical functions and root systems. In contrast to the means used in Definition 2 there are no theoretical objections to generalize these new means to $p \neq 2$. The further investigations seems to be of interest.

3.1. Quasi-norms by non-euclidean local means

A distribution f on X is called to be K-invariant if $f(\psi_k) = f(\psi)$ for every $\psi \in C_0^\infty(X)$ and every $k \in K$, here $\psi_k(x) = \psi(k^{-1} \cdot x)$.

Lemma 2 *Let h be a K-invariant distribution on X with compact support. Then*

(i) *$\mathcal{H}h$ can be extended to a W-invariant entire holomorphic function on $\mathfrak{a}^\star_{\mathbf{C}}$ satisfying the following estimates: there exist a constant $R \geq 0$ and the integer $m \geq 0$ such that*

$$\sup_{\lambda \in \mathfrak{a}^\star_{\mathbf{C}}} (1 + \|\lambda\|)^{-m} \exp(-R\|\Im\lambda\|)|\mathcal{H}h| < \infty,$$

(ii) *if $f \in \mathcal{C}'(X)$ then $f \star h \in \mathcal{C}'(X)$ and $\mathcal{H}(f \star h) = \mathcal{H}h \cdot \mathcal{H}f$.*

Proof. The point (i) is proved in [7]. From (i) and the definition of $\mathcal{Z}(\mathfrak{a}^\star \times B)$ it follows that $\psi\mathcal{H}h \in \mathcal{Z}(\mathfrak{a}^\star \times B)$ provided that $\psi \in \mathcal{Z}(\mathfrak{a}^\star \times B)$ and that $\psi_n\mathcal{H}h$ converges to $\psi\mathcal{H}h$ in $\mathcal{Z}(\mathfrak{a}^\star \times B)$ if ψ_n converges to ψ. Therefore $\mathcal{H}h \cdot \mathcal{H}f \in \mathcal{Z}'(\mathfrak{a}^\star \times B)$. We have

$$\begin{aligned}(\mathcal{H}h\mathcal{H}f)(\psi) &= |W|^{-1} \times \int_G d\tilde{f}(g) \int_G \int_{\mathfrak{a}^\star \times B} \psi(\lambda, b) e^{(-i\lambda+\rho)A(gK,b)} \\ &\qquad \cdot e^{(-i\lambda+\rho)A(yK,g^{-1}(b))} |c(\lambda)|^{-2} d\lambda db d\tilde{h}(y) \\ &= |W|^{-1} \int_G d\tilde{f}(g) \int_G \int_{\mathfrak{a}^\star \times B} \psi(\lambda, b) \\ &\qquad \cdot e^{(-i\lambda+\rho)A(gyK,b)} |c(\lambda)|^{-2} d\lambda db d\tilde{h}(y)\end{aligned}$$

$$= \int_G \int_G (\mathcal{H}_o^{-1}(gy) d\tilde{h}(y) d\tilde{f}(g) = (f \star h)(\mathcal{H}_o^{-1}\psi).$$

This proves the lemma. □

Let $\Omega(x, r)$ denote a geodesic ball centered at $x \in X$ with radius r.

Theorem 2 *Let Δ be the Laplace–Beltrami operator on X and let $\Gamma = \Delta + |\rho|^2$. Let k_0, k be K-invariant C^∞-functions on X satisfying the following conditions:*

$$\operatorname{supp} k_0 \subset \Omega(o,1),\ \operatorname{supp} k \subset \Omega(o,1),\ \mathcal{H}k_0(0) \neq 0\ \mathcal{H}k(0) \neq 0. \tag{29}$$

Let $k^N = \Gamma^N k$ and $\kappa^N = \mathcal{A}(k^N)$. Moreover, let $k_t^N = t^{-l}\mathcal{A}^{-1}(\kappa_t^N)$, $0 < t < 1$, where $\kappa_t^N(\lambda) = \kappa^N(t^{-1}\lambda)$ and $l = \dim \mathfrak{a}$.

If $0 < q \leq \infty$, $-\infty < s < \infty$ and $2N > s$, $N \in \mathbf{N}$ then

$$\|f \star k_0 \mid L_2(X)\| + \left(\int_0^1 t^{-sq} \|f \star k_t^N \mid L_2(X)\|^q \frac{dt}{t}\right)^{1/q} \tag{30}$$

is an equivalent norm in $B_q^s(X)$.

Proof. Almost all follows immediately from Theorem 1. The Abel transform $\mathcal{A}$ is an isomorphism of the space $C_0^\infty(X)_K$ of K-invariant C^∞-function on X with compact support onto the space $C_0^\infty(\mathfrak{a})_W$ of W-invariant C^∞-function on a with compact support. (cf. [8]). Therefore the functions k_t^N are well defined. They are C^∞-functions with compact support and $\operatorname{supp} k_t^N \subset \Omega(o, t)$ (the support conservation property — cf. [3]).

Let $\phi = \mathcal{H}k^N$ and $\phi_0 = \mathcal{H}k_0$. Then ϕ and ϕ_0 are W-invariant entire analytical functions of uniformly exponential type on $\mathfrak{a}^\star$ (the Paley–Wiener theorem for spherical Fourier transform — cf. [3]) and $\phi(\lambda) = |\lambda|^{2N}\mathcal{H}(k)(\lambda)$. The functions ϕ and ϕ_0 satisfy the conditions (21)–(23) with immaterial changes. In particular there is a ball $B(0, 2\varepsilon)$ in $\mathfrak{a}^\star$ such that $|\phi_0(\lambda)| > 0$ if $\lambda \in B(0, 2\varepsilon)$ and $|\phi(\lambda)| > 0$ if $\lambda \in B(0, 2\varepsilon)$, $\lambda \neq 0$. But Theorem 1 remains true if we use the ball $B(0, \varepsilon)$ instead of $B(0, 2)$ in the assumptions of the theorem cf. [15]. Moreover $\mathcal{H}^{-1}\phi(t\cdot) = k_t^N$. Thus the theorem follows from Theorem 1 and the above lemma. □

3.2. Means via the spherical functions

A vector H is a regular element of $\mathfrak{a}$ i.e. $\lambda(H) \neq 0$ for any $\lambda \in \Sigma_+$.

Theorem 3 *Let $H_0 \in \mathfrak{a}$ be regular. Let*

$$\psi_{H_0}(\lambda) = \varphi(-\lambda : \exp H_0 \cdot o) - \varphi(0 : \exp H_0 \cdot o)\,, \quad \lambda \in \mathfrak{a}^\star\,.$$

Let $0 < s < N$ and $0 < q \leq \infty$. Then the expression

$$\|f \mid L_2(X)\| + \left(\int_0^1 t^{-sq} \|\mathcal{H}^{-1}\psi_{H_0}^N(t\cdot)\mathcal{H}f \mid L_2(X)\|^q \frac{dt}{t}\right)^{1/q} \tag{31}$$

is an equivalent quasi-norm in $B_q^s(X)$.

Proof. The function ψ_{H_0} is an entire analytical function on $\mathfrak{a}_{\mathbf{C}}^\star$ and $\psi_{H_0}(0) = 0$. It is not difficult to see that $\psi_{H_0}(\lambda) \neq 0$ if $\lambda \neq 0$. In fact

$$\begin{aligned}\psi_{H_0}(\lambda) &= \int_K e^{(-i\lambda+\rho)A(\exp H_0, kM)} dk \\ &\quad - \int_K |e^{(-i\lambda+\rho)A(\exp H_0, kM)}| dk.\end{aligned}$$

But an integral of the continuous complex-valued function is equaled to the integral of the absolute value of the function if and only if the function is real-valued and nonnegative which is impossible for the function $k \mapsto e^{(-i\lambda+\rho)A(\exp H_0, kM)}$ if $\lambda \neq 0$ because of the definition of $\lambda A(\exp H_0, kM)$. Thus $\psi_{H_0}^N$ satisfy the (21) of Theorem 1. Moreover it is known that there exists the constant $C > 0$ such that $|\psi_{H_0}(\lambda)| < C\varphi(0 : \exp H_0 \cdot o)$ if $\lambda \in \mathfrak{a}^\star$ (cf. [3]). Now the theorem follows from Theorem 1 with $s_0 = 0$, $s_1 = N$, $\phi_0 = 1$ and $\phi(\lambda) = \psi_{H_0}^N(\lambda)$. □

Now we try to give the local version of the last theorem. We defined a distribution $\mathcal{D}_{H_0}$ by

$$\mathcal{D}_{H_0}(\phi) = \int_K \phi(k \exp H_0 \cdot o) - \phi(o) e^{\rho A(\exp H_0 \cdot o, kM)} dk.$$

If f is a suitable function on X then $(f \star \mathcal{D}_{H_0})(gK) = \int_K f(gk \exp H_0 \cdot o) dk - f(gK)\varphi(o : \exp H_0 \cdot o)$ It should be clear that the distribution $\mathcal{D}_{H_0}$ is K-invariant, $\operatorname{supp} \mathcal{D}_{H_0} \subset \Omega(o, \|H_0\|)$ and

$$(\mathcal{H}\mathcal{D}_{H_0})(\lambda) = \varphi(\lambda : \exp H_0 \cdot o) - \varphi(0 : \exp H_0 \cdot o).$$

Now, we calculate $T_t = \mathcal{H}^{-1}(\lambda \mapsto \varphi(t\lambda : (\exp -H_0) \cdot o))$. T_t is a K-invariant distribution with compact support on X. It is known that there exists a function $F_0 \in L_1(\mathfrak{a}, dH)$ such that

$$\int_K f(H((\exp H_0)k))dk = \int_{\mathfrak{a}} f(H)F_0(H)dH, \quad e^{-\rho}F_0 \in L_1(\mathfrak{a}, dH),$$

cf. [11, Th.IV.10.11)]. So $\varphi(\lambda, (\exp -H_0)\cdot o) = \int_{\mathfrak{a}} e^{i\lambda(H)}e^{-\rho(H)}F_0(H)dH$. Let $\psi \in \mathcal{C}(X)$ be K-invariant and let $a_0 = \exp(-H_0)$. Then

$$\begin{aligned}
&T_t(\psi)\\
&= |W|^{-1}\int_{\mathfrak{a}^\star} \varphi(t\lambda : a_0\cdot o)\mathcal{H}_o\psi(\lambda)|c(\lambda)|^{-2}d\lambda\\
&= |W|^{-1}\int_{\mathfrak{a}^\star} t^{-l}\mathcal{F}_o(e^{-\rho(t^{-1}\cdot)}F_0(t^{-1}\cdot)\star\mathcal{A}\psi)(\lambda)|c(\lambda)|^{-2}d\lambda\\
&= |W|^{-1}\int_{\mathfrak{a}^\star} \mathcal{F}_o\Big(\int_K e^{-\rho(H(a_0k))}\mathcal{A}\psi(\cdot - tH(a_0k))dk\Big)(\lambda)|c(\lambda)|^{-2}d\lambda\\
&= |W|^{-1}\int_{\mathfrak{a}^\star}\int_K e^{-\rho H(a_0k)}e^{it\lambda H(a_0k)}\mathcal{H}_o\psi(\lambda)dk|c(\lambda)|^{-2}d\lambda\\
&= \int_K e^{(t-1)\rho H(a_ok)}\psi(\exp(-tH(a_oh))\cdot o)dk.
\end{aligned}$$

T_t is a K-invariant distribution with compact support therefore for every $\psi \in C^\infty(X)$ we have $T_t(\psi) = T_t(\psi^\sharp)$ where $\psi^\sharp(x) = \int_K \psi(k\cdot x)dk$. Let

$$\mathcal{D}_{H_0,t}(\psi) = \int_K e^{(t-1)\rho H(a_ok)}\psi^\sharp(\exp(-tH(a_oh))\cdot o)dk - \psi(o)\varphi(0 : \exp H_0).$$

$\psi \in C^\infty(X)$. Then $\mathcal{D}_{H_0,t}$ is a K-invariant distribution with compact support, $\mathcal{H}(\mathcal{D}_{H_0,t})(\lambda) = \psi_{H_0}(t\lambda)$. Thus we have proved the following corollary.

Corollary 1 *Let $0 < s < N$ and $0 < q \le \infty$. Let $\mathcal{D}^N_{H_0,t}$ denote the following distribution $\mathcal{D}^N_{H_0,t} = \mathcal{D}_{H_0,t}\star\cdots\star\mathcal{D}_{H_0,t}$ (N times). Then the expression*

$$\|f \mid L_2(X)\| + \left(\int_0^1 t^{-sq}\|f\star\mathcal{D}^N_{H_0,t} \mid L_2(X)\|^q\frac{dt}{t}\right)^{1/q} \tag{32}$$

is an equivalent quasi-norm in $B^s_q(X)$.

3.3. Means via the root systems

We recall that a positive restricted root $\lambda \in \Sigma_+$ is called simple if it can not be written $\lambda = \alpha + \beta$ with $\alpha, \beta \in \Sigma_+$. Let $\mathcal{S} = \{\alpha_1 \ldots, \alpha_l\}$ be the set of all simple restricted roots corresponding to the given Weyl chamber $\mathfrak{a}_+$. Then $l = \dim\mathfrak{a}$, $< \alpha_i, \alpha_j >< 0$ for $i \neq j$, $span\mathcal{S} = \mathfrak{a}^\star$ (cf. [11, VII§3]). The action of the Weyl group on the set of the Weyl chambers is simple transitive. Therefore each $w \in W$ appoints the set $\mathcal{S}_w$ of simple roots. We define the following functions:

$$\psi_1(\lambda) = \sum_{w\in W} e^{i<\lambda,\alpha_w>} - |W|, \quad \text{where } \alpha_w = \sum_{\alpha\in\mathcal{S}_w} \alpha, \tag{33}$$

$$\psi_2(\lambda) = \sum_{w\in W} \sum_{\alpha\in\mathcal{S}_w} e^{i<\lambda,\alpha>} - l|W|, \ \ \lambda \in \mathfrak{a}^\star_{\mathbf{C}}. \tag{34}$$

Both the functions are holomorphic on $\mathfrak{a}^\star$, bounded if $\lambda \in \mathfrak{a}^\star$ and $\psi_1(0) = \psi_2(0) = 0$. Moreover the functions are W-invariant since the Weyl group is generated by orthogonal reflections with respect to hyperplanes in the scalar product space $(\mathfrak{a}, <,>)$ and any two sets $\mathcal{S}_{w_1}$ and $\mathcal{S}_{w_2}$ are conjugate under the unique element of the Weyl group. We prove that there is a neighborhood V_j of 0 such that $\psi_j(\lambda) \neq 0$ if $\lambda \in V_j \setminus \{0\}$, $j = 1, 2$. We start with the following elementary fact: if $b_k \in \mathbf{R}$, $k = 1, \ldots m$ then $\sum_{k=1}^m e^{ib_k} = m$ if and only if $b_k \in 2\pi\mathbf{Z}$. Thus, $\psi_1(\lambda) = 0$ if and only if $< \lambda, \alpha_w >\in 2\pi\mathbf{Z}$ for every $w \in W$. The map $F : \mathfrak{a}^\star \ni \lambda \mapsto (< \lambda, \alpha_{w_k} >)_{k=1}^m \in \mathbf{R}^m$, $m = |W|$, is linear and continuous therefore the set $V_1 = F^{-1}(\{(a_k) : |a_k| < \pi, k = 1, \ldots, m\})$ is an open neighborhood of 0 in $\mathfrak{a}^\star$. If $\lambda \in V_1$ and $\psi_1(\lambda) = 0$ then $| < \lambda, \alpha_w > | < \pi$ and $< \lambda, \alpha_w >\in 2\pi\mathbf{Z}$. So $< \lambda, \alpha_w >= 0$ for every $w \in W$. But the vectors α_w span $\mathfrak{a}^\star$ therefore $\lambda = 0$. The proof for ψ_2 is the same.

The next theorem follows from the above consideration and Theorem 1.

Theorem 4 *Let $0 < s < N$, $0 < q \leq \infty$ and $j = 1, 2$. Then the expressions*

$$\|f \mid L_2(X)\| + \left(\int_0^1 t^{-sq} \|\mathcal{H}^{-1}\psi_j(t\cdot)\mathcal{H}f \mid L_2(X)\|^q \frac{dt}{t}\right)^{1/q} \tag{35}$$

are equivalent quasi-norms in $B^s_q(X)$.

Let H_w be the unique element of $\mathfrak{a}$ such that $\alpha_w(H) =< H, H_w >$, $H \in \mathfrak{a}$. If the Lie group G has only one conjugacy class of Cartan subgroups then the function $|c(\lambda)|^{-1}$ is a polynomial without constant term (cf. [11, p.448]). For these symmetric spaces $X = G/K$ we have the following local version of the last theorem.

Corollary 2 *Let G have only one conjugacy class of Cartan subgroups. Let D_c be the differential operator on A corresponding to polynomial $|c(\lambda)|^{-2}$ via the Euclidean Fourier transform. Let*

$$\mathcal{D}_{1,t}(\psi) = |W| \sum_{w\in W} D_c \mathcal{A}\psi^\sharp(-tH_w) - \psi(o),$$

$$\mathcal{D}_{2,t}(\psi) = |W| \sum_{w\in W} \sum_{\alpha\in\mathcal{S}_w} D_c \mathcal{A}\psi^\sharp(-tH_\alpha) - \psi(o).$$

and $\mathcal{D}_{j,t}^N = \mathcal{D}_{j,t}\star\cdots\star\mathcal{D}_{j,t}$ (N times), $t \in (0,1)$, $N \in \mathbf{N}$. Then the expressions

$$\|f \mid L_2(X)\| + \left(\int_0^1 t^{-sq} \|f \star \mathcal{D}_{j,t}^N \mid L_2(X)\|^q \frac{dt}{t}\right)^{1/q}, \tag{36}$$

$j = 1,2$ are equivalent quasi-norms in $B_q^s(X)$ with $0 < s < N$, and $0 < q < \infty$.

Proof. Let $\psi \in \mathcal{C}(X)_K$. The inversion formula for the Helgason–Fourier transform implies

$$\begin{aligned}&\left(\mathcal{H}^{-1}\big(\sum_{w\in W} e^{i<t\lambda,\alpha_w>} - |W|\big)\right)(\psi)\\ &= |W|^{-1}\cdot \sum_{w\in W}\int_{\mathfrak{a}^\star} (e^{i\lambda(tH_w-1)}(\mathcal{F}_o \circ \mathcal{A}\psi)|c(\lambda)|^{-2} d\lambda\\ &= |W|^{-1} \sum_{w\in W} D_c\mathcal{A}\psi(-tH_w) - \psi(o).\end{aligned}$$

This and the above theorem prove the corollary for $j = 1$. The proof for $j = 2$ is similar. □

References

[1] Anker J.-Ph., *Handling the inverse of spherical Fourier transform.* In: Harmonic Analysis on Reductive Groups, ed. W.H. Baker and P. Sally, Birkhäser Basel and Boston (1991), 51–56.

[2] Anker J.-Ph., *L_p-Fourier multipliers on Riemannian symmetric spaces of the non-compact type.* Ann. Math. **132** (1990), 597–628.

[3] Anker J.-Ph., *The spherical Fourier transform of Rapidly decreasing functions. A simple proof of a characterization due to Harish-Chandra, Helgason, Trombi, and Varadarajan.* J. Func. Anal. **96** (1991), 331–349.

[4] Aubin T., *Nonlinear Analysis on Manifolds.Monge-Ampere Equations.* Springer Verlag, New York (1982).

[5] Aubin T., *Espaces de Sobolev sur les variétés Riemanniennes.* Bull. Sci. Math. **100** (1976), 149–173.

[6] Eguchi M., *Asymptotic expansions of Eisenstein integrals and Fourier transform on symmetric spaces.* J. Funct. Anal. **34** (1979), 164–216.

[7] Equchi M., Hashizume M. and Okamoto K., *The Paley-Wiener theorem for distributions on symmetric spaces.* Hiroshima Math. J. **3** (1973), 109–120.

[8] Gangolli R.A. and Varadarajan V.S., *Harmonic analysis of spherical functions on real reductive groups.* Ergebnisse Math. Grenzgeb. 101, Springer-Verlag, (1988).

[9] Helgason S., *A duality for symmetric spaces with applications to group representations.* Adv. in MAth. **5** (1970), 1–154.

[10] Helgason S., *Differential Geometry, Lie Groups and Symmetric Spaces.* Academic Press, (1978).

[11] Helgason S., *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions.* Academic Press, (1984).

[12] Helgason S., *The surjectivity of invariant differential operators on symmetric spaces I.* Ann. of Math. **98** (1973), 451–480.

[13] Skrzypczak L., *Function spaces of Sobolev type on Riemannian symmetric manifolds.* Forum Math. **3** (1991), 339–353.

[14] Strichartz R.S., *Analysis of the Laplacian on a complete Riemannian manifold.* J. Func. Anal. **52** (1983), 48–79.

[15] Triebel H., *Characterizations of Besov-Hardy-Sobolev spaces:a unified approach.* J. Approx. Th. **52** (1988), 162–203.

[16] Triebel H., *Characterizations of function spaces on a complete Riemannian manifold with bounded geometry.* Math. Nach. **130** (1987), 321–346.

[17] Triebel H., *How to measure smoothness of distributions on Riemannian symmetric manifolds and Lie groups?* Zeitsch. Anal. ihre Andwend. **7** (1988), 471–480.

[18] Triebel H., *Spaces of Besov-Hardy-Sobolev type on complete Riemannian manifolds.* Arkiv Mat. **24** (1986), 300–337.

[19] Triebel H., *Theory of Function Spaces.* Birkhäuser Verlag (1983).

[20] Triebel H., *Theory of Function Spaces II.* Birkhäuser Verlag (1992).

[21] Warner G., *Harmonic Analysis on Semi-Simple Lie Groups.* Vol. I, II, Springer-Verlag, (1972).

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