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# The principle of closeness of sufficiently large sets of $a$-points of meromorphic functions

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**Abstract.** A new version of the proximity principle is given for functions meromorphic in the unit disk.

*Key words*: value distribution theory, proximity property of $a$-points.

## Introduction. Value distribution of functions meromorphic in $\mathbb{C}$.

The classical theories of R. Nevanlinna and L. Ahlfors [7] describe the distribution of the $a$-points of functions meromorphic in $\mathbb{C}$. These theories give very precise information for most values of $a$; they do not say anything about the mutual arrangement of $a$-points for varying $a$. The mutual arrangement (m.a.) of $a$-points was considered in numerous articles devoted to the study of "cercles de remplissage" and to Julia and Borel lines. However, this research was concerned with the m.a. in relatively small portions of the plane. The papers [1], [2], [3] by the present author give a "general principle of the proximity of $a$-points". This principle also has some bearing on the classical value-distribution theory.

## 1. Value distribution of functions meromorphic in the unit disk $D$.

Results analogous to those of Nevanlinna-Ahlfors theory and the proximity principle are true in the case of functions meromorphic in the unit disk $D$, provided the spherical characteristic function $A(r)$ has a rather fast rate of growth:

$$\limsup_{r\to 1} A(r)(1-r)=\infty. \tag{1}$$

(See [3], [7]).

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If $A(r)$ has slow growth, i.e.

$$A(r) = O\left(\frac{1}{1-r}\right), \qquad r \to 1, \tag{2}$$

we have very few results, not only for m.a., but even for the number of $a$-points. In fact, in most cases (classes of bounded functions, $H^p$, Dirichlet and so on) we can only estimate Blaschke sums and we can not even compare the number of $a$-points in a subset of $D$ for different $a$. Indeed these numbers can be quite different, so that there is no general "proximity principle" here. However if we consider the m.a. of values $a$ for which the set $w^{-1}(a)$ is "sufficiently large" we obtain a new version of the proximity principle for functions meromorphic in the unit disk. This version is given in section 2. The proofs are given in section 3.

## 2.1. Preliminaries

Let $D(r) = \{z : |z| < r\}$, $w(z)$ be meromorphic function in the unit disk $D = D(1)$. Let $\Gamma$ be a smooth Jourdan curve in $\mathbb{C}$ passing through the given points $a_1, b_1, a_2, b_2 \dots, a_n, b_n$ in this order, $n < \infty$. We denote by $z_i(a)$ the $a$-points of function $w$ and $A = A(\Gamma, r)$ the totality of all $z_i(w)$ in $D(r)$ with counting multiplicities, where $w \in a_1, b_1, a_2, b_2 \dots, a_n, b_n$. We assume that

a) $\Gamma$ has a parametric representation

$$\Gamma = \{\zeta : \zeta = \zeta(s) \qquad (0 \le s \le L)\},$$

where $\zeta(s)$ is a smooth function of the Euclidean arc length $s$ of $\Gamma$ and

$$v(\Gamma) = \int_0^L |d \arg z'(s)| < \infty.$$

We may also assume that

b) $\Gamma$ does not pass through any of the finite number of points $z \in D(r)$, $z \notin A$ at which $w'(z) = 0$.

We denote by $\Gamma_{a_\nu, b_\nu}$ the part of curve lying between points $a_\nu$ and $b_\nu$, $\nu = 1, 2, \dots, n$. Then the part of the set $w^{-1}(\Gamma_{a_\nu, b_\nu})$ lying in $D(r)$ consist of some curves of type $L_1$, $L_2$, $L_3$, which due to assumption b), are completely determined by the following rules.

**A)** Every curve of type $L_1$ connects a point $z_j(a_\nu)$ with a point $z_{j^*}(b_\nu)$ (we shall denote these $b_\nu$-points by $z_j(b_\nu)$); correspondingly $|z_j(a_\nu) - z_j(b_\nu)|$ is less than the length of the curve;

**B)** Every curve of type $L_2$ connects a point $z_i(a_\nu)$ (or $z_i(b_\nu)$ ) with a point on boundary $D(r)$; correspondingly the quantity $(r - |z_i(a_\nu|)$ (or $r - |z_i(b_\nu)|)$ is less than the length of the curve;

**C)** The closure of every curve of type $L_3$ does not involve any point $z_i(a_\nu)$ or $z_i(b_\nu)$.

We recall that every $a_\nu$-point or $b_\nu$-point above is enumerated counting multiplicities.

Let $\tilde{A} = \tilde{A}(\Gamma, r)$ be totality of all pairs $(z_j(a_\nu), z_j(b_\nu))$ from A) and

$$P(r, \tilde{A}) = \sum_{(z_j(a_\nu), z_j(b_\nu)) \in \tilde{A}} |z_j(a_\nu) - z_j(b_\nu)|. \tag{3}$$

We will call $P(r, \tilde{A})$ the closeness function of $a$- and $b$-points.

Let $A^*$ be totality of all points $z_i(a_\nu)$ and $z_i(b_\nu)$ from B) and let

$$b(r, A^*) = \sum_{z_j \in A^*} (r - |z_j|). \tag{4}$$

We will call $b(r, A^*)$ function of lonely $a$- and $b$-points. Obviously

$$P(r, \tilde{A}) + b(r, A^*) \leq L(r, \Gamma), \tag{5}$$

where $L(r, \Gamma)$ is the total Euclidean length of the curves $w^{-1}(\Gamma)$ in $D(r)$.

We shall measure the density of $A(\Gamma, r)$ by the function

$$b(r, A) = \sum_{z \in A} (r - |z|).$$

A suitable analogue of the Nevanlinna characteristic function $T$ in this problem and in related ones is

$$B(r, w) = \int\int_{D(r)} \left| \frac{w''(z)}{w'(z)} \right| d\sigma,$$

where $d\sigma$ is the Euclidean area element in $\mathbb{C}$.

## 2.2. We shall need

**Theorem A** (*The "tangent variation principle", see* [1] *and* [4].) *Let $w(z)$ be a meromorphic function in the closure of $D(r)$. And let $\Gamma$ be a smooth Jordan curve in $\mathbb{C}$ with $v(\Gamma) < \infty$. Then*

$$L(r, \Gamma) \leq 3(v(\Gamma) + 1)(B(r) + 2\pi r). \tag{6}$$

We shall also use the beautiful theorem of W.K. Hayman and G.J-M. Wu [6]:

**Theorem B** *Let* $w(z) = z + \alpha_2 z^2 + \cdots$ *be holomorphic and univalent in the unit disk* $D$ *and let* $\Gamma$ *be a circle or a straight line. Then the length* $L(1,\Gamma)$ *of* $w^{-1}(\Gamma)$ *satisfies*

$$L(1,\Gamma) < const < 10^{35}.$$

**Definition** Let $w$ be meromorphic in $z : |z| < R$, $R \leq \infty$. Let $\epsilon(r) : (0, R) \to (\epsilon(0), 0)$, $\epsilon(r) < 1$, be a non-increasing continuous function tending to 0 as $r \to \infty$. We call the set $A$ *sufficiently large* for given $r$, $0 < r < R$, if

$$b(r, A)\epsilon(r) \geq 3(v(\Gamma) + 1)((B(r) + 2\pi r). \tag{7}$$

Obviously for any meromorphic function in $D(R)$ and given $r < R$ we can indicate a curve $\Gamma$ and a sufficienly large set $A(\Gamma, r)$.

## 2.3. Statement of Results

**Theorem 1** *Let* $w$ *be a function meromorphic in* $z : |z| < R$, $R \leq \infty$. *If* $A = \tilde{A} \cup A^*$ *is suffciently large for given* $r < R$ *then*

$$\frac{b(r, A^*)}{b(r, A)} \leq \epsilon(r), \tag{8}$$

*and*

$$\frac{P(r, \tilde{A})}{b(r, \tilde{A})} \leq \epsilon(r). \tag{9}$$

Theorem 1 has a simple meaning. If $\epsilon(r)$ is small, then, by (8) most of the $a$- and $b$-points belong to $\tilde{A}$ and, by (9), the $a$-and $b$-points in $\tilde{A}$ are close to one other.

**Theorem 2** *Let* $w$ *be meromorphic function in* $D$ *satisfying* (2). *Suppose* $\tilde{A}$, $A^*$, $\Gamma$ *are determined as above in* (3) *and* (4). *Then*

$$P(r, \tilde{A}) + b(r, A^*) \leq L(r, \Gamma) = O\left(\frac{1}{1-r}\log\frac{1}{1-r}\right) \quad (r \to 1-0). \tag{10}$$

**Theorem 3** *Let* $w = z + \alpha_2 z^2 + \cdots$ *be univalent in the unit disk and let*

$\Gamma$ *be a circle or a straight line. Then*

$$P(r, \tilde{A}) + b(r, A^*) \leq const < \infty. \tag{11}$$

### 3.1. Proof of Theorem 1

By (6), (7), Theorem A and the definition of "sufficiently large"

$$P(r, \tilde{A}) + b(r, A^*) \leq b(r, A)\epsilon(r), \tag{12}$$

which proves (8). (9) follows from (12) on replacing $b(r, A)$ by $b(r, \tilde{A}) + b(r, A^*)$.

### 3.2. Proof of Theorem 2

Let $r' = (1 + 2r)/3$. By the inequalities (5.24)–(5.26) of [1],

$$B(r) = O\left(\frac{1}{1-r}T(r', w')\right). \tag{13}$$

By a well known inequality,

$$T(r', w') = O(2T(r', w) + m(r', w'/w)).$$

If $r_1' = (2 + r)/3$, then, by the Lemma of the Logarithmic Derivative,

$$T(r', w') = O\left(2T(r, w) + \log T(r_1', w) + \log \frac{1}{r_1' - r'}\right), \quad r \to 1.$$

Therefore by (13)

$$B(r) = O\left(\frac{1}{1-r}\left[T(r_1', w) + \log \frac{1}{1-r}\right]\right), \quad r \to 1. \tag{14}$$

By [5, Theorem 6.2], (2) implies that

$$T(r, w) = O\left(\log \frac{1}{1-r}\right), \quad r \to 1,$$

Theorem 2 now follows from (14).

### 3.3. Proof of Theorem 3

By Theorem B,

$$L(r, \Gamma) < L(1, \Gamma) < const$$

and Theorem 3 is a consequence of (5).

### 3.4. Proof of Theorem 4

By Theorem A it is enough to prove (12). This was already done in the proof of Theorem 2.

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