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On Kato's square root problem

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Abstract. We consider abstract versions,

$$H=-\sum_{i,j=1}^{n}A_{i}c_{ij}A_{j}+\sum_{i=1}^{n}(c_{i}A_{i}+A_{i}c_{i}^{\prime})+c_{0},$$

of second-order partial differential operators defined by sectorial forms on a Hilbert space $\mathcal{H}$. The A_i are closed skew-symmetric operators with a common dense domain $\mathcal{H}_1$ and the c_{ij}, c_i etc. are bounded operators on $\mathcal{H}$ with the real part of the matrix $C=(c_{ij})$ strictly positive-definite.

We assume that $D(L)\subseteq\bigcap_{i,j=1}^{n}D(A_{i}A_{j})$ where $L=-\sum_{i=1}^{n}A_{i}^{2}$ is defined as a form on $\mathcal{H}_1\times\mathcal{H}_1$. We further assume the c_{ij} are bounded operators on one of the Sobolev spaces $\mathcal{H}_{\gamma}=D((I+L)^{\gamma/2})$, $\gamma\in\langle 0,1\rangle$, equipped with the graph norm. Then we prove that

$$D((\lambda I+H)^{1/2})=D((\lambda I+H^{*})^{1/2})=\mathcal{H}_{1} \tag{1}$$

for all large $\lambda\in\mathbf{R}$.

As a corollary we deduce that in any unitary representation of a Lie group all second-order subelliptic operators in divergence form with Hölder continuous principal coefficients satisfy (1).

Let K be a closed maximal accretive, regular accretive, sectorial operator on the Hilbert space $\mathcal{H}$ with associated regular sesquilinear form k and $\operatorname{Re}K$ the closed maximal accretive operator associated with the real part of k. Kato [Kat1], Theorem 3.1, proved that $D(K^{\delta})=D(K^{*\delta})=D((\operatorname{Re}K)^{\delta})$ for all $\delta\in[0,1/2\rangle$ but Lions [Lio] subsequently gave an example of a closed maximal accretive operator for which $D(K^{1/2})\neq D(K^{*1/2})$. Then Kato [Kat2], Theorems 1 and 2, proved that $D(K^{1/2})=D(K^{*1/2})$ if, and only if, both $D(K^{1/2})\subseteq D(k)$ and $D(K^{*1/2})\subseteq D(k)$. More generally $D(K^{1/2})\subseteq D(k)$ if and only if $D(k)\subseteq D(K^{*1/2})$ with a similar equivalence if K and K^* are interchanged. Therefore the identity of any two of the sets $D(K^{1/2})$, $D(K^{*1/2})$, $D(k)$ implies the identity of all three. Establishing that a particular operator K satisfies these last identities has become known as Kato's square root problem, or the Kato problem.

Kato's initial interest in these questions was motivated by problems of

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evolution equations and much subsequent attention has been devoted to the Kato problem for strongly elliptic second-order operators with complex measurable coefficients in divergence form on $L_2(\mathbf{R}^d; dx)$ or on a subspace corresponding to a subdomain $\Omega \subset \mathbf{R}^d$. The problem has proved remarkably intractable but it has been solved under some special additional assumptions. For example the one-dimensional case was solved by Coifman, McIntosh and Meyer [CMM] in 1982 and in 1985 McIntosh [McI2] showed that the problem can be solved if the coefficients are Hölder continuous. A survey of the situation up to 1990 is given by McIntosh in [McI1] and a more recent update in [AuT]. This latter paper establishes the equivalence of the Kato problem with several other classical problems of harmonic analysis and illustrates the difficulties of its solution.

Our purpose in this note is to solve the Kato problem for an abstract class of second-order operators under a mild regularity condition on the principal coefficients. In particular we extend the results of McIntosh [McI2] by quite different arguments which rely on interpolation theory. Indeed we draw analogous conclusions to McIntosh for operators associated with an arbitrary unitary representation of a Lie group.

Let $A_1, \ldots, A_n$ be closed skew-symmetric operators on the Hilbert space $\mathcal{H}$ such that $\mathcal{H}_1 = \bigcap_{i=1}^n D(A_i)$ is norm-dense. Define the corresponding **Laplacian** L as the positive self-adjoint operator associated with the form l with domain $\mathcal{H}_1$ given by

$$l(\varphi) = \sum_{i=1}^n \|A_i\varphi\|^2 .$$

Then

$$\bigcap_{i=1}^n D(A_i) = \mathcal{H}_1 = D((\lambda I + L)^{1/2}) \tag{2}$$

for all $\lambda \geq 0$ by [Kat3], Theorem VI.2.23. In particular $\mathcal{H}_1$ is a Banach space with respect to the norm

$$\|\varphi\|_1 = \|(I + L)^{1/2}\varphi\| = \Big(\|\varphi\|^2 + \sum_{i=1}^n \|A_i\varphi\|^2\Big)^{1/2} .$$

Next introduce the corresponding Sobolev spaces $\mathcal{H}_\gamma$, $\gamma \in \mathbf{R}$, as $\mathcal{H}_\gamma =$

$D((I+L)^{\gamma/2})$, if $\gamma > 0$, with the graph norm

$$\|\varphi\|_\gamma = \|(I+L)^{\gamma/2}\varphi\|, \tag{3}$$

and as the completion of $(I+L)^{\gamma/2}\mathcal{H}$ with respect to the norm (3) if $\gamma \leq 0$. Then $\mathcal{H}_{-\gamma}$ is the dual of $\mathcal{H}_\gamma$. Since L is self-adjoint the Sobolev spaces form a scale of complex interpolation spaces.

The class of operators we analyze are defined by sectorial forms h on $\mathcal{H}_1 \times \mathcal{H}_1$ with values

$$\begin{aligned} h(\psi,\varphi) &= \sum_{i,j=1}^n (A_i\psi, c_{ij}A_j\varphi) \\ &\qquad + \sum_{i=1}^n \Big((\psi, c_iA_i\varphi) - (A_i\psi, c_i'\varphi)\Big) + (\psi, c_0\varphi) \end{aligned} \tag{4}$$

where c_{ij}, c_i, c_i' and c_0 are bounded operators on $\mathcal{H}$ with the real part of the matrix $C = (c_{ij})$ of principal coefficients strictly positive-definite, i.e.,

$$\sum_{i,j=1}^n \mathrm{Re}(\varphi_i, c_{ij}\varphi_j) \geq \mu \sum_{i=1}^n \|\varphi_i\|^2$$

for some $\mu > 0$ and all $\varphi_1, \ldots, \varphi_n \in \mathcal{H}$. Forms of this type will be called **subelliptic**. The positive-definiteness condition, i.e., the subellipticity, ensures that h is sectorial and closed on $\mathcal{H}_1 \times \mathcal{H}_1$. Hence if H is the sectorial operator associated with h then $\lambda I + H$ is a closed maximal accretive, regularly accretive operator for all sufficiently large $\lambda \in \mathbf{R}$. It follows that $\lambda I + H$ has a bounded H_∞-functional calculus and bounded imaginary powers: $\|(\lambda I + H)^{is}\| \leq e^{\pi|s|/2}$ for all $s \in \mathbf{R}$ and all sufficiently large λ. A proof of these facts can be found, for example, in [ADM]. One of the consequences of the bounded imaginary powers is the fractional powers are well-defined and form a scale of complex interpolation spaces. For example,

$$[D((\lambda I + H)^\alpha), D((\lambda I + H)^\beta)]_\theta = D((\lambda I + H)^{(1-\theta)\alpha + \theta\beta})$$

for all large λ, all $\alpha, \beta \geq 0$ with $\alpha \neq \beta$ and all $\theta \in \langle 0, 1\rangle$ (see [Tri], Theorem 1.15.2).

Our main result is the following.

Theorem 1 *Assume the regularity inclusion*

$$D(L) \subseteq \bigcap_{i,j=1}^{n} D(A_i A_j). \tag{5}$$

Let H be the closed sectorial operator associated with the subelliptic form (4) *and suppose the c_{ij} and their adjoints c_{ij}^* are bounded operators on the Sobolev space $\mathcal{H}_\gamma = D((I+L)^{\gamma/2})$ for some $\gamma \in \langle 0,1\rangle$. Then*

$$D((\lambda I + H)^{1/2}) = D((\lambda I + H^*)^{1/2}) = \mathcal{H}_1$$

for all large $\lambda \in \mathbf{R}$.

If the matrix of principal coefficients $C = (c_{ij})$ is self-adjoint, i.e., if $c_{ij} = c_{ji}^*$ for all $i,j \in \{1,\ldots,n\}$, there is no need for the regularity assumptions of the theorem. Then the principal part $H_0 = -\sum_{i,j=1}^n A_i c_{ij} A_j$ of H is positive, self-adjoint, and $D((\lambda I + H_0)^{1/2}) = D((\lambda I + H_0^*)^{1/2}) = \mathcal{H}_1$ for all $\lambda \geq 0$ by [Kat3], Theorem VI.2.23. This conclusion can then be extended to H, at least for large positive values of λ, by the interpolation–perturbation argument used at the end of the following proof. Thus the difficulty in the theorem occurs when the principal coefficients are not self-adjoint. Then the assumptions, $C\mathcal{H}_\gamma \subseteq \mathcal{H}_\gamma$ and $C^*\mathcal{H}_\gamma \subseteq \mathcal{H}_\gamma$, reflect a form of smoothness of the action of the operators c_{ij} by the following reasoning.

First remark that the value of γ in the assumption is not of particular significance. If c is a bounded operator on $\mathcal{H}$ and in addition bounded on $\mathcal{H}_\gamma$ for some $\gamma \in \langle 0,1\rangle$ then it is bounded on $\mathcal{H}_\delta$, for all $\delta \in \langle 0,\gamma\rangle$, by complex interpolation. Secondly, let c be a bounded operator on $\mathcal{H}$ with norm $\|c\|_{\mathcal{H}}$ and T the 'heat' semigroup generated by L on $\mathcal{H}$. Then for c and c^* to be bounded on $\mathcal{H}_\gamma$ it suffices that one has bounds

$$\|T_t c - cT_t\| \leq a t^{\nu/2} e^{\omega t} \tag{6}$$

for some $\nu > \gamma > 0$, some $\omega \geq 0$ and all $t > 0$. This follows because

$$(\lambda I + L)^{\gamma/2} = c_\gamma^{-1} \int_0^\infty dt\, t^{-1-\gamma/2}(I - e^{-\lambda t}T_t)$$

for all $\lambda > 0$ where $c_\gamma = \int_0^\infty dt\, t^{-1-\gamma/2}(1-e^{-t})$. Hence

$$(\lambda I + L)^{\gamma/2} c - c(\lambda I + L)^{\gamma/2} = c_\gamma^{-1} \int_0^\infty dt\, t^{-1-\gamma/2} e^{-\lambda t}(cT_t - T_t c)$$

and the bounds (6) give

$$\begin{aligned}
&|((\lambda I+L)^{\gamma/2}\psi, c\varphi) - (\psi, c(\lambda I+L)^{\gamma/2}\varphi)|\\
&\qquad\leq a c_\gamma^{-1}\|\psi\|\cdot\|\varphi\|\int_0^\infty dt\, t^{-1+(\nu-\gamma)/2}e^{-(\lambda-\omega)t}\\
&\qquad\leq a_{\gamma,\lambda}\|\psi\|\cdot\|\varphi\|
\end{aligned}$$

for all $\varphi,\psi\in\mathcal{H}_\gamma$ where $a_{\gamma,\lambda}$ is finite for all large λ whenever $\nu>\gamma$. It follows immediately that $c\varphi\in\mathcal{H}_\gamma$ and

$$\|c\varphi\|_\gamma\leq\|c\|_{\mathcal{H}}\|\varphi\|_\gamma+a_{\gamma,\lambda}\|\varphi\|.$$

Thus c is bounded on $\mathcal{H}_\gamma$. Since T is self-adjoint the bounds (6) are also valid for c^* and then c^* is bounded on $\mathcal{H}_\gamma$ by the same argument.

Proof of Theorem 1. We first prove the theorem for the principal part H_0 of H and subsequently extend the result to H by an interpolation–perturbation argument.

First, since

$$D(H_0^{1/2})=[\mathcal{H},D(H_0)]_{1/2}=[\mathcal{H},D(I+H_0)]_{1/2}=D((I+H_0)^{1/2})$$

it suffices to establish the result for the operator $I+H_0$.

Secondly, fix $\psi\in D(H_0^*)\subseteq\mathcal{H}_1$ and $\varphi\in\mathcal{H}_{1+\gamma}\subset\mathcal{H}_1$. Then $(I+H_0^*)^{-(1-\gamma)/2}\psi\in D(H_0^*)\subseteq\mathcal{H}_1$ and

$$\begin{aligned}
&((I+H_0^*)^{(1+\gamma)/2}\psi,\varphi)\\
&\quad=((I+H_0^*)(I+H_0^*)^{-(1-\gamma)/2}\psi,\varphi) \qquad (7)\\
&\quad=\sum_{i,j=1}^n(A_i(I+H_0^*)^{-(1-\gamma)/2}\psi, c_{ij}A_j\varphi)+((I+H_0^*)^{-(1-\gamma)/2}\psi,\varphi).
\end{aligned}$$

Now we aim to bound the terms on the right hand side of (8) by use of the Sobolev norms $\|\cdot\|_{-\gamma}$ and $\|\cdot\|_\gamma$. This estimation is based on the following observation.

Lemma 2 *The A_i, $i\in\{1,\ldots,n\}$, are bounded operators from $\mathcal{H}_{1+\delta}$ to $\mathcal{H}_\delta$ for each $\delta\in[-1,1]$.*

Proof. The space $\mathcal{K}=\bigcap_{i,j=1}^n D(A_iA_j)$ equipped with the norm

$$\varphi\mapsto\sup_{1\leq i,j\leq n}\|A_iA_j\varphi\|+\sup_{1\leq i\leq n}\|A_i\varphi\|+\|\varphi\|$$

is a Banach space since all the operators A_i are closed. Then the regularity hypothesis (5) gives the set inclusion $D(I+L) \subseteq \mathcal{K}$. But since $D(I+L)$ is a Banach space with respect to the norm $\varphi \mapsto \|(I+L)\varphi\|$ it follows from the closed graph theorem that the inclusion is continuous, i.e., there exists an $a > 0$ such that

$$\sup_{1\leq i,j\leq n} \|A_i A_j \varphi\| + \sup_{1\leq i\leq n} \|A_i\varphi\| + \|\varphi\| \leq a\|(I+L)\varphi\|$$

for all $\varphi \in D(L)$. These bounds, together with the Kato identity (2), imply that one has bounds $\|A_i\varphi\|_1 \leq a'\|\varphi\|_2$ for all $\varphi \in \mathcal{H}_2$ and $i \in \{1,\ldots,n\}$. Hence the A_i are bounded operators from $\mathcal{H}_2$ into $\mathcal{H}_1$. On the other hand $D((I+L)^{1/2}) \subseteq D(A_i)$ and hence the operators $A_i(I+L)^{-1/2}$ are bounded on $\mathcal{H}$. By duality the operators $(I+L)^{-1/2}A_i$ extend to bounded operators on $\mathcal{H}$ and therefore the A_i are bounded from $\mathcal{H}$ into $\mathcal{H}_{-1}$. Since the $\mathcal{H}_\gamma$ form a scale of complex interpolation spaces the statement of the lemma follows by interpolation. □

Now we return to the estimation of the right hand side of (8).

Since $\gamma \in \langle 0,1\rangle$ it follows that $\delta = (1-\gamma)/2 \in \langle 0,1/2\rangle$ and $(I+H_0^*)^{-\delta}$ is a continuous operator from $\mathcal{H}$ into $D((I+H_0^*)^\delta)$. But by [Kat1], Theorem 3.1, and complex interpolation one deduces that

$$\begin{aligned} D((I+H_0^*)^\delta) &= D((\mathrm{Re}(I+H_0^*))^\delta) \\ &= [\mathcal{H}, D((\mathrm{Re}(I+H_0^*))^{1/2})]_{2\delta} = [\mathcal{H},\mathcal{H}_1]_{2\delta} = \mathcal{H}_{2\delta}, \end{aligned}$$

since $\mathrm{Re}(I+H_0^*)$ is self-adjoint. Therefore $(I+H_0^*)^{-(1-\gamma)/2}$ is a continuous operator from $\mathcal{H}$ into $\mathcal{H}_{1-\gamma}$. Then by Lemma 2 the $A_i(I+H_0^*)^{-(1-\gamma)/2}$ are continuous operators from $\mathcal{H}$ into $\mathcal{H}_{-\gamma}$. Thus one has bounds

$$\|A_i(I+H_0^*)^{-(1-\gamma)/2}\psi\|_{-\gamma} \leq a\|\psi\|$$

for all $\psi \in \mathcal{H}$. Alternatively, by Lemma 2, the A_j are continuous operators from $\mathcal{H}_{1+\gamma}$ into $\mathcal{H}_\gamma$ and, by assumption, the c_{ij} are continuous on $\mathcal{H}_\gamma$. Therefore one has bounds

$$\|c_{ij}A_j\varphi\|_\gamma \leq a'\|\varphi\|_{1+\gamma}$$

for all $\varphi \in \mathcal{H}_{1+\gamma}$. Then (8) gives

$$\begin{aligned}|((I+H_0^*)^{(1+\gamma)/2}\psi,\varphi)| &\le \sum_{i,j=1}^{n} \|A_i(I+H_0^*)^{-(1-\gamma)/2}\psi\|_{-\gamma}\cdot\|c_{ij}A_j\varphi\|_\gamma \\ &\qquad + \|(I+H_0^*)^{-(1-\gamma)/2}\psi\|\cdot\|\varphi\| \\ &\le b\|\psi\|\cdot\|\varphi\|_{1+\gamma}\end{aligned}$$

for some $b > 0$ and all $\psi \in D(H_0^*)$ and $\varphi \in \mathcal{H}_{1+\gamma}$. Here we have used the foregoing estimates and the bounds $\|(I+H_0^*)^{-(1-\gamma)/2}\psi\| \le \|\psi\|$ and $\|\varphi\| \le \|\varphi\|_{1+\gamma}$. Since $D(H_0^*)$ is a core of $(I+H_0^*)^{(1+\gamma)/2}$ one concludes that $\mathcal{H}_{1+\gamma} \subseteq D((I+H_0)^{(1+\gamma)/2})$ and

$$\|(I+H_0)^{(1+\gamma)/2}\varphi\| \le b\|\varphi\|_{1+\gamma} \tag{8}$$

for all $\varphi \in \mathcal{H}_{1+\gamma}$. But then

$$\begin{aligned}\mathcal{H}_1 &= [\mathcal{H},\mathcal{H}_{1+\gamma}]_{1/(1+\gamma)} \subseteq [\mathcal{H}, D((I+H_0)^{(1+\gamma)/2})]_{1/(1+\gamma)} \\ &= D((I+H_0)^{1/2}).\end{aligned}$$

Now H_0^* is an operator analogous to H_0, with c_{ij} replaced by c_{ji}^*, and the same arguments apply. Therefore one has two inclusions

$$\mathcal{H}_1 \subseteq D((I+H_0)^{1/2}), \quad \mathcal{H}_1 \subseteq D((I+H_0^*)^{1/2}).$$

But $I + H_0$ is a closed maximal accretive operator associated with a form whose domain is $\mathcal{H}_1$. Therefore, from the result of Kato cited in the introduction, [Kat2], Theorem 1, one concludes that $\mathcal{H}_1 = D((I+H_0)^{1/2}) = D((I+H_0^*)^{1/2})$. Thus the desired identities are established for the principal part of H. Next consider the addition of lower order terms.

First let h_1 denote the form obtained from h by setting $c_i' = 0$ and H_1 the corresponding closed sectorial operator. Then

$$h_1(\psi,\varphi) = h_0(\psi,\varphi) + (\psi, V\varphi)$$

for all $\psi,\varphi \in \mathcal{H}_1$ where h_0 is the form associated with the principal part H_0 and V is the operator $\sum_{i=1}^n c_iA_i + c_0$ with $D(V) = \mathcal{H}_1$. But $D(H_1)$ consists of those $\varphi \in \mathcal{H}_1$ for which there is an $a > 0$ such that $|h_1(\psi,\varphi)| \le a\|\psi\|$ for all $\psi \in \mathcal{H}_1$. The domain $D(H_0)$ is defined similarly, relative to h_0. It follows immediately that $D(H_1) = D(H_0)$. Hence

$$\begin{aligned}D((\lambda I + H_1)^{1/2}) &= [\mathcal{H}, D(H_1)]_{1/2} \\ &= [\mathcal{H}, D(H_0)]_{1/2} = D((\lambda I + H_0)^{1/2})\end{aligned}$$

for sufficiently large λ. Therefore, by the foregoing, one has $D((\lambda I + H_1)^{1/2}) = \mathcal{H}_1$. But $K = \lambda I + H_1$ is a closed maximal accretive operator corresponding to a form k with $D(k) = \mathcal{H}_1$. Thus $D(K^{1/2}) = D(k)$ and again invoking [Kat2], Theorem 1, one concludes that $D(K^{*1/2}) = D(k)$. Therefore $D((\lambda I + H_1^*)^{1/2}) = \mathcal{H}_1$.

Finally

$$h^*(\varphi, \psi) = h_1^*(\varphi, \psi) - \sum_{i=1}^{n} (c_i' \varphi, A_i \psi)$$

for all $\varphi, \psi \in \mathcal{H}_1$ where h^* and h_1^* are the forms associated with H^* and H_1^*, respectively. Then repetition of the foregoing argument gives $D((\lambda I + H^*)^{1/2}) = D((\lambda I + H_1^*)^{1/2}) = \mathcal{H}_1$ and another application of Theorem 1 in [Kat2] yields $D((\lambda I + H)^{1/2}) = \mathcal{H}_1$. □

Theorem 1 has a simple implication for subelliptic operators associated with a unitary representation of a Lie group because the basic regularity properties are a direct consequence of unitarity.

Let $(\mathcal{H}, G, U)$ denote a representation of the Lie group G by unitary operators $g \mapsto U(g)$ on the Hilbert space $\mathcal{H}$. Further let $a_1, \ldots, a_n$ be elements of the Lie algebra $\mathfrak{g}$ of G. Denote the skew-adjoint generators of the one-parameter unitary groups $t \mapsto U(\exp(-ta_i))$ by $A_1, \ldots, A_n$, i.e, the A_i are the representatives of the a_i in the derived representation of the Lie algebra. Then the C^1-subspace $\mathcal{H}_1$ corresponding to the A_i is automatically dense in $\mathcal{H}$ because it contains the dense subspace of C^∞-elements of the representation. Moreover, if the $a_1, \ldots, a_n$ form a vector space basis of $\mathfrak{g}$ then the regularity property (5) is a result of Nelson [Nel] (a simple proof is given in [Rob], Section I.6, page 53). More generally, if the $a_1, \ldots, a_n$ are a Lie algebraic basis of $\mathfrak{g}$ then (5) is established in [ElR], Theorem 7.2.IV. In light of these observations one has the following conclusion.

Corollary 3 *Let $(\mathcal{H}, G, U)$ denote a unitary representation of a Lie group G and $A_1, \ldots, A_n$ the skew-adjoint representatives of an algebraic basis $a_1, \ldots, a_n$ of the Lie algebra of G. Let H be the closed sectorial operator associated with the subelliptic form* (4) *and the A_i and suppose the c_{ij} and c_{ij}^* are bounded operators on the Sobolev space $\mathcal{H}_\gamma = D((I + L)^{\gamma/2})$ for some $\gamma \in \langle 0, 1 \rangle$.*

Then $D((\lambda I + H)^{1/2}) = D((\lambda I + H^*)^{1/2}) = \mathcal{H}_1$ for all large $\lambda \in \mathbf{R}$.

Again it is worth noting that the assumption that a bounded operator c on $\mathcal{H}$ is also bounded on $\mathcal{H}_\gamma$ is a type of Hölder continuity. It follows, for example, if c satisfies bounds

$$\|U(g)cU(g)^{-1} - c\| \le a|g|^\nu \tag{9}$$

for some $\nu > \gamma$ and all $g \in G$ with $|g| \le 1$ where $|\cdot|$ denotes the subelliptic distance to the identity element corresponding to the basis $a_1, \ldots, a_n$ (see, for example, [Rob], Section IV.4). These bounds imply the boundedness of c and c^* on $\mathcal{H}_\gamma$ by the following reasoning.

The action of T, the semigroup generated by L on $\mathcal{H}$, is given by a kernel K,

$$T_t = \int_G dg K_t(g)U(g)$$

where dg denotes left invariant Haar measure. This kernel is positive and satisfies Gaussian bounds

$$0 \le K_t(g) \le at^{-D/2}e^{\omega t}e^{-b|g|^2t^{-1}} \tag{10}$$

with D the local subelliptic dimension (see [Rob], Section IV.4). Therefore

$$T_tc - cT_t = \int_G dg K_t(g)(U(g)c - cU(g))$$

and the bounds (9), which extend to all $g \in G$, together with (10), immediately give estimates

$$\|T_tc - cT_t\| \le at^{\nu/2} \int_G dg\, t^{-D/2}e^{\omega t}e^{-b|g|^2t^{-1}}(|g|^2t^{-1})^{\nu/2}.$$

The integral, however, is bounded by a factor $a'e^{\omega' t}$ and hence one concludes that

$$\|T_tc - cT_t\| \le at^{\nu/2}e^{\omega t}$$

for some $a > 0$, $\omega \ge 0$ and all $t > 0$. Then the boundedness of c and c^* on $\mathcal{H}_\gamma$ for each $\gamma \in \langle 0, \nu\rangle$ follows from the discussion following Theorem 1.

Thus if the c_{ij} are operators which act by multiplication by Hölder continuous functions then the corollary applies. This is a general Lie group version of McIntosh's result [McI1] for $\mathbf{R}^n$. But if one specializes to Euclidean space one can draw more general conclusions. For example, let $\mathcal{H} = L_2(\Omega; dx)$ for some open set $\Omega \subseteq \mathbf{R}^n$ and set $A_i = \partial_i = \partial/\partial x_i$, the partial differential operators with Dirichlet boundary conditions. Then L is the Dirichlet Laplacian and the regularity property (5) is valid. Therefore Theorem 1 applies to Dirichlet operators

$$H = -\sum_{i,j=1}^{n} \partial_i c_{ij} \partial_j + \sum_{i=1}^{n} (c_i \partial_i + \partial_i c_i') + c_0$$

with coefficients in the bounded operators on $L_2(\Omega; dx)$. The only restraints are the ellipticity condition and the 'Hölder continuity' on the principal coefficients c_{ij}. The theorem also has applications to operators on other manifolds as long as the C^2-regularity condition (5) is satisfied.

There is one natural question which is not resolved by the foregoing arguments.

It follows from [Kat1], Theorem 3.1 that for each subelliptic operator H given by a subelliptic form (4) one has

$$D((\lambda I + H)^{\alpha}) = \mathcal{H}_{2\alpha}$$

for all large λ and all $\alpha \in [0, 1/2\rangle$. This conclusion does not need any regularity of the coefficients c_{ij} or the Laplacian. But Theorem 1 establishes that the regularity condition (5) together with the boundedness of the c_{ij} and c_{ij}^* on $\mathcal{H}_\gamma$ ensures the stronger conclusion

$$D((\lambda I + H)^{1/2}) = \mathcal{H}_1 .$$

In the course of the proof, however, we also deduced in (8) that

$$D((\lambda I + H)^{\alpha}) \supseteq \mathcal{H}_{2\alpha} \tag{11}$$

for all $\alpha \in \langle 1/2, (1+\gamma)/2]$ and H a pure second-order subelliptic operator satisfying the assumptions of Theorem 1. On the other hand, if H is an operator with $c_i = 0$ for all i but the c_i' are possibly non-zero then the additional terms in (8) can be dealt with as before. So (11) is valid for all operators with the c_i equal to zero. But then one can add the terms $c_i A_i$ by the perturbation–interpolation argument. Thus one arrives at the following conclusion.

Proposition 4 *Assume the regularity inclusion*

$$D(L) \subseteq \bigcap_{i,j=1}^{n} D(A_i A_j) .$$

Let H be the closed sectorial operator associated with the subelliptic form (4) and suppose the c_{ij} are bounded operators on the Sobolev space $\mathcal{H}_\gamma = D((I+L)^{\gamma/2})$ for some $\gamma \in \langle 0, 1 \rangle$. Then

$$D((\lambda I + H)^\alpha) \supseteq \mathcal{H}_{2\alpha}$$

for all large $\lambda \in \mathbf{R}$ and all $\alpha \in [0, (1+\gamma)/2]$, with equality if $\alpha \in [0, 1/2]$.

It is, however, unclear whether the hypotheses of Theorem 1 imply that these the containments are identities for $\alpha > 1/2$. Probably some additional regularity is required.

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Note Added in Proof After this paper was completed we learned that the interpolation–perturbation argument used to complete the proof of Theorem 1 occurred earlier in a paper of P. Auscher and P. Tchamitchian, Rev. Ibero. Mat. **8** (1992) 149–199.

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