Hokkaido Mathematical Journal Vol. 26 (1997) p. 323–335

First variation of holomorphic forms and some applications

Bahman KHANEDANI and Tatsuo SUWA

(Received April 12, 1996)

Abstract. We study various local invariants associated with a singular holomorphic foliation on a complex surface admitting a possibly singular invariant curve. We establish the relation among them and prove/reprove formulas relating the total sum of these invariants to some global invariants of the foliation and the invariant curve.

Key words: singular holomorphic foliations, invariant curves, indices.

For a holomorphic vector field v on a complex surface leaving a non-singular curve C invariant, C. Camacho and P. Sad [CS] introduced the index of v relative to C and proved an index formula, which says that the total sum of the indices is equal to the Chern number of the normal bundle of C. After the work of a number of authors, the theory has been generalized to the case of singular invariant curves in [S], and further, to the higher dimensional case in [LS]. In [S], the index formula was proved by taking desingularization of the curve and reducing to the case of non-singular invariant curves, while the proof in [LS] involves the Chern-Weil theory, the vanishing theorem and so forth. In this article, we first give a direct proof of the index theorem for a singular foliation $\mathcal{F}$ on a complex surface leaving a (possibly singular) compact curve C invariant by explicitly computing the Chern class of the normal bundle of C (Theorem 1.2).

We then consider "exponent forms" for holomorphic 1-forms defining the foliation $\mathcal{F}$ and define the "variation" of $\mathcal{F}$ relative to C at a singular point as the residue of an exponent form along the link of the singularity in C. This turns out to be a localized class of the (co)normal bundle of the foliation (Theorem 2.2). We extend the notion of the "multiplicity" of a vector field v along a (locally) irreducible invariant curve [CLS] to the case of possibly reducible curves so that it coincides with the "Schwartz index" [SS] of the restriction of v to the curve. After establishing the relation among

---

1991 Mathematics Subject Classification : Primary 14B05, 14H20, 32L30, 32S65; Secondary 57R20, 58F18.

these invariants in Lemma 2.3, we give a formula for the total sum of the (Schwartz) indices in Theorem 2.6, which is the "Poincaré-Hopf theorem" for a singular foliation, with possibly non-trivial tangent bundle, on a singular curve.

In the final section, we discuss the geometric meaning of the variation and give an alternative proof of the fact that the index of $\mathcal{F}$ relative to C represents the first order term of the holonomy along the link of the singularity in C, which was shown earlier in [S].

The first named author would like to thank S. Shahshahani for encouragement and advice and the Institute for Studies in Theoretical Physics and Mathematics for financial support. The second named author would also like to thank S. Shahshahani for useful conversations.

1. The index formula

We generally use the notation and the definitions in [S]. First we consider everything in a neighborhood of the origin 0 in $\mathbb{C}^2 = \{(x, y)\}$. Let v be a germ of holomorphic vector field at 0 with (at most) an isolated singularity at 0 and ω a germ of holomorphic 1-form with an isolated singularity at 0 which annihilates v. More explicitly, if $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ with a and b germs of holomorphic functions at 0, we may set $\omega = b\,dx - a\,dy$. Also, let C be a germ of reduced curve with defining function f. We quote Lemma (1.1) in [S]:

Lemma 1.1 *The vector field v leaves C invariant if and only if there exist germs of holomorphic functions g and h and a germ of holomorphic 1-form η such that h and f are relatively prime and that*

$$g\omega = hdf + f\eta. \tag{1.1}$$

The lemma is proved in [Li] when f is irreducible. Note that if ω is non-singular at 0, C is also non-singular at 0 and, by a suitable choice of f, we may set $\eta = 0$. Denoting by $\mathcal{F}$ the foliation defined by v (or ω), we define the index of $\mathcal{F}$ relative to C at 0 by

$$\mathrm{Ind}_0(\mathcal{F}; C) = \frac{\sqrt{-1}}{2\pi} \int_L \frac{\eta}{h},$$

where L denotes the link of the singularity 0 in C with natural orientation. When f is irreducible, this coincides with the one defined in [Li]. See [S]

Proposition (1.4) for their relation in the general case.

Now let X be a (non-singular) complex surface. Recall that a (co)dimension one (singular) foliation $\mathcal{F}$ on X is defined by a system $\{(U_\lambda, \omega_\lambda, \varphi_{\lambda\mu})\}$, where

(i) $\{U_\lambda\}$ is an open covering of X,

(ii) for each λ, ω_λ is a (not identically zero) holomorphic 1-form on U_λ and

(iii) for each pair (λ, μ), $\varphi_{\lambda\mu}$ is a non-vanishing holomorphic function on $U_\lambda \cap U_\mu$ with $\omega_\mu = \varphi_{\lambda\mu}\omega_\lambda$.

The singular set $S(\mathcal{F})$ of $\mathcal{F}$ is defined to be the union of the singular sets of the ω_λ's. We assume that $S(\mathcal{F})$ consists of isolated points hereafter.

Theorem 1.2 *For a (co)dimension one foliation $\mathcal{F}$ on X and a compact reduced curve C in X which is invariant by $\mathcal{F}$, we have*

$$\sum_{p\in S} \operatorname{Ind}_p(\mathcal{F}; C) = C \cdot C,$$

where S denotes the set of singular points of $\mathcal{F}$ on C and $C \cdot C$ the self-intersection number of C.

This is proved in [S] Theorem (2.1) and the higher dimensional case is in [LS]. Here we give a simple direct proof.

Proof. We let $S = \{p_1, \dots, p_s\}$ and take a system $\{(U_\lambda, \omega_\lambda, \varphi_{\lambda\mu})\}$ as above so that it further satisfies:

(iv) C is defined by f_λ on U_λ,

(v) for each p_i, there is only one U_{λ_i} with $p_i \in U_{\lambda_i}$ and $U_{\lambda_i} \cap U_{\lambda_j} = \varnothing$, if $i \neq j$.

If we set $f_{\lambda\mu} = \frac{f_\lambda}{f_\mu}$ on $U_\lambda \cap U_\mu$, then the cocycle $\{f_{\lambda\mu}\}$ defines the line bundle L_C on X associated with the divisor C. We compute $c_1(L_C) \frown [C] = \int_C c_1(L_C)$ in two ways. First, since $c_1(L_C)$ is the Poincaré dual to the homology class [C], we see that it is equal to the self-intersection number $C \cdot C$. Next we compute it directly. If we let $\{\rho_\lambda\}$ be a partition of unity subordinate to $\{U_\lambda\}$, we have

$$c_1(L_C)|_{U_\lambda} = \frac{\sqrt{-1}}{2\pi} \sum_\mu d(\rho_\mu \, d\log f_{\mu\lambda}).$$

On each U_λ, we have a decomposition

$$g_\lambda \omega_\lambda = h_\lambda \, df_\lambda + f_\lambda \eta_\lambda \tag{1.1$_\lambda$}$$

as (1.1). We may assume that $\eta_\lambda = 0$ for $\lambda \neq \lambda_i$. Evaluation of the both sides of the identity (1.1_λ) at each point of $U_\lambda \cap C$ gives

$$g_\lambda \omega_\lambda = h_\lambda \, df_\lambda. \tag{1.2_λ}$$

Also, from $dg_\lambda \wedge \omega_\lambda + g_\lambda d\omega_\lambda = (dh_\lambda - \eta_\lambda) \wedge df_\lambda + f_\lambda \, d\eta_\lambda$ and (1.2_λ), we have, at each point of $U_\lambda \cap C$,

$$d\omega_\lambda = \left(-\frac{\eta_\lambda}{h_\lambda} + d\log \frac{h_\lambda}{g_\lambda} \right) \wedge \omega_\lambda. \tag{1.3_λ}$$

From (1.2_λ) and (1.2_μ), we have, in $U_\lambda \cap U_\mu \cap C$,

$$\frac{h_\mu}{g_\mu} = f_{\lambda\mu} \varphi_{\lambda\mu} \frac{h_\lambda}{g_\lambda}. \tag{1.4}$$

Also, from (1.3_λ) and (1.3_μ), we have, in $U_\lambda \cap U_\mu \cap C$,

$$d\log \varphi_{\lambda\mu} = \frac{\eta_\lambda}{h_\lambda} - \frac{\eta_\mu}{h_\mu} + d\log \frac{h_\mu}{g_\mu} - d\log \frac{h_\lambda}{g_\lambda}. \tag{1.5}$$

Hence from (1.4) and (1.5), we have, at each point of $U_\lambda \cap U_\mu \cap C$,

$$d\log f_{\mu\lambda} = \frac{\eta_\lambda}{h_\lambda} - \frac{\eta_\mu}{h_\mu}. \tag{1.6}$$

Let $C' = C - \mathrm{Sing}(C)$ be the set of regular points of C (note that $\mathrm{Sing}(C) \subset S$). Then, from (1.6), we have

$$c_1(L_C)|_{U_\lambda \cap C'} = \frac{\sqrt{-1}}{2\pi} \sum_\mu d\rho_\mu \wedge \left(\frac{\eta_\lambda}{h_\lambda} - \frac{\eta_\mu}{h_\mu} \right) = -\frac{\sqrt{-1}}{2\pi} \sum_\mu d\rho_\mu \wedge \frac{\eta_\mu}{h_\mu}.$$

Since $\eta_\lambda = 0$ for $\lambda \neq \lambda_i$, we have

$$\int_C c_1(L_C) = \int_{C'} c_1(L_C) = \sum_{i=1}^{s} \int_{U_{\lambda_i} \cap C'} c_1(L_C).$$

We denote by D_{λ_i} a disk in U_{λ_i} with center p_i such that $\rho_{\lambda_i} \equiv 1$ on D_{λ_i}. Note that $\partial D_{\lambda_i} \cap C = L_{\lambda_i}$, the link of C at p_i. Then we have

$$\begin{aligned} \int_{U_{\lambda_i} \cap C'} c_1(L_C) &= -\frac{\sqrt{-1}}{2\pi} \int_{U_{\lambda_i} \cap C'} d\rho_{\lambda_i} \wedge \frac{\eta_{\lambda_i}}{h_{\lambda_i}} \\ &= -\frac{\sqrt{-1}}{2\pi} \int_{(U_{\lambda_i} - D_{\lambda_i}) \cap C'} d\rho_{\lambda_i} \wedge \frac{\eta_{\lambda_i}}{h_{\lambda_i}} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{\sqrt{-1}}{2\pi}\int_{(U_{\lambda_i}-D_{\lambda_i})\cap C'} d\left(\rho_{\lambda_i}\frac{\eta_{\lambda_i}}{h_{\lambda_i}}\right) \\
&= \frac{\sqrt{-1}}{2\pi}\int_{L_{\lambda_i}} \rho_{\lambda_i}\frac{\eta_{\lambda_i}}{h_{\lambda_i}} \\
&= \frac{\sqrt{-1}}{2\pi}\int_{L_{\lambda_i}} \frac{\eta_{\lambda_i}}{h_{\lambda_i}} = \mathrm{Ind}_{p_i}(\mathcal{F}; C).
\end{aligned}
$$

□

2. Exponent forms

Suppose $\mathcal{F}$ is a germ of foliation at 0 in $\mathbb{C}^2$ with defining 1-form ω (or vector field v) and C a germ of reduced curve with defining function f which is invariant by $\mathcal{F}$. In a neighborhood of a non-singular point, there exists a holomorphic 1-form α such that $d\omega = \alpha \wedge \omega$. If α' is another such 1-form, we have $\alpha' \equiv \alpha$ on every leaf. Thus in a neighborhood of 0 (away from 0) there exists a holomophic multi-valued 1-form α such that $d\omega = \alpha \wedge \omega$ and that its restriction to each leaf is single-valued. We call α an *exponent form* for ω. We consider the residue of α along C;

$$
\mathrm{Res}_0(\alpha|_C) = \frac{1}{2\pi\sqrt{-1}}\int_L \alpha,
$$

where L is the link of 0 in C as before.

Lemma 2.1 *The residue* $\mathrm{Res}_0(\alpha|_C)$ *is an invariant of the foliation.*

Proof. Suppose $\omega' = \varphi\omega$ with φ a non-vanishing holomorphic function. We have

$$
d\omega' = d\varphi \wedge \omega + \varphi d\omega = d\varphi \wedge \omega + \varphi\alpha \wedge \omega = (\alpha + d\log\varphi) \wedge \omega'.
$$

Since φ is non-vanishing, we obtain $\int_L(\alpha + d\log\varphi) = \int_L \alpha$. □

In view of the above lemma, we set

$$
\mathrm{Var}_0(\mathcal{F}; C) = \mathrm{Res}_0(\alpha|_C)
$$

and call it the *variation* of $\mathcal{F}$ relative to C at 0. Note that if $C = \bigcup_{i=1}^r C_i$ is the irreducible decomposition of C at 0, $\mathcal{F}$ leaves each component C_i

invariant and we have

$$\mathrm{Var}_0(\mathcal{F};C)=\sum_{i=1}^{r}\mathrm{Var}_0(\mathcal{F};C_i). \tag{2.1}$$

Now we go back to the global situation as in Theorem 1.2 and suppose the foliation $\mathcal{F}$ is defined on a complex surface X by a system $\{(U_\lambda,\omega_\lambda,\varphi_{\lambda\mu})\}$. Let T^*X denote the (holomorphic) cotangent bundle of X and F the line bundle defined by the cocycle $\{\varphi_{\lambda\mu}\}$. Then we have a bundle map on X;

$$F\xrightarrow{\omega}T^*X,$$

which is injective on $X-S(\mathcal{F})$. We call F the conormal bundle of the foliation $\mathcal{F}$.

Theorem 2.2 *In the above situation, if C is a compact curve in X invariant by $\mathcal{F}$, we have*

$$\sum_{p\in S}\mathrm{Var}_p(\mathcal{F};C)=-c_1(F)\frown[C].$$

Proof. Take a system $\{(U_\lambda,\omega_\lambda,\varphi_{\lambda\mu})\}$ defining $\mathcal{F}$ so that it satisfies also (iv) and (v) in the proof of Theorem 1.2. Let α_λ be an exponent form for ω_λ. For $\lambda\neq\lambda_i$, we may set $\alpha_\lambda=0$, since we may choose a closed form as ω_λ. As in Theorem 1.2, we have

$$c_1(F)|_{U_\lambda}=\frac{\sqrt{-1}}{2\pi}\sum_{\mu}d(\rho_\mu\,d\log\varphi_{\mu\lambda}).$$

In $U_\lambda\cap U_\mu\cap C$, we have

$$d\log\varphi_{\lambda\mu}=\alpha_\lambda-\alpha_\mu$$

and the rest is done similarly as for Theorem 1.2. □

Let C be a germ of reduced curve at 0 in $\mathbb{C}^2$ invariant by a foliation $\mathcal{F}$ defined by v. If C is irreducible, then one defines, following [CLS], the *multiplicity* of v along C at 0 to be the topological index of $v|_C$ at 0, where C is seen as being homeomorphic to a two dimensional disk. Since it is also an invariant of the foliation $\mathcal{F}$, we denote it by $\mathrm{Ind}_0(\mathcal{F}_C)$. In general, let $C=\bigcup_{i=1}^r C_i$ be the irreducible decomposition of C at 0. We define

$\mathrm{Ind}_0(\mathcal{F}_C)$ by

$$\mathrm{Ind}_0(\mathcal{F}_C) = \sum_{i=1}^{r} \mathrm{Ind}_0(\mathcal{F}_{C_i}) - r + 1 \tag{2.2}$$

and call it the *index* of the restriction of $\mathcal{F}$ to C at 0. Note that it coincides with the "Schwartz index" of $v|_C$ at 0 in the sense of [SS]. Recall that the Milnor number $\mu_0(C)$ of C at 0 is given by $\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]_0$, the intersection number of the curves defined by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at 0.

Lemma 2.3 *We have*

$$\mathrm{Ind}_0(\mathcal{F}_C) = \mathrm{Var}_0(\mathcal{F}; C) - \mathrm{Ind}_0(\mathcal{F}; C) + \mu_0(C).$$

Proof. First we prove the lemma when C is irreducible. If we take a decomposition as in Lemma 1.1, at each point of C we have (see (1.3))

$$d\omega = \left(-\frac{\eta}{h} + d\log\frac{h}{g}\right) \wedge \omega.$$

Hence we get

$$\mathrm{Var}_0(\mathcal{F}; C) = \mathrm{Ind}_0(\mathcal{F}; C) + [h, f]_0 - [g, f]_0. \tag{2.3}$$

Now, by a suitable choice of coordinates (x, y) of $\mathbb{C}^2$, we may set $g = \frac{\partial f}{\partial y}$ and $h = -a$, when we write $v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ (see the proof of Lemma (1.1) in [S]). By [CLS] Proposition 3, $\mathrm{Ind}_0(\mathcal{F}_C)$ is computed as follows. Let $\pi : (D, 0) \to (C, 0)$ be a Puiseux parametrization. Then the vector field V in $D = \{t\}$ with $\pi_* V = v|_C$ is given by $V = \frac{a}{\dot{x}}\frac{d}{dt}$, $\dot{x} = \frac{dx}{dt}$. Thus

$$\mathrm{Ind}_0(\mathcal{F}_C) = [h, f]_0 - [x, f]_0 + 1. \tag{2.4}$$

On the other hand, we know from [Li] (8) that

$$\mu_0(C) = \left[\frac{\partial f}{\partial y}, f\right]_0 - [x, f]_0 + 1. \tag{2.5}$$

and the formula follows from (2.3), (2.4) and (2.5). Next, in general, if $C = \bigcup_{i=1}^{r} C_i$ is the irreducible decomposition of C, we have ([S] (1.11))

$$\mathrm{Ind}_0(\mathcal{F}; C) - \mu_0(C) = \sum_{i=1}^{r} (\mathrm{Ind}_0(\mathcal{F}; C_i) - \mu_0(C_i)) + r - 1.$$

Hence the lemma follows from the formula for the irreducible case together with (2.1) and (2.2). □

Remark 2.4. Let $\mathcal{F}^\circ$ be the foliation defined by df. Then, since we may set $\alpha = 0$ we have $\mathrm{Var}_0(\mathcal{F}^\circ; C) = 0$. Also, since we may set $\eta = 0$ in (1.1), we have $\mathrm{Ind}_0(\mathcal{F}^\circ; C) = 0$ and $\mathrm{Ind}_0(\mathcal{F}^\circ; C_i) = -\sum_{j \neq i}(C_i \cdot C_j)_0$ ([S] Proposition (1.4). Note that $\mathrm{Ind}_0(\mathcal{F}^\circ; C, C_i) = 0$ in the notation used there). Thus, by Lemma 2.3, we have

$$\mathrm{Ind}_0(\mathcal{F}^\circ_C) = \mu_0(C) \quad \text{and} \quad \mathrm{Ind}_0(\mathcal{F}^\circ_{C_i}) = \mu_0(C_i) + \sum_{j \neq i}(C_i \cdot C_j)_0.$$

The first equality also follows from the fact that the vector field defining $\mathcal{F}^\circ$ is tangent to the nearby Milnor fibers of f and has no singularities on the fiber ([SS] Proposition 5.3). The second equality shows that $\mathrm{Ind}_0(\mathcal{F}^\circ_{C_i})$ coincides with $c_0(C, C_i)$ in [S] (1.8). If we set $c_0(C) = \sum_{i=1}^r c_0(C, C_i)$, it is related to the Milnor number by $c_0(C) = \mu_0(C) + r - 1$ ([S] (1.9)).

The above remark may be used to prove the "adjunction formula" as follows, although we should note that the argument is essentially equivalent to the one in [K]. Let C be a compact (reduced) curve in a surface X. We take a covering $\{U_\lambda\}$ of X by coordinate neighborhoods with coordinates (x_λ, y_λ) so that C is defined by $f_\lambda = 0$ in U_λ. Let $\mathcal{F}^\circ_\lambda$ be the foliation on U_λ defined by df_λ. Then it is defined by the vector field $v_\lambda = \frac{\partial f_\lambda}{\partial y_\lambda}\frac{\partial}{\partial x_\lambda} - \frac{\partial f_\lambda}{\partial x_\lambda}\frac{\partial}{\partial y_\lambda}$. By computation, we see that, in $U_\lambda \cap U_\mu \cap C$,

$$v_\lambda = f_{\lambda\mu}\kappa_{\lambda\mu}v_\mu,$$

where $\kappa_{\lambda\mu} = \det \frac{\partial(x_\mu, y_\mu)}{\partial(x_\lambda, y_\lambda)}$, the Jacobian of (x_μ, y_μ) with respect to (x_λ, y_λ). Thus, if we let $\pi : \tilde{C} \to C \subset X$ be a resolution of C, the collection $\{v_\lambda|_C\}$ determines a section of the line bundle $\pi^*(L_C \otimes K_X) \otimes T\tilde{C}$, where K_X denotes the canonical bundle of X and $T\tilde{C}$ the tangent bundle of $\tilde{C}$. Hence from the second equality in Remark 2.4, we have the adjunction formula

$$\chi(\tilde{C}) = -K_X \cdot C - C \cdot C + \sum_{p \in S} c_p(C),$$

where $\chi(\tilde{C})$ denotes the Euler number of $\tilde{C}$ and $K_X \cdot C = c_1(K_X) \frown [C]$. Since the Euler number $\chi(C)$ of C is given by $\chi(C) = \chi(\tilde{C}) - \sum_{p \in S}(r_p - 1)$

with r_p the number of local branches of C at p, we have

$$\chi(C) = -K_X \cdot C - C \cdot C + \sum_{p \in S} \mu_p(C), \tag{2.6}$$

which is a special case of the formula in [SS] Theorem 5.5.

From Theorem 1.2 and (2.6), we have the following formula, which is a modified form of the one in [S] Theorem (2.5).

Theorem 2.5 *Let X, $\mathcal{F}$ and C be as in Theorem* 1.2. *We have*

$$\sum_{p \in S} (\mathrm{Ind}_p(\mathcal{F}; C) - \mu_p(C)) = -K_X \cdot C - \chi(C).$$

Now we recall that a foliation $\mathcal{F}$ on a complex surface X is also defined by a system $\{(U_\lambda, v_\lambda, \varepsilon_{\lambda\mu})\}$, where
(i) $\{U_\lambda\}$ is an open covering of X,
(ii)$'$ for each λ, v_λ is a (not identically zero) holomorphic vector field on U_λ and
(iii)$'$ for each pair (λ, μ), $\varepsilon_{\lambda\mu}$ is a non-vanishing holomorphic function on $U_\lambda \cap U_\mu$ with $v_\mu = \varepsilon_{\lambda\mu} v_\lambda$.

A system $\{(U_\lambda, \omega_\lambda, \varphi_{\lambda\mu})\}$ of 1-forms and a system $\{(U_\lambda, v_\lambda, \varepsilon_{\lambda\mu})\}$ of vector fields define the same foliation $\mathcal{F}$ if, for each λ, ω_λ and v_λ have isolated singularities and they annihilate each other. Suppose this is the case. Then the singular set $S(\mathcal{F})$ of $\mathcal{F}$ coincides with the union of the singular sets of the v_λ's. Let TX denote the tangent bundle of X and E the line bundle defined by the the cocycle $\{\varepsilon_{\lambda\mu}\}$. Then we have a bundle map on X;

$$E \xrightarrow{v} TX,$$

which is injective on $X - S(\mathcal{F})$. We call E the tangent bundle of the foliation $\mathcal{F}$. By a straightforward computation using the explicit relation between the forms and the vector fields defining $\mathcal{F}$, we have

$$F = E \otimes K_X.$$

Therefore, from Lemma 2.3 and Theorems 2.2 and 2.5, we have

Theorem 2.6 *For a foliation $\mathcal{F}$ on a complex surface X leaving a compact* *curve C invariant, we have*

$$\sum_{p\in S} \mathrm{Ind}_0(\mathcal{F}_C) = \chi(C) - c_1(E) \frown [C].$$

In particular, if $\mathcal{F}$ is defined by a global vector field, then, since E becomes trivial,

$$\sum_{p\in S} \mathrm{Ind}_0(\mathcal{F}_C) = \chi(C).$$

The second formula above is a special case of the Poincaré-Hopf theorem for singular varieties ([SS] Theorem 5.4). Also, when C is non-singular, the right hand side of the first formula above is equal to the Chern number of the normal sheaf of the foliation induced from $\mathcal{F}$ on C (cf. [BB]).

We finish this section with a remark on the topological invariance of some invariants associated with holomorphic foliations. Recall that the Milnor number is a topological invariant [Lê] and that the local intersection number of two analytic curves is also a topological invariant [GH]. We say that two foliations are topologically equivalent if there is a homeomorphism between the ambient spaces preserving the singular sets and the leaves. Let $\mathcal{F}$ be a foliation on a surface leaving a curve C invariant. If C is irreducible at a point p, it is shown that $\mathrm{Ind}_p(\mathcal{F}_C)$ is a topological invariant of holomorphic foliations [CLS]. Hence, by (2.2), it is a topological invariant in general. Thus, from Theorems 1.2, 2.2 and 2.6 and Lemma 2.3, we have;

Proposition 2.7 *For a foliation $\mathcal{F}$ on a surface X admitting a compact invariant curve C, $c_1(F) \frown [C]$ and $c_1(E) \frown [C]$ are topological invariants.*

Note that, in [GSV], it is already shown that $c_1(E)$ is a topological invariant of a dimension one foliation.

3. Relation with holonomy

Let $\mathcal{F}$ be a foliation on a complex surface and γ a loop in a leaf of $\mathcal{F}$. Suppose for the moment that $\mathcal{F}$ is defined by a *closed* multi-valued 1-form ω in a neighborhood of γ. Fixing a point p_0 on γ, let ω_0 be the restriction of a branch of ω to a neighborhood of p_0 and let ω_1 be the branch obtained after one revolution around γ. Then there exists a holomorphic function φ defined in a neighborhood of x_0 so that $\varphi\omega_1 = \omega_0$. Recall that the multiplier of $\mathcal{F}$ relative to γ is the derivative of the holonomy mapping at its basepoint.

Lemma 3.1 *In the above situation, the multiplier is given by $\varphi(p_0)$.*

Proof. Let p be a point in γ. Since ω is assumed to be closed, there is a biholomorphic map ζ_p, by the Frobenius theorem (or simply by 'straightening out'), from an open neighborhood U_p of p onto a neighborhood of 0 in $\mathbb{C}^2 = \{(x, y)\}$, $\zeta_p(p) = 0$, such that $\zeta_p^* dy = \omega|_{U_p}$. By compactness of γ, there is a finite set of charts $\{(U_i, \zeta_i)\}$, $i = 0, \cdots, n$, with $p_0 \in U_0 \cap U_n$, $U_i \cap U_{i+1} \neq \varnothing$, $\zeta_0^* dy = \omega_0$, and $\zeta_i^* dy$ equal to the restriction of the branch of ω to U_i obtained by analytic continuation along γ. We have $\zeta_i^* dy = \zeta_{i+1}^* dy$ in the common domain, from which we deduce that the second coordinate of $(\zeta_{i+1} \circ \zeta_i^{-1})(x, y)$ is y. Now $\zeta_0^* dy = \omega_0 = \varphi\omega_1 = \varphi\zeta_n^* dy$, and writing $\zeta_0 \circ \zeta_n^{-1} = (x', y')$, we see that $\varphi \circ \zeta_n^{-1}$ is equal to $\frac{\partial y'}{\partial y}$ and $\frac{\partial y'}{\partial x} = 0$. □

Suppose $\mathcal{F}$ is defined by a holomorphic 1-form ω in a neighborhood of γ. Then one can write $d\omega = \alpha \wedge \omega$, where α is a multi-valued 1-form in a neighborhood of γ, and the restriction of α to every leaf is single-valued.

Theorem 3.2 *The multiplier of $\mathcal{F}$ relative to γ is given by* $\exp\left(\int_\gamma \alpha\right)$.

Proof. We have $d\omega = \alpha \wedge \omega$ as above. Let Γ be a local transversal at a point p_0 of γ. Denote by h the backward projection on Γ along the leaves, defined in a neighborhood of γ. For p in a neighborhood of γ, define:

$$g(p) = \exp\left(-\int_{h(p)}^{p} \alpha\right),$$

where integration is performed along a curve from $h(p)$ to p on the leaf going through p which defines the holonomy. Since any two such curves are homotopic, the integration is well-defined. We have

$$d(g\omega) = dg \wedge \omega + g\, d\omega = -g \cdot d\left(\int_{h(p)}^{p} \alpha\right) \wedge \omega + g\alpha \wedge \omega.$$

Now we take a biholomorphic map ζ from a neighborhood of p_0 onto a neighborhood of 0 in $\mathbb{C}^2 = \{(x, y)\}$ such that $\zeta^* dy$ defines the foliation $\mathcal{F}$ in a neighborhood of p_0. Writing $\alpha = \zeta^*(k_1 dx + k_2 dy)$, we have, for p in a neighborhood of p_0, $\int_{h(p)}^{p} \alpha = \int_0^{x(p)} k_1 dx$ so that:

$$d\left(\int_{h(p)}^{p} \alpha\right) = \zeta^* d\left(\int_0^{x(p)} k_1 dx\right) = \zeta^*\left(k_1 dx + \left(\int_0^{x(p)} \frac{\partial k_1}{\partial y} dx\right) dy\right).$$

Therefore using analytic continuation we obtain:

$$d\Big(\int_{h(p)}^{p} \alpha\Big) \wedge \omega = \alpha \wedge \omega.$$

Then

$$d(g\omega) = -g\alpha \wedge \omega + g\alpha \wedge \omega = 0.$$

Applying Lemma 3.1 to the closed multi-valued 1-form $g\omega$, we obtain that the multiplier is $g(p_0)^{-1} = \exp(\int_\gamma \alpha)$, as desired. □

Now let $\mathcal{F}$ be a germ of foliation at 0 in $\mathbb{C}^2$ and C a germ of reduced and irreducible curve which is invariant by $\mathcal{F}$. Since $\mathrm{Ind}_0(\mathcal{F}_C)$ and $\mu_0(C)$ are integers, from Lemma 2.3 we obtain the following result, which is proved in [S] Proposition (3.1) by different approach.

Corollary 3.3 *The quantity* $\exp(2\pi\sqrt{-1}\,\mathrm{Ind}_0(\mathcal{F}, C))$ *gives the multiplier of* $\mathcal{F}$ *relative to the link of the singularity* 0 *in* C.

Note: After the preparation of the manuscript, the recent preprint of M. Brunella [B] was brought to our attention. Theorem 2.2 above together with Theorem 1.2 and Lemma 2.3 implies the first formula in [B] Lemme 3 and Theorem 2.6 is equivalent to the second formula there. We note that the formulas in [B] are given under the assumption that the ambient surface be compact, which is not necessary in this article.

References

[BB] Baum P. and Bott R., *Singularities of holomorphic foliations.* J. Differential Geom. **7** (1972), 279–342.

[B] Brunella M., *Feuilletages holomorphes sur les surfaces complexes compactes.* preprint.

[CLS] Camacho C., Lins Neto A. and Sad P., *Topological invariants and equidesingularization for holomorphic vector fields.* J. Differential Geom. **20** (1984), 143–174.

[CS] Camacho C. and Sad P., *Invariant varieties through singularities of holomorphic vector fields.* Ann. of Math. **115** (1982), 579–595.

[GSV] Gómez-Mont X., Seade J. and Verjovsky A., *The index of a holomorphic flow with an isolated singularity.* Math. Ann. **291** (1991), 737–751.

[GH] Griffiths P. and Harris J., *Principles of Algebraic Geometry.* John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1978.

[K] Kodaira K., *On compact complex analytic surfaces I.* Ann. of Math. **71** (1960), 111–152.

[Lê] Lê D.-T., *Topologie des singularités des hypersurfaces complexes.* Singularités à Cargèse, Astérisque 7/8, Soc. Math. de France, 1973, pp. 171–182.

[LS] Lehmann D. and Suwa T., *Residues of holomorphic vector fields relative to singular invariant subvarieties.* J. Differential Geom. **42** (1995), 165–192.

[Li] Lins Neto A., *Algebraic solutions of polynomial differential equatuins and foliations in dimension two.* Holomorphic Dynamics, Mexico 1986, Lecture Notes in Mathematics 1345, Springer-Verlag, New York, Heidelberg, Berlin, 1988, pp. 192–232.

[SS] Seade J. and Suwa T., *An adjunction formula for local complete intersections.* preprint.

[Su] Suwa T., *Indices of holomorphic vector fields relative to invariant curves on surfaces.* Proc. of the Amer. Math. Soc. **123** (1995), 2989–2997.

Bahman Khanedani
Institute for Studies in Theoretical
Physics and Mathematics (IPM)
P.O. Box 19395-1795
Tehran, Iran
E-mail: khandani@rose.ipm.ac.ir

Tatsuo Suwa
Department of Mathematics
Hokkaido University
Sapporo 060, Japan
E-mail: suwa@math.hokudai.ac.jp