Hokkaido Mathematical Journal Vol. 41 (2012) p. 441–461

Generalized Mannheim curves in Minkowski space-time E_1^4

Soley Ersoy, Murat Tosun and Hiroo Matsuda

(Received January 31, 2011)

Abstract. In this paper, the definition of generalized spacelike Mannheim curve in Minkowski space-time E_1^4 is given. The necessary and sufficient conditions for the generalized spacelike Mannheim curve are obtained. Also, some characterizations of Mannheim curve are given.

Key words: Mannheim curve, Minkowski space-time.

1. Introduction

The curves are a fundamental structure of differential geometry. An increasing interest of the theory of curves makes a development of special curves to be examined. A way to classification and characterization of curves is the relationship between the Frenet vectors of the curves. For example, Saint Venant proposed the question whether upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal of the given curve in 1845. This question was answered by Bertrand in 1850. He showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients exists between the first and second curvatures of the given original curve. The pairs of curves of this kind have been called Bertrand partner curves or more commonly Bertrand curves [4], [7], [14]. There are many works related with Bertrand curves in the Euclidean space and Minkowski space, [11]–[2]. Also, generalized Bertrand curves in Euclidean 4- space are defined and characterized in [9]. Another kind of associated curve have been called Mannheim curve and Mannheim partner curve. The notion of Mannheim curves was discovered by A. Mannheim in 1878. These curves in Euclidean 3-space are characterized in terms of the curvature and torsion as follows: A space curve is a Mannheim curve if and only if its curvature κ and torsion τ satisfy the relation

$$\kappa(s) = \alpha\big(\kappa^2(s) + \tau^2(s)\big)$$

2010 Mathematics Subject Classification : 53B30, 53A35, 53A04.

for some constant α. The articles concerning Mannheim curves are rather few. In [3], a remarkable class of Mannheim curves is studied. General Mannheim curves in the Euclidean 3-space are obtained in [15]. Mannheim partner curves in Euclidean 3-space and Minkowski 3-space are studied and the necessary and sufficient conditions for the Mannheim partner curves are obtained in [8], [13]. Recently, Mannheim curves are generalized and some characterizations and examples of generalized Mannheim curves in Euclidean 4-space E^4 are given by [10].

In this paper, we study the generalized spacelike Mannheim partner curves in 4−dimensional Minkowski space-time. We will give the necessary and sufficient conditions for the generalized spacelike Mannheim partner curves.

2. Preliminaries

The basic concepts of the theory of curves in Minkowski space-time E_1^4 are briefly presented in this section. A more complete elementary treatment can be found in [12]. Minkowski space-time E_1^4 is an Euclidean space provided with the standard flat metric given by

$$\langle\ ,\ \rangle = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system in $\mathbb{R}^4$.

Since $\langle\ ,\ \rangle$ is an indefinite metric, recall that a vector $\mathbf{v} \in E_1^4$ can have one of the three causal characters; it can be spacelike if $\langle \mathbf{v}, \mathbf{v}\rangle > 0$ or $\mathbf{v} = \mathbf{0}$, timelike if $\langle \mathbf{v}, \mathbf{v}\rangle < 0$ and null (lightlike) if $\langle \mathbf{v}, \mathbf{v}\rangle = 0$ and $\mathbf{v} \neq \mathbf{0}$. Similarly, an arbitrary curve $\mathbf{c} = \mathbf{c}(s)$ in E^4 can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors $\mathbf{c}'(s)$ are, respectively, spacelike, timelike or null. The norm of $\mathbf{v} \in E_1^4$ is given by $\|\mathbf{v}\| = \sqrt{|\langle \mathbf{v}, \mathbf{v}\rangle|}$. If $\|\mathbf{c}'(s)\| = \sqrt{|\langle \mathbf{c}'(s), \mathbf{c}'(s)\rangle|} \neq 0$ for all $s \in L$, then C is a regular curve in E_1^4. A spacelike (timelike) regular curve C is parameterized by arc-length parameter s which is given by $\mathbf{c} : L \to E_1^4$, then the tangent vector $\mathbf{c}'(s)$ along C has unit length, that is,

$$\langle \mathbf{c}(s), \mathbf{c}(s)\rangle = 1, \quad (\langle \mathbf{c}(s), \mathbf{c}(s)\rangle = -1)$$

for all $s \in L$.

Hereafter, curves are considered spacelike and regular C^∞ curves in E_1^4.

Let $\mathbf{e}_1(s) = \mathbf{c}'(s)$ for all $s \in L$, then the vector field $\mathbf{e}_1(s)$ is spacelike and it is called spacelike unit tangent vector field on C.

The spacelike curve C is called special spacelike Frenet curve if there exist three smooth functions k_1, k_2, k_3 on C and smooth non-null frame field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ along the curve C. Also, the functions k_1, k_2, and k_3 are called the first, the second, and the third curvature function on C, respectively. For the C^∞ special spacelike Frenet curve C, the following Frenet formula is hold

$$\begin{bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \\ \mathbf{e}'_4 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ \mu_1 k_1 & 0 & k_2 & 0 \\ 0 & \mu_2 k_2 & 0 & k_3 \\ 0 & 0 & \mu_3 k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{bmatrix}$$

where $\mu_i = \mp 1$, $1 \leq i \leq 3$, [12].

Due to characters of Frenet vector fields of the spacelike curve C, μ_i $(1 \leq i \leq 3)$ are defined as in the following three subcases;

Case 1: If $\mathbf{e}_4$ is timelike, then μ_i, $1 \leq i \leq 3$ are

$$\mu_1 = \mu_2 = -1, \quad \mu_3 = 1$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}_4$ are mutually orthogonal vector fields satisfying equations

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = \langle \mathbf{e}_3, \mathbf{e}_3 \rangle = 1, \quad \langle \mathbf{e}_4, \mathbf{e}_4 \rangle = -1.$$

Case 2: If $\mathbf{e}_3$ is timelike, then μ_i, $1 \leq i \leq 3$ are

$$\mu_1 = -1, \quad \mu_2 = \mu_3 = 1$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}_4$ are mutually orthogonal vector fields satisfying equations

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = \langle \mathbf{e}_4, \mathbf{e}_4 \rangle = 1, \quad \langle \mathbf{e}_3, \mathbf{e}_3 \rangle = -1.$$

Case 3: If $\mathbf{e}_2$ is timelike, then μ_i, $1 \leq i \leq 3$ are

$$\mu_1 = \mu_2 = 1, \quad \mu_3 = -1$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{e}_4$ are mutually orthogonal vector fields satisfying equations

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbf{e}_3, \mathbf{e}_3 \rangle = \langle \mathbf{e}_4, \mathbf{e}_4 \rangle = 1, \quad \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = -1.$$

For $s \in L$, the non-null frame field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and curvature functions k_1 and k_2 are determined as follows

$$\begin{aligned}
&1^{\text{st}} \text{ step} && \mathbf{e}_1(s) = \mathbf{c}'(s) \\
&2^{\text{nd}} \text{ step} && k_1(s) = \|\mathbf{e}'_1(s)\| > 0 \\
& && \mathbf{e}_2(s) = \frac{1}{k_1(s)} \mathbf{e}'_1(s) \\
&3^{\text{rd}} \text{ step} && k_2(s) = \|\mathbf{e}'_2(s) - \mu_1 k_1(s) \mathbf{e}_1(s)\| > 0 \\
& && \mathbf{e}_3(s) = \frac{1}{k_2(s)} (\mathbf{e}'_2(s) - \mu_1 k_1(s) \mathbf{e}_1(s)) \\
&4^{\text{th}} \text{ step} && e_4(s) = \varepsilon \frac{1}{\|\mathbf{e}'_3(s) - \mu_2 k_2(s) \mathbf{e}_2(s)\|} (\mathbf{e}'_3(s) - \mu_2 k_2(s) \mathbf{e}_2(s))
\end{aligned}$$

where ε is taken -1 or $+1$ to make $+1$ the determinant of $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, that is, the non-null orthonormal frame field is of positive orientation. The function k_3 is determined by

$$k_3(s) = \langle \mathbf{e}'_3(s), \mathbf{e}_4(s) \rangle \neq 0.$$

So the function k_3 never vanishes.

In order to make sure that the spacelike curve C is a special spacelike Frenet curve, above steps must be checked, from 1^{st} step to 4^{th} step, for $s \in L$.

At the each point of spacelike curve C, a line ℓ_1 in the direction of $\mathbf{e}_2$ is called the first normal line, a line ℓ_2 in the direction of $\mathbf{e}_3$ is called the second normal line and a line ℓ_3 in the direction of $\mathbf{e}_4$ is called the third normal line.

Note that, according to three different case of spacelike curve C, ℓ_3, ℓ_2 and ℓ_1 can be timelike, respectively, which are called second binormal, first binormal and principal normal line at the each point of the spacelike curve C.

3. Generalized spacelike Mannheim curves in E_1^4

In E^4 the Bertrand curves and Mannheim curves are generalized by [9] and [10], respectively. In these regards, we have investigate generalization of spacelike Mannheim curves Minkowski space in E_1^4.

Definition 3.1 A special spacelike curve C in E_1^4 is a generalized spacelike Mannheim curve if there exists a special spacelike Frenet curve C^* in E_1^4 such that the first normal line at each point of C is included in the plane generated by the second normal line and the third normal line of C^* at the corresponding point under ϕ. Here ϕ is a bijection from C to C^*. The curve C^* is called the generalized spacelike Mannheim mate curve of C.

By the definition, a generalized Mannheim mate curve C^* is given by

$$\mathbf{c}^*(s) = \mathbf{c}(s) + \alpha(s)\mathbf{e}_2(s), \quad s \in L \tag{3.1}$$

where α is a smooth function on L. Generally, the parameter s isn't an arc-length of C^*. Let s^* be the arc-length of C^* defined by

$$s^* = \int_0^s \left\| \frac{d\mathbf{c}^*(s)}{ds} \right\| ds.$$

If a smooth function $f : L \to L^*$ is given by $f(s) = s^*$, then

$$\begin{aligned} \frac{d\mathbf{c}^*(s)}{ds} &= \mathbf{e}_1(s) + \alpha'(s)\mathbf{e}_2(s) + \alpha(s)\mu_1 k_1(s)\mathbf{e}_1(s) + \alpha(s)k_2(s)\mathbf{e}_3(s) \\ &= (1 + \mu_1\alpha(s)k_1(s))\mathbf{e}_1(s) + \alpha'(s)\mathbf{e}_2(s) + \alpha(s)k_2(s)\mathbf{e}_3(s). \end{aligned}$$

for $\forall s \in L$. Thus, we have

$$\begin{aligned} f'(s) = \frac{ds^*}{ds} &= \left\| \frac{d\mathbf{c}^{*(s)}}{ds} \right\| \\ &= \sqrt{\left|(1 + \mu_1\alpha(s)k_1(s))^2 + \varepsilon_2(\alpha'(s))^2 + \varepsilon_3(\alpha(s)k_2(s))^2\right|} \end{aligned}$$

where $\varepsilon_i = \begin{cases} -1, & \mathbf{e}_i \text{ is timelike} \\ 1, & \mathbf{e}_i \text{ is spacelike} \end{cases}$, for $2 \le i \le 4$.

This means that, in the Case 1, $\mathbf{e}_4$ is timelike and

$$f'(s) = \sqrt{\left|(1-\alpha(s)k_1(s))^2 + (\alpha'(s))^2 + (\alpha(s)k_2(s))^2\right|}$$

or in the Case 2, $\mathbf{e}_3$ is timelike and

$$f'(s) = \sqrt{\left|(1-\alpha(s)k_1(s))^2 + (\alpha'(s))^2 - (\alpha(s)k_2(s))^2\right|}$$

or in the Case 3, $\mathbf{e}_2$ is timelike and

$$f'(s) = \sqrt{\left|(1+\alpha(s)k_1(s))^2 - (\alpha'(s))^2 + (\alpha(s)k_2(s))^2\right|}.$$

The spacelike curve C^* with arc-length parameter s^* is

$$\begin{aligned} \mathbf{c}^* : L^* &\to E_1^4 \\ s^* &\to \mathbf{c}^*(s^*). \end{aligned}$$

For a bijection $\phi : C \to C^*$ defined by $\phi(\mathbf{c}(s)) = \mathbf{c}^*(f(s))$, the reparametrization of C^* is

$$\mathbf{c}^*(f(s)) = \mathbf{c}(s) + \alpha(s)\mathbf{e}_2(s)$$

where α is a smooth function on L.

Theorem 3.1 *If a special spacelike Frenet curve C in E_1^4 is a generalized spacelike Mannheim curve, then the first curvature function k_1 and the second curvature function k_2 of C satisfy the equality*

$$k_1(s) = -\alpha\big(\mu_1 k_1^2(s) + \mu_2 k_2^2(s)\big), \quad s \in L \tag{3.2}$$

where α is a constant number and $\mu_1 = \mu_2 = -1$ when $\mathbf{e}_4$ is timelike or $\mu_1 = -1$, $\mu_2 = 1$ when $\mathbf{e}_3$ is timelike or $\mu_1 = \mu_2 = 1$ when $\mathbf{e}_2$ is timelike.

Proof. Let C be a generalized spacelike Mannheim curve and C^* be the generalized spacelike Mannheim mate curve of C with the diagram;

$$\begin{array}{ccccc} & \mathbf{c} & & \mathbf{c}^* \\ & \vdots & & \vdots \\ f: & L & \longrightarrow & L^* \\ & \downarrow & & \downarrow \\ \phi: & E_1^4 & \longrightarrow & E_1^4. \end{array}$$

A smooth function f is defined by $f(s) = \int \left\| \frac{d\mathbf{c}^*(s)}{ds} \right\| ds = s^*$ and s^* is the arc-length parameter of C^*. Also ϕ is a bijection which is defined by $\phi(\mathbf{c}(s)) = \mathbf{c}^*(f(s))$. Thus, the spacelike curve C^* is reparametrized by

$$\mathbf{c}^*(f(s)) = \mathbf{c}(s) + \alpha(s)\mathbf{e}_2(s) \tag{3.3}$$

where α is a smooth function. By differentiating both sides of (3.3) with respect to s

$$f'(s)\mathbf{e}_1^*(f(s)) = (1 + \mu_1 \alpha(s) k_1(s))\mathbf{e}_1 + \alpha'(s)\mathbf{e}_2(s) + \alpha(s) k_2(s)\mathbf{e}_3(s) \tag{3.4}$$

is obtained.

On the other hand, since the first normal line at the each point of C is lying in the plane generated by the second normal line and the third normal line of C^* at the corresponding points under bijection ϕ, the vector field $\mathbf{e}_2(s)$ is given by

$$\mathbf{e}_2(s) = g(s)\mathbf{e}_3^*(f(s)) + h(s)\mathbf{e}_4^*(f(s))$$

where g and h are some smooth functions on L. If we take into consideration

$$\left\langle \mathbf{e}_1^*(f(s)),\, g(s)\mathbf{e}_3^*(f(s)) + h(s)\mathbf{e}_4^*(f(s)) \right\rangle = 0$$

and the equation (3.4), then we have $\alpha'(s) = 0$. So we rewrite the equation (3.4) as

$$f'(s)\mathbf{e}_1^*(f(s)) = (1 + \mu_1 \alpha k_1(s))\mathbf{e}_1(s) + \alpha k_2(s)\mathbf{e}_3(s), \tag{3.5}$$

that is,

$$\mathbf{e}_1^*(f(s)) = \frac{(1 + \mu_1 \alpha k_1(s))}{f'(s)}\mathbf{e}_1(s) + \frac{\alpha k_2(s)}{f'(s)}\mathbf{e}_3(s)$$

where

$$f'(s) = \sqrt{\left|(1 + \mu_1 \alpha k_1(s))^2 + \varepsilon_3(\alpha k_2(s))^2\right|}, \quad \varepsilon_3 = \begin{cases} -1, & \mathbf{e}_3 \text{ is timelike}, \\ 1, & \mathbf{e}_3 \text{ is spacelike}. \end{cases}$$

By taking differentiation both sides of the equations (3.5) with respect to s,

$$
\begin{aligned}
&f'(s)k_1^*(f(s))\mathbf{e}_2^*(f(s)) \\
&\quad= \left(\frac{1+\mu_1\alpha k_1(s)}{f'(s)}\right)'\mathbf{e}_1(s) + \left(\frac{(1+\mu_1\alpha k_1(s))k_1(s)+\mu_2\alpha(k_2(s))^2}{f'(s)}\right)\mathbf{e}_2(s) \\
&\qquad+ \left(\frac{\alpha k_2(s)}{f'(s)}\right)'\mathbf{e}_3(s) + \left(\frac{\alpha k_2(s)k_3(s)}{f'(s)}\right)\mathbf{e}_4(s) \qquad (3.6)
\end{aligned}
$$

is obtained for $s \in L$. Since

$$\left\langle \mathbf{e}_2^*(f(s)),\, g(s)\mathbf{e}_3^*(f(s)) + h(s)\mathbf{e}_4^*(f(s))\right\rangle = 0,$$

then in the equation (3.6) the coefficient of $\mathbf{e}_2(s)$ vanishes, that is,

$$(1+\mu_1\alpha k_1(s))k_1(s) + \mu_2\alpha(k_2(s))^2 = 0.$$

Thus, $k_1(s) = -\alpha(\mu_1 k_1^2(s) + \mu_2 k_2^2(s))$ is satisfied. This completes the proof. □

If we investigate the special cases separately, then we have

in the Case 1; $k_1(s) = \alpha\big(k_1^2(s) + k_2^2(s)\big)$,

in the Case 2; $k_1(s) = \alpha\big(k_1^2(s) - k_2^2(s)\big)$,

in the Case 3; $k_1(s) = -\alpha\big(k_1^2(s) + k_2^2(s)\big)$.

Theorem 3.2 *Let C be a special spacelike Frenet curve in E_1^4 whose curvature functions k_1 and k_2 are non-constant functions and satisfy the equality $k_1(s) = -\alpha(\mu_1 k_1^2(s) + \mu_2 k_2^2(s))$, where α is non-zero constant, for all $s \in L$. If the spacelike curve C^* given by*

$$\mathbf{c}^*(s) = \mathbf{c}(s) + \alpha\mathbf{e}_2(s)$$

is a special spacelike Frenet curve, then C^* is a generalized spacelike Mannheim mate curve of C.

Proof. The arc-length parameter of C^* is defined by

$$s^* = \int_0^s \left\|\frac{d\mathbf{c}^*(s)}{ds}\right\| ds$$

for all $s \in L$. Under the assumption of

$$k_1(s) = -\alpha\big(\mu_1 k_1^2(s) + \mu_2 k_2^2(s)\big)$$

and after calculations for all cases, separately, we obtain

in the Case 1; $f'(s) = \sqrt{|1 - \alpha k_1(s)|}$,
in the Case 2; $f'(s) = \sqrt{|1 - \alpha k_1(s)|}$,
in the Case 3; $f'(s) = \sqrt{|1 + \alpha k_1(s)|}$.

Thus, we can generalize

$$f'(s) = \sqrt{|1 + \mu_1 \alpha k_1(s)|}$$

for all $s \in L$.

By differentiating the equation $\mathbf{c}^*(f(s)) = \mathbf{c}(s) + \alpha \mathbf{e}_2(s)$ with respect to s, it is seen that

$$f'(s)\mathbf{e}_1^*(f(s)) = (1 + \mu_1 \alpha k_1(s))\mathbf{e}_1(s) + \alpha k_2(s)\mathbf{e}_3(s).$$

So, it is seen that

$$\mathbf{e}_1^*(f(s)) = \left(\frac{1 + \mu_1 \alpha k_1(s)}{\sqrt{|1 + \mu_1 \alpha k_1(s)|}}\mathbf{e}_1(s) + \frac{\alpha k_2(s)}{\sqrt{|1 + \mu_1 \alpha k_1(s)|}}\mathbf{e}_3(s)\right) \tag{3.7}$$

for $s \in L$.

The differentiation of the last equation with respect to s is

$$\begin{aligned} f'(s)k_1^*(f(s))\mathbf{e}_2^*(f(s)) &= \left(\sqrt{|1 + \mu_1 \alpha k_1(s)|}\right)'\mathbf{e}_1(s) + \left(\frac{(1 + \mu_1 \alpha k_1(s))k_1(s) + \mu_2 \alpha k_2^2(s)}{\sqrt{|1 + \mu_1 \alpha k_1(s)|}}\right)\mathbf{e}_2(s) \\ &\quad + \left(\frac{\alpha k_2(s)}{\sqrt{|1 + \mu_1 \alpha k_1(s)|}}\right)'\mathbf{e}_3(s) + \left(\frac{\alpha k_2(s) k_3(s)}{\sqrt{|1 + \mu_1 \alpha k_1(s)|}}\right)\mathbf{e}_4(s). \end{aligned} \tag{3.8}$$

According to our assumption,

$$\frac{(1 + \mu_1 \alpha k_1(s))k_1(s) + \mu_2 \alpha k_2^2(s)}{\sqrt{|1 + \mu_1 \alpha k_1(s)|}} = 0$$

is hold. Thus, the coefficient of $\mathbf{e}_2(s)$ in the equation (3.8) is zero. It is seen from the equation (3.8), $\mathbf{e}_2^*(f(s))$ is given by linear combination of $\mathbf{e}_1(s)$, $\mathbf{e}_3(s)$ and $\mathbf{e}_4(s)$. Also, from equation (3.7), $\mathbf{e}_1^*(f(s))$ is a linear combination of $\mathbf{e}_1(s)$ and $\mathbf{e}_3(s)$. Moreover, C^* is a special spacelike Frenet curve that the vector $\mathbf{e}_2(s)$ is given by linear combination of $\mathbf{e}_3^*(f(s))$ and $\mathbf{e}_4^*(f(s))$.

Therefore, the first normal line C lies in the plane generated by the second normal line and third normal line of C^* at the corresponding points under a bijection ϕ which is defined by $\phi(\mathbf{c}(s)) = \mathbf{c}^*(f(s))$. Thus, the proof of the theorem is completed. □

Remark 3.1 In 4-dimensional Minkowski space for a special spacelike Frenet curve C with curvature functions k_1 and k_2 satisfying

$$k_1(s) = -\alpha\big(\mu_1 k_1^2(s) + \mu_2 k_2^2(s)\big),$$

it is not clear that a smooth spacelike curve C^* given by (3.1) is a special Frenet curve. So, it is unknown whether the reverse of Theorem 3.1 is true or false.

Theorem 3.3 *Let C be a spacelike special curve in E_1^4 with non-zero third curvature function k_3. If there exists a spacelike special Frenet curve C^* in E_1^4 such that the first normal line of C is linearly dependent with the third normal line of C^* at the corresponding points $\mathbf{c}(s)$ and $\mathbf{c}^*(s)$, respectively, under a bijection $\phi : C \to C^*$, then the curvatures k_1 and k_2 of C are constant functions.*

Proof. Let C be a spacelike Frenet curve in E_1^4 with the Frenet frame field $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ and curvature functions k_1, k_2 and k_3. Also, we assume that C^* be a spacelike special Frenet curve in E_1^4 with the Frenet frame field $\{\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*, \mathbf{e}_4^*\}$ and curvature functions k_1^*, k_2^* and k_3^*.

Let the first normal line of C be linearly dependent with the third normal line of C^* at the corresponding points C and C^*, respectively. Then the parametrization of C^* is

$$\mathbf{c}^*(f(s)) = \mathbf{c}(s) + \alpha(s)\mathbf{e}_2(s) \tag{3.9}$$

for all $s \in L$. If s^* is the arc-length parameter of C^*, then

$$s^* = \int_0^s \sqrt{\left|(1+\mu_1\alpha k_1)^2 + \varepsilon_2(\alpha'(s)) + \varepsilon_3(\alpha(s)k_2(s))^2\right|}ds \tag{3.10}$$

where

$$\varepsilon_i = \begin{cases} -1, & \mathbf{e}_i \text{ is timelike} \\ 1, & \mathbf{e}_i \text{ is spacelike} \end{cases}, \quad \text{for } 2 \leq i \leq 4$$

and

$$f : L \to L^*$$
$$s \to f(s) = s^*.$$

Moreover, $\phi : C \to C^*$ is a bijection given by $\phi(\mathbf{c}(s)) = \mathbf{c}^*(f(s))$.

By differentiating the equation (3.9) with respect to s and using Frenet formulas, we have

$$f'(s)\mathbf{e}_1^*(f(s)) = (1 + \mu_1\alpha(s)k_1(s))\mathbf{e}_1(s) + \alpha'(s)\mathbf{e}_2(s) + \alpha(s)k_2(s)\mathbf{e}_3(s). \tag{3.11}$$

Since $\mathbf{e}_4^*(f(s)) = \mp\mathbf{e}_2(s)$, then

$$\langle f'(s)\mathbf{e}_1^*(f(s)), \mathbf{e}_4^*(f(s))\rangle = \langle(1 + \mu_1\alpha(s)k_1(s))\mathbf{e}_1(s) + \alpha'(s)\mathbf{e}_2(s) + \alpha(s)k_2(s)\mathbf{e}_3(s), \mp\mathbf{e}_2(s)\rangle,$$

that is,

$$0 = \mp\alpha'(s).$$

It is easily seen that α is a constant number from the last equation. Thus, hereafter we can denote $\alpha(s) = \alpha$, for all $s \in L$.

From the equation (3.10), we get

$$f'(s) = \sqrt{\left|(1 + \mu_1\alpha k_1(s))^2 + \varepsilon_3(\alpha k_2(s))^2\right|} > 0$$

where

$$\varepsilon_3 = \begin{cases} -1, & \mathbf{e}_i \text{ is timelike} \\ 1, & \mathbf{e}_i \text{ is spacelike} \end{cases}, \quad \text{for } 2 \leq i \leq 4.$$

Then, we rewrite the equation (3.11) as follows;

$$\mathbf{e}_1^*(f(s)) = \left(\frac{1+\mu_1\alpha k_1(s)}{f'(s)}\right)\mathbf{e}_1(s) + \left(\frac{\alpha k_2(s)}{f'(s)}\right)\mathbf{e}_3(s).$$

The differentiation of the last equation with respect to s is

$$\begin{aligned} & f'(s)k_1^*(f(s))\mathbf{e}_2^*(f(s)) \\ & \quad = \left(\frac{1+\mu_1\alpha k_1(s)}{f'(s)}\right)'\mathbf{e}_1(s) + \left(\frac{k_1(s)+\mu_1\alpha k_1^2(s)+\mu_2\alpha k_2^2(s)}{f'(s)}\right)\mathbf{e}_2(s) \\ & \qquad + \left(\frac{\alpha k_2(s)}{f'(s)}\right)'\mathbf{e}_3(s) + \left(\frac{\alpha k_2(s)k_3(s)}{f'(s)}\right)\mathbf{e}_4(s). \end{aligned} \tag{3.12}$$

Since $\langle f'(s)k_1^*(f(s))\mathbf{e}_2^*(f(s)),\, \mathbf{e}_4^*(f(s))\rangle = 0$ and $\mathbf{e}_4^*(f(s)) = \mp\mathbf{e}_2(s)$ for all $s \in L$, we obtain

$$k_1(s) + \mu_1\alpha k_1^2(s) + \mu_2\alpha k_2^2(s) = 0$$

is satisfied. Then,

$$\alpha = -\frac{k_1(s)}{\mu_1 k_1^2(s) + \mu_2 k_2^2(s)} \tag{3.13}$$

is a non-zero constant number. Thus, from the equation (3.12), it is seen that

$$\begin{aligned} \mathbf{e}_2^*(f(s)) = {} & \frac{1}{f'(s)K(s)}\left(\frac{1+\mu_1\alpha k_1(s)}{f'(s)}\right)'\mathbf{e}_1(s) + \frac{1}{f'(s)K(s)}\left(\frac{\alpha k_2(s)}{f'(s)}\right)\mathbf{e}_3(s) \\ & + \frac{1}{f'(s)K(s)}\left(\frac{\alpha k_2(s)k_3(s)}{f'(s)}\right)\mathbf{e}_4(s) \end{aligned}$$

where $K(s) = k_1^*(f(s))$ for all $s \in L$. By differentiating the last equation with respect to s, we obtain

$$
\begin{aligned}
&f'(s)\big[\mu_1 k_1^*(f(s))\mathbf{e}_1^*(f(s)) + k_2^*(f(s))\mathbf{e}_3^*(f(s))\big] \\
&\quad = \left(\frac{1}{f'(s)K(s)}\left(\frac{1+\mu_1\alpha k_1(s)}{f'(s)}\right)'\right)'\mathbf{e}_1(s) \\
&\qquad + \left(\frac{k_1(s)}{f'(s)K(s)}\left(\frac{1+\mu_1\alpha k_1(s)}{f'(s)}\right)' + \frac{\mu_2 k_2(s)}{f'(s)K(s)}\left(\frac{\alpha k_2(s)}{f'(s)}\right)'\right)\mathbf{e}_2(s) \\
&\qquad + \left(\left(\frac{1}{f'(s)K(s)}\left(\frac{\alpha k_2(s)}{f'(s)}\right)'\right)' + \frac{\mu_3 k_3(s)}{f'(s)K(s)}\left(\frac{\alpha k_2(s)k_3(s)}{f'(s)}\right)\right)\mathbf{e}_3(s) \\
&\qquad + \left(\left(\frac{1}{f'(s)K(s)}\left(\frac{\alpha k_2(s)k_3(s)}{f'(s)}\right)'\right)' + \frac{k_3(s)}{f'(s)K(s)}\left(\frac{\alpha k_2(s)}{f'(s)}\right)'\right)\mathbf{e}_4(s)
\end{aligned}
$$

for all $s \in L$. If we take into consideration

$$\big\langle f'(s)(\mu_1 k_1^*(f(s))\mathbf{e}_1^*(f(s)) + k_2^*(f(s))\mathbf{e}_3^*(f(s))),\, \mathbf{e}_4^*(f(s))\big\rangle = 0$$

and

$$\mathbf{e}_4^*(f(s)) = \mp\mathbf{e}_2(s),$$

then

$$
\begin{aligned}
&\mu_1\alpha k_1(s){k_1}'(s)f'(s) - k_1(s)(1+\mu_1\alpha k_1(s))f''(s) \\
&\qquad\qquad + \mu_2\alpha k_2(s){k_2}'(s)f'(s) - \mu_2\alpha k_2^2(s)f''(s) = 0.
\end{aligned}
$$

If we arrange the last equation, then we find

$$
\begin{aligned}
&\alpha\big(\mu_1 k_1(s){k'}_1(s) + \mu_2 k_2(s){k'}_2(s)\big)f'(s) \\
&\qquad\qquad - \big(k_1 + \alpha\big(\mu_1 k_1^2(s) + \mu_2 k_2^2(s)\big)\big)f''(s) = 0. \qquad (3.14)
\end{aligned}
$$

Moreover, the differentiation of the equation (3.13) with respect to s is

$$k_1'(s) + 2\alpha\big(\mu_1 k_1(s){k'}_1(s) + \mu_2 k_2(s){k'}_2(s)\big) = 0.$$

From the above equation, we see

$$-\frac{{k'}_1(s)}{2} = \alpha\big(\mu_1 k_1(s){k'}_1(s) + \mu_2 k_2(s){k'}_2(s)\big). \qquad (3.15)$$

If we substitute the equations (3.13) and (3.15) into the equation (3.14), we obtain

$$-\frac{k'_1(s)}{2} = 0.$$

Finally, we find that the first curvature function is constant (that is, positive constant).

Thus, from the equation (3.15) it is seen that the second curvature function k_2 is positive constant, too. This completes the proof. □

In [5], a formula of parametric equation of Mannheim curve is given in E^3. Moreover, the parametric equation of generalized Mannheim curve in E^4 is obtained in [10]. The following theorem gives a parametric representation of a generalized spacelike Mannheim curve with timelike second binormal vector in E_1^4.

Theorem 3.4 *Let C be a spacelike special curve defined by*

$$\mathbf{c}(u) = \begin{bmatrix} \alpha \int f(u) \sinh u du \\ \alpha \int f(u) \cosh u du \\ \alpha \int f(u) g(u) du \\ \alpha \int f(u) h(u) du \end{bmatrix}$$

for $u \in I \subset \mathbb{R}$. Here α is a non-zero constant number, $g : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ are any smooth functions and the positive valued smooth function $f : I \to \mathbb{R}$ is given by

$$\begin{aligned} f(u) &= (1 + g^2(u) + h^2(u))^{-3/2} \\ &\quad \times \big| -1 - g^2(u) - h^2(u) + \dot{g}^2(u) + \dot{h}^2(u) + (\dot{g}(u)h(u) - g(u)\dot{h}(u))^2 \big|^{-/2} \\ &\quad \times \big| \big(-1 - g^2(u) - h^2(u) + \dot{g}^2(u) + \dot{h}^2(u) + (\dot{g}(u)h(u) - g(u)\dot{h}(u))^2 \big)^3 \\ &\quad - (1 + g^2(u) + h^2(u))^3 \big[(g(u) - \ddot{g}(u))^2 + (h(u) - \ddot{h}(u))^2 \\ &\quad - \big((g(u)\dot{h}(u) - \dot{g}(u)h(u)) + (\dot{g}(u)\ddot{h}(u) - \ddot{g}(u)\dot{h}(u)) \big)^2 \\ &\quad + (g(u)\ddot{h}(u) - \ddot{g}(u)h)(u)^2 \big|, \end{aligned}$$

for $u \in I$. Then the curvature functions k_1 and k_2 of C satisfy

$$k_1(u) = \alpha\big(k_1^2(u) + k_2^2(u)\big)$$

at the each point $\mathbf{c}(u)$ of C.

Proof. Let C be a spacelike special curve defined by

$$\mathbf{c}(u) = \begin{bmatrix} \alpha \int f(u) \sinh u du \\ \alpha \int f(u) \cosh u du \\ \alpha \int f(u) g(u) du \\ \alpha \int f(u) h(u) du \end{bmatrix}, \quad u \in I \subset \mathbb{R}$$

where α is a non-zero constant number, g and h are any smooth functions. f is a positive valued smooth function. Thus, we obtain

$$\dot{\mathbf{c}}(u) = \begin{bmatrix} \alpha f(u) \sinh u \\ \alpha f(u) \cosh u \\ \alpha f(u) g(u) \\ \alpha f(u) h(u) \end{bmatrix}, \quad u \in I \subset \mathbb{R} \tag{3.16}$$

where the subscript dot (.) denotes the differentiation with respect to u.

The arc-length parameter s of C is given by

$$s = \psi(u) = \int_{u_0}^{u} \|\dot{\mathbf{c}}(u)\| du$$

where $\|\dot{\mathbf{c}}(u)\| = \alpha f(u)\sqrt{1 + g^2(u) + h^2(u)}$.

If φ denotes the inverse function of $\psi : I \to L \subset \mathbb{R}$, then $u = \varphi(s)$ and

$$\varphi'(s) = \left\| \left. \frac{d\mathbf{c}(u)}{du} \right|_{u=\varphi(s)} \right\|^{-1}, \quad s \in I$$

where the prime ($'$) denotes the differentiation with respect to s.

The unit tangent vector $\mathbf{e}_1(s)$ of the curve C at the each point $\mathbf{c}(\varphi(s))$ is given by

$$\mathbf{e}_1(s) = (1 + g^2(\varphi(s)) + h^2(\varphi(s)))^{-1/2} \begin{bmatrix} \sinh(\varphi(s)) \\ \cosh(\varphi(s)) \\ g(\varphi(s)) \\ h(\varphi(s)) \end{bmatrix} \tag{3.17}$$

for all $s \in L$. Some simplifying assumptions are made for the sake of brevity as follows;

$$\sinh := \sinh(\varphi(s)), \quad \cosh := \cosh(\varphi(s))$$

$$f := f(\varphi(s)), \quad g := g(\varphi(s)), \quad h := h(\varphi(s)),$$

$$\dot{g} := \dot{g}(\varphi(s)) = \left.\frac{dg(u)}{du}\right|_{u=\varphi(s)}, \quad \dot{h} := \dot{h}(\varphi(s)) = \left.\frac{dh(u)}{du}\right|_{u=\varphi(s)},$$

$$\ddot{g} := \ddot{g}(\varphi(s)) = \left.\frac{d^2g(u)}{du^2}\right|_{u=\varphi(s)}, \quad \ddot{h} := \ddot{h}(\varphi(s)) = \left.\frac{d^2h(u)}{du^2}\right|_{u=\varphi(s)},$$

$$\varphi' := \varphi'(s) = \left.\frac{d\varphi}{ds}\right|_{s},$$

$$A := 1 + g^2 + h^2, \quad B := g\dot{g} + h\dot{h}, \quad C := \dot{g}^2 + \dot{h}^2,$$

$$D := g\ddot{g} + h\ddot{h}, \quad E := \dot{g}\ddot{g} + \dot{h}\ddot{h}, \quad F := \ddot{g}^2 + \ddot{h}^2.$$

Then, we have

$$\dot{A} = 2B, \quad \dot{B} = C + D, \quad \dot{C} = 2E, \quad \varphi' = \alpha^{-1} f^{-1} A^{-1/2}.$$

So, we rewrite the equation (3.17) as

$$\mathbf{e}_1 := \mathbf{e}_1(s) = A^{-1/2} \begin{bmatrix} \sinh \\ \cosh \\ g \\ h \end{bmatrix}. \tag{3.18}$$

By differentiating the last equation with respect to s, we find

$$\mathbf{e'}_1 = \varphi' \begin{bmatrix} -\frac{1}{2}A^{-3/2}\dot{A}\sinh + A^{-1/2}\cosh \\ -\frac{1}{2}A^{-3/2}\dot{A}\cosh + A^{-1/2}\sinh \\ -\frac{1}{2}A^{-3/2}\dot{A}g + A^{-1/2}\dot{g} \\ -\frac{1}{2}A^{-3/2}\dot{A}h + A^{-1/2}\dot{h} \end{bmatrix},$$

that is,

$$\mathbf{e'}_1 = -\varphi' A^{-1/2} \begin{bmatrix} A^{-1}B\sinh - \cosh \\ A^{-1}B\cosh - \sinh \\ A^{-1}Bg - \dot{g} \\ A^{-1}Bh - \dot{h} \end{bmatrix}. \tag{3.19}$$

From the last equation, we obtain

$$k_1 := k_1(s) = \|\mathbf{e'}_1(s)\| = \varphi' A^{-1}|-A + AC - B^2|^{1/2}. \tag{3.20}$$

By the fact that $\mathbf{e}_2(s) = (k_1(s))^{-1}\mathbf{e'}_1(s)$, we have

$$\mathbf{e}_2 := \mathbf{e}_2(s) = -A^{1/2}|-A + AC - B^2|^{-1/2} \begin{bmatrix} A^{-1}B\sinh - \cosh \\ A^{-1}B\cosh - \sinh \\ A^{-1}Bg - \dot{g} \\ A^{-1}Bh - \dot{h} \end{bmatrix}.$$

In order to get second curvature function k_2, we need to calculate $k_2(s) = \|\mathbf{e'}_2(s) - \mu_1 k_1(s)\mathbf{e}_1(s)\|$. It is seen from the above equation $\langle \mathbf{e}_2(s), \mathbf{e}_2(s)\rangle = 1$, that is, $\mathbf{e}_2$ is spacelike. Thus, μ_1 is equal to -1 and $k_2(s) = \|\mathbf{e'}_2(s) + k_1(s)\mathbf{e}_1(s)\|$. After a long process of calculation, we have

$$\mathbf{e'}_2 + k_1\mathbf{e}_1 = \varphi' A^{-3/2}|-A + AC - B^2|^{-3/2} \begin{bmatrix} (P+Q)\sinh - R\cosh \\ (P+Q)\cosh - R\sinh \\ Pg - R\dot{g} + Q\ddot{g} \\ Ph - R\dot{h} + Q\ddot{h} \end{bmatrix} \tag{3.21}$$

where

$$\begin{aligned} P &= (-A + AC - B^2)^2 + (-A + AC - B^2)(B^2 - AC - AD) \\ &\quad + AB(-B + AE - BD), \\ Q &= A^2(-A + AC - B^2), \\ R &= A^2(-B + AE - BD). \end{aligned} \tag{3.22}$$

If we simplify P then we have

$$P = A^2(1 - C + BE + D - CD).$$

Thus, we rewrite the equations (3.22) and (3.23) as

$$\mathbf{e}'_2 + k_1\mathbf{e}_1 = \varphi' A^{1/2}|-A + AC - B^2|^{-3/2} \begin{bmatrix} (\tilde{P} + \tilde{Q})\sinh - \tilde{R}\cosh \\ (\tilde{P} + \tilde{Q})\cosh - \tilde{R}\sinh \\ \tilde{P}g - \tilde{R}\dot{g} + \tilde{Q}\ddot{g} \\ \tilde{P}h - \tilde{R}\dot{h} + \tilde{Q}\ddot{h} \end{bmatrix} \tag{3.23}$$

where

$$\begin{aligned} \tilde{P} &= 1 - C + BE + D - CD, \\ \tilde{Q} &= -A + AC - B^2, \\ \tilde{R} &= -B + AE - BD. \end{aligned} \tag{3.24}$$

Consequently, from the equations (3.24) and (3.25), we find

$$\begin{aligned} &\|\mathbf{e}'_2 + k_1\mathbf{e}_1\|^2 \\ &\quad = (\varphi')^2 A|-A + AC - B^2|^{-3} \\ &\qquad \times \big|(\tilde{P} + \tilde{Q})^2 - \tilde{R}^2 + \tilde{P}^2(g^2 + h^2) + \tilde{R}^2\big(\dot{g}^2 + \dot{h}^2\big) + \tilde{Q}^2\big(\ddot{g}^2 + \ddot{h}^2\big) \\ &\qquad\quad - 2\tilde{P}\tilde{R}\big(g\dot{g} + h\dot{h}\big) - 2\tilde{R}\tilde{Q}\big(\dot{g}\ddot{g} + \dot{h}\ddot{h}\big) + 2\tilde{P}\tilde{Q}\big(g\ddot{g} + h\ddot{h}\big)\big|. \end{aligned}$$

If we substitute the abbreviations into the last equation, we get

$$\|\mathbf{e}'_2 + k_1\mathbf{e}_1\|^2$$
$$= (\varphi')^2 A|-A+AC-B^2|^{-3}$$
$$\times \big|\tilde{P}^2 A + 2\tilde{P}\tilde{Q} + \tilde{Q}^2 - \tilde{R}^2 + \tilde{R}^2 C + \tilde{Q}^2 F - 2\tilde{P}\tilde{R}B - 2\tilde{R}\tilde{Q}E + 2\tilde{P}\tilde{Q}D\big|.$$

After substituting the equation (3.24) into the last equation and simplifying it, we have

$$k_2^2 = \|\mathbf{e}'_2 + k_1\mathbf{e}_1\|^2$$
$$= (\varphi')^2 A|-A+AC-B^2|^{-2}$$
$$\times \big|(-A+AC-B^2)(1+F) + (1-C)(1+D)^2 + 2BE(1+D) - AE^2\big|.$$

Moreover, from the equation (3.20) it is seen that

$$k_1^2 = (\varphi')^2 A^{-2}|-A+AC-B^2|.$$

The last two equation gives us

$$k_1^2 + k_2^2 = (\varphi')^2 A^{-2}|-A+AC-B^2|^{-2}$$
$$\times \big|(-A+AC-B^2)^3 + A^3\big((-A+AC-B^2)(1+F)$$
$$+ (1-C)(1+D)^2 + 2BE(1+D) - AE^2\big)\big|.$$

By the fact $\varphi' = \alpha^{-1} f^{-1} A^{-1/2}$, we obtain

$$k_1^2 + k_2^2 = \alpha^{-2} f^{-2} A^{-3}\big| -A+AC-B^2|^{-2}$$
$$\times \big|(-A+AC-B^2)^3 + A^3\big((-A+AC-B^2)(1+F)$$
$$+ (1-C)(1+D)^2 + 2BE(1+D) - AE^2\big)\big|. \quad (3.25)$$

and

$$k_1 = \alpha^{-1} f^{-1} A^{-3/2}(-A+AC-B^2)^{1/2}. \quad (3.26)$$

According to our assumption,

$$
\begin{aligned}
f = {} & (1+g^2+h^2)^{-3/2} \left| -1-g^2-h^2+\dot{g}^2+\dot{h}^2+\big(\dot{g}h-g\dot{h}\big)^2 \right|^{-5/2} \\
& \times \Big| (-1-g^2-h^2+\dot{g}^2+\dot{h}^2+\big(\dot{g}h-g\dot{h}\big)^2\big)^3 \\
& \quad - (1+g^2+h^2)^3 \Big(\big(g-\ddot{g}\big)^2+\big(h-\ddot{h}\big)^2 \\
& \qquad\qquad - \big((g\dot{h}-\dot{g}h)+(\dot{g}\ddot{h}-\ddot{g}\dot{h}) \big)^2 + \big(g\ddot{h}-\ddot{g}h\big)^2 \Big) \Big|,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
f = {} & A^{-3/2} \left| -A+AC-B^2 \right|^{-5/2} \\
& \times \Big| (-A+AC-B^2)^3 \\
& \quad + A^3 \big((1+F)+(1-C)(1+D)^2+2BE(1+D)-AE^2 \big) \Big|.
\end{aligned}
$$

If we substitute the above equations (3.25) and (3.26), we obtain

$$k_1 = \alpha(k_1^2 + k_2^2).$$

The proof is completed. □

In the above equation $\mu_1 = \mu_2 = -1$ which is the special Case 1. This formula is the parametric equation of generalized spacelike Mannheim curve with timelike second binormal vector in the Minkowski space-time E_1^4.

References

[1] Balgetir H., Bektaș M. and Ergüt M., *Bertrand Curves for Nonnull Curves in 3-Dimensional Lorentzian Space*. Hadronic J. **27** (2004), 229–236.

[2] Balgetir H., Bektaș M. and Inoguchi J., *Null Bertrand curves in Minkowski 3-space and their characterizations*. Note di Matematica **23**(1) (2004/2005), 7–13.

[3] Blum R., *A remarkable class of Mannheim curves*. Canad. Math. Bull. **9** (1966), 223–228.

[4] Do Carmo M. P., *Differential Geometry of Curves and Surfaces*, Pearson Education 1976.

[5] Eisenhart L. P. A., *Treatise on the Differential Geometry of Curves and Surfaces*, New York, Dover 1960.

[6] Ekmekçi N. and İlarslan, K., *On Bertrand curves and their characterization*. Differ. Geom. Dyn. Syst. (electronic) **3**(2) (2001), 17–24.

[7] Kuhnel W., *Differential geometry*, Curves-Surfaces-Manifolds, Braunschweig, Wiesbaden 1999.

[8] Liu H. and Wang F., *Mannheim Partner curves in* 3*-space*. Journal of Geometry **88** (2008), 120–126.

[9] Matsuda H. and Yorozu S., *Notes on Bertrand curves*. Yokohama Math. J. **50**(1–2) (2003), 41–58.

[10] Matsuda H. and Yorozu S., *On generalized Mannheim curves in Euclidean* 4*-space*, (in English). Nihonkai Math. J. **20**(1) (2009), 33–56.

[11] Nádeník Z., *Bertrand curves in five-dimensional space*, (in Russian). Czech. Math. J. **2**(77) (1952), 57–87.

[12] O'Neill B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York 1983.

[13] Orbay K. and Kasap E., *On Mannheim Partner Curves in* E^3. Int. J. of Phys. Sci. **4**(5) (2009), 261–264.

[14] Struik D. J., *Lectures on Classical Differential Geometry* (Second Ed.), Addison-Wesley, Reading, Massachusetts 1961.

[15] Tigano O., *Sulla determinazione delle curve di Mannheim*. Matematiche Catania **3** (1948), 25–29.

S. Ersoy
Department of Mathematics
Sakarya University
Sakarya, Turkey
E-mail: sersoy@sakarya.edu.tr

M. Tosun
Department of Mathematics
Sakarya University
Sakarya, Turkey
E-mail: tosun@sakarya.edu.tr

H. Matsuda
Department Mathematics
Kanazawa Medical University
Uchinada, Ishikawa, 920-02, Japan
E-mail: matsuda@kanazawa-med.ac.jp