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Some elliptic fibrations arising from free rigid body dynamics

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Abstract. An elliptic fibration over $P_3(\mathbb{C})$, naively arising from the Euler equation for free rigid body dynamics, is studied from the viewpoint of complex algebraic geometry. With this elliptic fibration, associated is an elliptic fibration in Weierstraß normal form, whose generic fibres are isomorphic to those of the original fibration. This normal form is desingularized in a canonical manner. It is shown that there is a four-to-one meromorphic mapping from the naive elliptic fibration to the Weierstraß mormal form. The latter fibration is also shown to be bimeromorphic to the family of spectral curves arising from the corresponding Manakov equation.

Key words: Free rigid body, elliptic fibration, Weierstraß normal form, quadrics intersection, spectral curve.

1. Introduction

The motion of a free rigid body can be described by the Euler equation

$$\frac{\mathsf{d}p}{\mathsf{d}t} = p \times \left(\mathsf{A}^{-1}p\right),$$

posed on the angular momentum vector $p \in \mathbb{R}^3$. Here, A stands for a positive-definite symmetric 3×3 matrix, which is called the inertia tensor of the rigid body, from the mechanical point of view. This dynamics possesses two first integrals, the energy and the squared norm of the angular momentum. The integral curve of the system is contained in the intersection of the two quadric level surfaces defined by these two first integrals, which is, in general, a real elliptic curve, consisting of two connected components. The integral curve coincides with one of them. Thus, the analysis of the dynamical system turns out to be the study of the geometry of the intersection of two quadric surfaces. If one pays attention to the branching phenomena of the dynamics (*cf.* [7]), it is natural to study the family of the integral curves of the dynamics parameterized by the eigenvalues (principal axes) of

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A. In fact, the main result of the present paper concerns the compactification of this elliptic fibration, which can be regarded as a description of the asymptotic behavior of the solutions of the Euler equation for a free rigid body around the critical values of the parameter from the mechanical point of view.

In connection with the theory of integrable Hamiltonian systems, one usually considers the Manakov equation, which is the Lax equation associated with the original Euler equation. The spectral curve associated with this Lax equation is again an elliptic curve, so that we obtain another family of elliptic curves with the same parameters.

The present paper deals with these families of elliptic curves appearing in the free rigid body dynamics from the complex algebro-geometric point of view. In Section 2, there are briefly reviewed some mechanical aspects of free rigid bodies, and then, in Section 3, one is led to the four-fold in the direct product $P_3(\mathbb{C}) \times P_3(\mathbb{C})$ of complex projective spaces defined by the following equations:

$$\begin{cases} ax^2 + by^2 + cz^2 + dw^2 = 0, \\ \qquad x^2 + y^2 + z^2 + w^2 = 0, \end{cases}$$

for $((a : b : c : d), (x : y : z : w)) \in P_3(\mathbb{C}) \times P_3(\mathbb{C})$. This four-fold naturally has the structure of elliptic fibration over $P_3(\mathbb{C})$ with respect to the projection to the first component, although it is not flat. This elliptic fibration is called the naive elliptic fibration in the present paper. It is shown that the naive elliptic fibration admits no section.

The naive elliptic fibration admits an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ as some special sign changes of the coordinates x, y, z, and w, which is free over the regular fibres, so that the quotient variety is also an elliptic fibration with the same base space. In Section 4, this quotient variety is desingularized in an appropriate way.

The main technical argument of this research is centered on Section 5. There has been made many researches on elliptic fibrations from the viewpoint of algebraic geometry since the comprehensive study of elliptic surfaces by K. Kodaira [12]. Especially, [9] and [16] treat elliptic fibrations over compact surfaces. (See also [17] [18].) Roughly speaking, these general theories assert that one can bring an elliptic fibration into the Weierstraß normal form, if it admits a global section. Since the quotient fibration has a meromorphic section, one obtains the Weierstraß normal form and desingularizes this form, so that it may, regarded as an elliptic fibration, have only singular fibres appearing in Kodaira's list of singular fibres of elliptic surfaces, as was required in [16]. (In fact, the desingularization can be obtained from an elliptic fibration over the complex projective plane.) At the end of Section 5, there is given an explicit meromorphic mapping of the quotient fibration onto the Weierstraß normal form, so that the naive elliptic fibration is mapped onto the last elliptic fibration by a four-to-one meromorphic mapping.

In the last section, Section 6, it is shown that the Weierstraß normal form and the family of the spectral curves associated with the Manakov equation are essentially the same. In fact, there is given an explicit bimeromorphic mapping between them. The Weierstraß normal form describes the family of spectral curves, although the Weierstraß normal form is introduced purely from the viewpoint of complex algebraic geometry.

It should be pointed out that the preceding studies on the integrable Hamiltonian systems from the viewpoint of algebraic geometry have been mainly concentrated on the spectral curves and their Jacobian varieties (*cf.* [2] [4] [11]), but it might be expected that the study of the integrable Hamiltonian systems would be enriched more from the algebro-geometric point of view. The authors hope that the present paper will serve as an example of such trials.

2. Review of Free Rigid Body Dynamics

A free rigid body is a spinning top whose fixed point coincide with the centre of mass. This problem is equivalent to the study of the geodesic flow over the three-dimensional rotation group $SO(3,\mathbb{R})$ with respect to the left-invariant metrics, defined by the shape of the rigid body. Moreover, using the canonical symplectic form Θ on $T^*SO(3,\mathbb{R})$, the geodesic flow can be reformulated as a Hamiltonian system $(T^*SO(3,\mathbb{R}),\Theta,\tilde{H})$ with respect to the Hamiltonian $\tilde{H}$, naturally defined by the left-invariant metric on $SO(3,\mathbb{R})$. In fact, we can regard the metric as a function on $T^*SO(3,\mathbb{R})$ canonically.

The tangent bundle $TSO(3,\mathbb{R})$ and the cotangent bundle $T^*SO(3,\mathbb{R})$ can be identified with the product $SO(3,\mathbb{R})\times\mathfrak{so}(3,\mathbb{R})$ and $SO(3,\mathbb{R})\times\mathfrak{so}(3,\mathbb{R})^*$, respectively, through the left-trivializations:

$$TSO(3,\mathbb{R}) \ni (g, X_g) \to (g, L_{g^{-1}*}X_g) \in SO(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R}),$$

and

$$T^*SO(3,\mathbb{R}) \ni (g, \alpha_g) \to (g, {L_g}^*\alpha_g) \in SO(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R})^*,$$

where L_g denotes the left-action by $g \in SO(3,\mathbb{R}) : a \mapsto ga$ for $a \in SO(3,\mathbb{R})$. The left-invariant metric $\tilde{H}$ on $SO(3,\mathbb{R})$ induces the Euclidean metric on $\mathfrak{so}(3,\mathbb{R})$, which can be realized by a real symmetric 3×3 matrix A through the Lie algebra isomorphism between $\mathfrak{so}(3,\mathbb{R})$ and $\mathbb{R}^3$ equipped with the ordinary exterior product $\times$. Using the natural projection $\pi : T^*SO(3,\mathbb{R}) \to SO(3,\mathbb{R})$, the canonical one form θ on $T^*SO(3,\mathbb{R})$ is given by $\theta(\tilde{X})|_{(g,\alpha)} = \alpha_g(\pi_*\tilde{X})$, where $\tilde{X} \in T_{(g,\alpha_g)}(T^*SO(3,\mathbb{R}))$. The canonical symplectic form Θ on $T^*SO(3,\mathbb{R})$ is defined to be $\Theta = -\mathsf{d}\theta$. The Hamiltonian vector field $\Xi_{\tilde{H}}$ of the system $(T^*SO(3,\mathbb{R}), \Theta, \tilde{H})$ is defined through $\mathsf{d}\tilde{H} = \iota_{\Xi_{\tilde{H}}}\Theta$, where ι denotes the interior product by a vector field. Identifying $T_{(g,\alpha_g)}(T^*SO(3,\mathbb{R}))$ with $T_gSO(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R})^*$, the Hamiltonian vector field $\Xi_{\tilde{H}}$ can be decomposed as $\Xi_{\tilde{H}} = ({\Xi_{\tilde{H}}}', {\Xi_{\tilde{H}}}'')$, where ${\Xi_{\tilde{H}}}' \in T_gSO(3,\mathbb{R})$ and ${\Xi_{\tilde{H}}}'' \in \mathfrak{so}(3,\mathbb{R})^*$. We have the following proposition. For the proof, see [1, p. 315, Proposition 4.4.1].

Proposition 1 *Regarding the differential* $\mathsf{d}\tilde{H}$ *on* $\mathfrak{so}(3,\mathbb{R})^*$ *as an element of* $\mathfrak{so}(3,\mathbb{R})$, *we have*

$$\Xi'_{\tilde{H}} = L_{g*}\mathsf{d}\tilde{H},$$
$$\Xi''_{\tilde{H}} = -\mathrm{ad}^*_{(\mathsf{d}\tilde{H})_\Pi}\Pi,$$

where $\Pi = L_g^*\alpha_g \in \mathfrak{so}(3,\mathbb{R})^*$.

Identifying $\mathfrak{so}(3,\mathbb{R})$ with $\mathfrak{so}(3,\mathbb{R})^*$ through the standard metric, we obtain the Euler equation of a free rigid body

$$\frac{\mathsf{d}\Pi}{\mathsf{d}t} = [\Pi, \Omega],$$

where $\Pi, \Omega \in \mathfrak{so}(3,\mathbb{R})$, and where $\Omega = (\mathsf{d}\tilde{H})_\Pi$ is the image of Π through the symmetric linear endomorphism of $\mathfrak{so}(3,\mathbb{R})$ defined by $\tilde{H}$. Equivalently, the Lie algebra isomorphism $R : \mathfrak{so}(3,\mathbb{R}) \xrightarrow{\sim} (\mathbb{R}^3, \times)$ transforms the Euler equation into

$$\frac{\mathsf{d}p}{\mathsf{d}t} = p \times (\mathsf{A}^{-1}p),$$

where $p = R(\Pi) \in \mathbb{R}^3$. Note that $\Omega = (\mathsf{d}\tilde{H})_\Pi = R^{-1}(\mathsf{A}^{-1}(R(\Pi)))$ and that there is a unique real symmetric matrix J, with which $\Pi = \mathsf{J}\Omega + \Omega\mathsf{J}$. In fact, we can assume that $\mathsf{A} = \mathrm{diag}(I_1, I_2, I_3)$. Then, we can set $\mathsf{J} = \mathrm{diag}(J_1, J_2, J_3)$, such that $I_1 = J_2 + J_3$, $I_2 = J_3 + J_1$, and $I_3 = J_1 + J_2$.

It is obviously checked that the Euler equation has two first integrals: the energy $H(p) = \frac{1}{2}p^{\mathrm{T}}\mathsf{A}^{-1}p$ and the half of the squared norm of the angular momentum $L(p) = \frac{1}{2}p^{\mathrm{T}}p$. The flow of the Euler equation is contained in the intersection of the two quadric surfaces

$$\begin{cases} \dfrac{1}{I_1}p_1^2 + \dfrac{1}{I_2}p_2^2 + \dfrac{1}{I_3}p_3^2 = 2h, \\ p_1^2 + p_2^2 + p_3^2 = 2l, \end{cases}$$

where $p = (p_1, p_2, p_3)^{\mathrm{T}}$, and where h and l are determined by the initial conditions. These equations can be transformed into

$$\begin{cases} ax^2 + by^2 + cz^2 + dw^2 = 0, \\ x^2 + y^2 + z^2 + w^2 = 0, \end{cases} \tag{1}$$

by putting $p_1 = \sqrt{-2l}\frac{x}{w}$, $p_2 = \sqrt{-2l}\frac{y}{w}$, $p_3 = \sqrt{-2l}\frac{z}{w}$, $I_1 = \frac{1}{a}$, $I_2 = \frac{1}{b}$, $I_3 = \frac{1}{c}$, and $\frac{h}{l} = d$. We consider Eqs. (1) in the next section from the complex algebro-geometric point of view.

On the other hand, it is well-known that the equation (called the Manakov equation)

$$\frac{\mathsf{d}}{\mathsf{d}t}(\Pi + \lambda\mathsf{J}^2) = [\Pi + \lambda\mathsf{J}^2, \Omega + \lambda\mathsf{J}]$$

is equivalent to the Euler equation, where λ is a complex parameter (*cf.* [13]). Its spectral curve is the set of all eigenvalues μ of the matrices $\Pi + \lambda\mathsf{J}^2$ parameterized by λ:

$$\det(\Pi + \lambda\mathsf{J}^2 - \mu\mathsf{E}) = 0, \tag{2}$$

where $(\lambda, \mu) \in \mathbb{C}^2$, and where E denotes the unit matrix. It can be shown that the generic integral curves of the Euler equation are the real part of the corresponding spectral curves on which the flows are linearized. We mention the family of the spectral curves (2) in Section 6.

Remark 1 We make a comment on the complete integrability of the dynamics for a free rigid body. Taking the momentum mapping $\varphi : T^*SO(3,\mathbb{R}) \cong SO(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R})^* \to \mathfrak{so}(3,\mathbb{R})^*$ given by $(g,\Pi) \mapsto \mathrm{Ad}^*_{g^{-1}}\Pi$ with $(g,\Pi) \in SO(3,\mathbb{R}) \times \mathfrak{so}(3,\mathbb{R})^*$, we have the following commutative diagram:

$$\begin{tikzcd}
\varphi^{-1}(\Pi) \arrow[r, hook] \arrow[d] & T^*SO(3,\mathbb{R}) \arrow[r, "\tilde{H}"] \arrow[d, "\varphi"] & \mathbb{R} \\
\mathcal{S}_\Pi \arrow[r, hook] & \mathfrak{so}(3,\mathbb{R})^* \arrow[r, "H"] & \mathbb{R}
\end{tikzcd}$$

Here, the coadjoint orbit $\mathcal{S}_\Pi$ through $\Pi \in \mathfrak{so}(3,\mathbb{R})^*$ is diffeomorphic to the quotient of the momentum manifold $\varphi^{-1}(\Pi)$ by the action of the stabilizer $SO(3,\mathbb{R})_\Pi \subset SO(3,\mathbb{R})$ at the point $\Pi \in \mathfrak{so}(3,\mathbb{R})^*$. Since the Hamiltonian system $(T^*SO(3,\mathbb{R}), \Theta, \tilde{H})$ is left-invariant, the Marsden-Weinstein reduction procedure [15] provides the reduced Hamiltonian system $(\mathcal{S}_\Pi, \omega_\Pi, H|_{\mathcal{S}_\Pi})$ on the coadjoint orbit through Π equipped with the natural symplectic form (called Kirrilov-Kostant-Souriau form) ω_Π. This reduced system is completely integrable in the sense of Liouville-Arnold [3], since the coadjoint orbits are of dimension two or of dimension zero.

3. Naive Elliptic Fibration

In this section, we study the algebraic variety defined through Eqs. (1) appearing in the previous section as the integral curve of the Euler equation up to component.

Let $(x : y : z : w)$ and $(a : b : c : d)$ be homogeneous coordinates of the complex projective space $P_3(\mathbb{C})$. It is known that the intersection of generic two quadric surfaces in $P_3(\mathbb{C})$ is an elliptic curve. Indeed, we have the following proposition. For another proof, see [4].

Proposition 2 *If a, b, c, and d are mutually distinct, then the variety C in $P_3(\mathbb{C})$ defined through Eqs.* (1) *is an elliptic curve, which has the four* *branch points a, b, c, and d as a double covering over the projective line $P_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.*

Proof. Under the hypothesis, the variety C is an algebraic curve. We consider the curve C_0 defined through the equations

$$\begin{cases} aX + bY + cZ + dW = 0, \\ \quad X + Y + Z + W = 0, \end{cases}$$

where $(X : Y : Z : W)$ is the homogeneous coordinates of $P_3(\mathbb{C})$. It is obvious that C_0 is a smooth rational curve. There are four points on C_0 where one of the coordinates is zero:

$$\begin{aligned} (X : Y : Z : W) = & \,(b - c : c - a : a - b : 0), \\ & \,(d - b : a - d : 0 : b - a), \\ & \,(d - c : 0 : a - d : c - a), \\ & \,(0 : c - d : d - b : b - c). \end{aligned} \tag{3}$$

The curve C is the inverse image of C_0 through the mapping $P_3(\mathbb{C}) \ni (x : y : z : w) \mapsto (x^2 : y^2 : z^2 : w^2) \in P_3(\mathbb{C})$ and the mapping $C \to C_0$ is a covering of degree eight with the covering group $G = \{\pm 1\}^4 / \{\pm(1, 1, 1, 1)\}$. Here, we assume that the group $\{\pm 1\}^4$ acts on $P_3(\mathbb{C})$ as

$$P_3(\mathbb{C}) \ni (x : y : z : w) \mapsto (s_1 x : s_2 y : s_3 z : s_4 w) \in P_3(\mathbb{C}), \tag{4}$$

where $(s_1, s_2, s_3, s_4) \in \{\pm 1\}^4$, and where the subgroup $\{\pm(1, 1, 1, 1)\}$ acts on $P_3(\mathbb{C})$ trivially. The branch points of the covering $C \to C_0$ are the four points in (3).

Let $G \to \{\pm 1\}$ be the group homomorphism induced by the multiplication of all four components: $\{\pm 1\}^4 \ni (s_1, s_2, s_3, s_4) \mapsto s_1 s_2 s_3 s_4 \in \{\pm 1\}$. Taking the kernel of this group homomorphism, we have the exact sequence $1 \to N \to G \to \{\pm 1\} \to 1$. The action by N on C can easily be shown to be fixed-point-free.

We obtain the sequence of the covering spaces

$$C \to C' \to C_0$$

corresponding to the above exact sequence. The second covering $C' \to C_0$ is a double-covering with four branch points (3), so that C' is an elliptic curve. There exists a lattice $L \cong \mathbb{Z}^2$, such that $C' \cong \mathbb{C}/L$, where we can naturally identify one of the branch points in C' with the zero of the Abelian group $\mathbb{C}/L$. Since the covering $C \to C'$ is unramified, there is a unique subgroup L_0 of $L = \pi_1(C')$, such that C is isomorphic to $\mathbb{C}/L_0$ and that $L/L_0 \cong N \cong \{\pm 1\}^2$, which implies $L_0 = 2L$. In other words, the mapping $C \to C'$ is a surjective homomorphism with the kernel $L/2L$, which can be identified with the group of two-torsions of C. This will be called the canonical four-to-one isogeny, later. Taking the last two components of the homogeneous coordinates of the branch points (3) as the homogeneous coordinates of $P_1(\mathbb{C})$, for example, the branch locus can be considered to be $(1:0)$ $(0:1)$, $\left(\frac{a-d}{c-a} : 1\right)$, and $\left(\frac{d-b}{b-c} : 1\right)$, which can be transformed into a, b, c, and d through an appropriate linear fraction, since the cross ratios of the both quadruplets coincide with $\frac{a-d}{a-c} \cdot \frac{b-c}{b-d}$. □

Next, we consider the algebraic variety F defined through Eqs. (1) as a four-dimensional subvariety of $P_3(\mathbb{C}) \times P_3(\mathbb{C})$.

Lemma 1 *The variety F is a smooth four-fold.*

This lemma is verified by a straightforward calculation of Jacobian matrix. Moreover, we have the following proposition.

Proposition 3 *The variety F is rational, i.e. bimeromorphic to $P_4(\mathbb{C})$.*

Proof. Taking the homogeneous coordinates $((a:b:c:d),(x:y:z:w)) \in P_3(\mathbb{C}) \times P_3(\mathbb{C})$, it can easily be checked that the projection onto the second component induces the projection from F onto the quadric surface $x^2+y^2+z^2+w^2=0$ in $P_3(\mathbb{C})$. Then, F possesses the structure of $P_2(\mathbb{C})$-bundle over $P_1(\mathbb{C}) \times P_1(\mathbb{C})$, which is bimeromorphic to $P_4(\mathbb{C})$. □

On the other hand, the projection onto the first component $P_3(\mathbb{C}) \times P_3(\mathbb{C}) \ni ((a:b:c:d),(x:y:z:w)) \mapsto (a:b:c:d) \in P_3(\mathbb{C})$ makes F an elliptic fibration over $P_3(\mathbb{C})$. This is a direct consequence of Proposition 2. (An elliptic fibration is defined to be a proper surjective holomorphic mapping between complex spaces with generic fibres being elliptic curves. We sometimes refer to its total space as an elliptic fibration, as well.) Although the four-fold itself is rational, it is not so simple to investigate the elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$. In fact, over the six planes

$$a-b=0,\ a-c=0,\ a-d=0,\ b-c=0,\ b-d=0,\ c-d=0,$$

the fibres are not smooth elliptic curves, and even over the point $(a:b:c:d)=(1:1:1:1)$, the fibre is not a curve, but a quadric surface in $P_3(\mathbb{C})$. In particular, the elliptic fibration π_F is not flat. Here, we mention the classification of the fibres of π_F, which is essentially given in [6]:

Classification of the fibres of π_F

1. In the case where the coordinates a, b, c, and d are distinct, the fibres $\pi_F^{-1}(a:b:c:d)$ are smooth elliptic curves by Proposition 2.
2. In the case where only two of the coordinates a, b, c, and d are equal, the fibre consists of two smooth rational curves intersecting at two points. This is a singular fibre of type I_2 in Kodaira's notation (*cf.* [12] or [5]).
3. In the case where two of the coordinates a, b, c, and d are equal and that the other two are also equal without further coincidence, the fibre consists of four smooth rational curves intersecting cyclically. This is a singular fibre of type I_4 in Kodaira's notation.
4. In the case where three of the coordinates a, b, c, and d are equal without further coincidence, the fibre is a smooth rational curve, as a point set, but with multiplicity two. This singular fibre is not included in the list of singular fibres of elliptic surfaces by Kodaira.
5. In the case where all the coordinates a, b, c, and d are equal, Eqs. (1) define a space quadric surface $x^2+y^2+z^2+w^2=0$.

Remark 2 Here, we briefly mention the relation between the classification of singular fibres of the naive elliptic fibration π_F and the branching phenomena of the dynamical system of the Euler equation. It is well-known that there appear two pairs of stable equilibria on the generic coadjoint orbit in $\mathfrak{so}(3)^*$, where we assume that I_1, I_2, and I_3 are distinct. (See, e.g., the front cover of [14].) These two pairs of simply closed curves correspond to the real part of the singular fibre of type I_2. (Note that we have to take care of the term $\sqrt{-2l}$, when we determine the real structure. See the transformation to obtain Eqs.(1).) There are also four heteroclinic orbits connecting the two unstable equilibria on the coadjoint orbit. The union of these four orbits, including the unstable equilibria, corresponds to the real part of the singular fibre of type I_2. Clearly, the generic integral curves appear in pair on the coadjoint orbit and they correspond to the real part of the regular fibres of the naive elliptic fibration. On the other hand, a rigid body is called a symmetric top, when two of the principal axes I_1, I_2, and I_3 are equal. In this case, the pair of generic integral curves, which are symmetric with respect to one of the three coordinate planes, and each of which is a circle, corresponds to the real part of the singular fibre of type I_2. The intersection of the coadjoint orbit and the coordinate plane, which is a circle, corresponds to the real part of the singular fibre appearing as a rational curve with multiplicity two. Further, there is one pair of stable equilibria on the coadjoint orbit, for the symmetric top. This pair corresponds to the real part of the singular fibre of type I_4. The two-dimensional fibre corresponds to the case where $I_1 = I_2 = I_3$.

The following proposition states another difficult aspect of the elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$.

Proposition 4 *There exists no local holomorphic section of π_F on any neighbourhood of the point $(a : b : c : d) = (1 : 1 : 1 : 1)$.*

Proof. We concentrate ourselves in the coordinate neighbourhood with $d \neq 0$ and $w \neq 0$ of $P_3(\mathbb{C}) \times P_3(\mathbb{C})$, where we put $d = 1$ and $w = 1$, and where we regard $((a, b, c), (x, y, z))$ as the inhomogeneous coordinates. Set $a' = a - 1$, $b' = b - 1$, and $c' = c - 1$. The equations defining F can be expressed as

$$\begin{cases} a'x^2 + b'y^2 + c'z^2 = 0, \\ x^2 + y^2 + z^2 + 1 = 0. \end{cases}$$

Assume that there is a holomorphic section near a fixed point $(a', b', c') = (a'_0, b'_0, c'_0)$. Then, the intersection of the section and the fibre over $(a', b', c') = (a'_0, b'_0, c'_0)$ can be determined by the following equations:

$$\begin{cases} F_0(a', b', c', x, y, z) = 0 \\ a'x^2 + b'y^2 + c'z^2 = 0, \\ x^2 + y^2 + z^2 + 1 = 0, \\ a' = a'_0, \\ b' = b'_0, \\ c' = c'_0, \end{cases}$$

where F_0 is a locally defined holomorphic function. The Jacobian matrix of these equations is

$$\begin{pmatrix} \dfrac{\partial F_0}{\partial a'} & \dfrac{\partial F_0}{\partial b'} & \dfrac{\partial F_0}{\partial c'} & \dfrac{\partial F_0}{\partial x} & \dfrac{\partial F_0}{\partial y} & \dfrac{\partial F_0}{\partial z} \\ x^2 & y^2 & z^2 & 2a'x & 2b'y & 2c'z \\ 0 & 0 & 0 & 2x & 2y & 2z \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since $F_0 = 0$ defines the image of the section, it can be observed that the determinant of this Jacobian matrix vanishes only at $a' = b' = c' = 0$. This is impossible, because the determinant is a holomorphic function of (a', b', c'). □

A general theory of the elliptic fibrations in [17] asserts that an elliptic fibration with a global meromorphic section has an elliptic fibration in Weierstraß normal form within its bimeromorphic equivalent class. As to our naive elliptic fibration, we have the following proposition.

Proposition 5 *The naive elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$ has no meromorphic section.*

Proof. The proposition can be shown by the famous Greek argument as follows: Assume that there would be a meromorphic section of π_F. Then, by the GAGA principle, this section should be a rational section, so that we would have four homogeneous polynomials s_0, s_1, s_2, s_3 in a, b, c, d, which would have no common divisor and which would satisfy the following:

$$\begin{cases} s_0^2 + s_1^2 + s_2^2 + s_3^2 = 0, \\ as_0^2 + bs_1^2 + cs_2^2 + ds_3^2 = 0. \end{cases} \tag{5}$$

From this, we would have

$$(a-d)s_0^2 + (b-d)s_1^2 + (c-d)s_2^3 = 0. \tag{6}$$

If one of s_0 and s_1 could be a multiple of $c - d$, the other should be also divided by $c - d$, so that s_2 should be by (6). Further, s_3 would be divisible by $c-d$ from (5). This contradicts to our assumption. Now, we assume that neither s_0 nor s_1 be divisible by $c-d$. Let $\check{s}_0$ and $\check{s}_1$ be the polynomials in a, b, c obtained from s_0 and s_1 by setting $d=c$. These polynomials would satisfy

$$(a-c)\check{s}_0^2 + (b-c)\check{s}_1^2 = 0. \tag{7}$$

This implies $\check{s}_0 = \check{s}_1 = 0$. For, otherwise, both $\check{s}_0$ and $\check{s}_1$ would be non-zero, so that $\left(\frac{\check{s}_1}{\check{s}_0}\right)^2 = \frac{a-c}{c-b}$ is a non-zero element of the rational function field in a, b, c. It is further to be noted that the polynomial ring in a, b, c is a UFD and that the notion of prime factorization naturally extends to the quotient field. Since $a-c$ and $c-b$ are obviously prime, the prime factorization of $\left(\frac{\check{s}_1}{\check{s}_0}\right)^2$ should coincide with $\frac{a-c}{c-b}$. On the other hand, this should be the double of the prime factorization of $\frac{\check{s}_1}{\check{s}_0}$, which is clearly a contradiction. From (6), s_2 would be divisible by $c-d$. Then, Eq.(5) would imply that s_3 would be divisible by $c-d$. This is again a contradiction. $\square$

From this proposition, we will rather form a quotient fibration without changing the base space in the next section, and show that it has global meromorphic sections.

4. Quotient Elliptic Fibration

Let $P_4(\mathbb{C}; 2:1:1:1:1)$ be the four-dimensional weighted projective space of weight $(2:1:1:1:1)$ and $(P:X:Y:Z:W)$ its weighted homogeneous coordinates. Note that the weighted projective space $P_4(\mathbb{C}; 2:1:1:1:1)$ is the quotient manifold of $\mathbb{C}^5 \setminus \{0\}$ by the action of $\mathbb{C}^*$ as $\mathbb{C}^5 \setminus \{0\} \ni (P, X, Y, Z, W) \mapsto (u^2P, uX, uY, uZ, uW) \in \mathbb{C}^5 \setminus \{0\}$, where $u \in \mathbb{C}^*$. We consider the four-dimensional variety Q in $P_4(\mathbb{C}; 2:1:1:1:1)$ $\times P_3(\mathbb{C})$ defined through

$$\begin{cases} P^2 = XYZW, \\ \quad 0 = aX + bY + cZ + dW, \\ \quad 0 = X + Y + Z + W, \end{cases} \tag{8}$$

where $(a:b:c:d)$ is the homogeneous coordinates of $P_3(\mathbb{C})$ as in the previous section.

The relation between the four-dimensional varieties F and Q can be described by the mapping $P_3(\mathbb{C}) \times P_3(\mathbb{C}) \ni ((x:y:z:w),(a:b:c:d)) \mapsto ((P:X:Y:Z:W),(a:b:c:d)) \in P_4(\mathbb{C};2:1:1:1:1) \times P_3(\mathbb{C})$, where $P = xyzw$, $X = x^2$, $Y = y^2$, $Z = z^2$, $W = w^2$. Indeed, this holomorphic mapping induces a surjective mapping $f : F \to Q$. We recall the group $G = \{\pm 1\}^4/\{\pm(1,1,1,1)\}$ and the exact sequence $1 \to N \to G \to \{\pm 1\} \to 1$, appearing in the proof of Proposition 2. Since the action of G on $P_3(\mathbb{C})$ is induced by that of $\{\pm 1\}^4$ in (4), it is obvious that the mapping $f : F \to Q$ is a realization of the quotient of F by N, *i.e.* $Q \cong F/N$.

We mention the singularities of the quotient variety Q. Though the ambient space $P_4(\mathbb{C};2:1:1:1:1) \times P_3(\mathbb{C})$ of Q itself has a singularity where $(P:X:Y:Z:W) = (1:0:0:0:0)$, the variety Q does not pass through it. The variety Q, however, has its own singularity.

Proposition 6 *The singularity of the variety Q consists of the six surfaces*

$$\begin{aligned}
&P=0, \quad X+Y=0, \quad Z=W=0, \quad a=b;\\
&P=0, \quad X+Z=0, \quad Y=W=0, \quad a=c;\\
&P=0, \quad X+W=0, \quad Y=Z=0, \quad a=d;\\
&P=0, \quad Y+Z=0, \quad X=W=0, \quad b=c;\\
&P=0, \quad Y+W=0, \quad X=Z=0, \quad b=d;\\
&P=0, \quad Z+W=0, \quad X=Y=0, \quad c=d,
\end{aligned} \tag{9}$$

which is the image of those points of F where the action by N has non-trivial stabilizer.

The proof of this proposition can also be performed by a straightforward calculation of the Jacobian matrix. Here, we give the list of the points on F, where the stabilizer of the action by N is non-trivial.

$(x:y:z:w)$	$(a:b:c:d)$	stabilizer
$(1:\pm\sqrt{-1}:0:0)$	$a=b$	$\langle(1,1,-1,-1)\rangle$
$(1:0:\pm\sqrt{-1}:0)$	$a=c$	$\langle(1,-1,1,-1)\rangle$
$(1:0:0:\pm\sqrt{-1})$	$a=d$	$\langle(-1,1,1,-1)\rangle$
$(0:1:\pm\sqrt{-1}:0)$	$b=c$	$\langle(1,-1,-1,1)\rangle$
$(0:1:0:\pm\sqrt{-1})$	$b=d$	$\langle(1,-1,1,-1)\rangle$
$(0:0:1:\pm\sqrt{-1})$	$c=d$	$\langle(1,1,-1,-1)\rangle$

Note that the six surfaces in (9) do not intersect with each other. This can be verified easily. Since the action by N preserves the fibres of the elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$, and since the quotients of the regular fibres by N are also smooth elliptic curves as shown in the proof of Proposition 2, the quotient variety Q is an elliptic fibration over $P_3(\mathbb{C})$, which we denote by $\pi_Q : Q \to P_3(\mathbb{C})$.

We explain the types of the fibres of π_Q. Except the fibre over $(a : b : c : d) = (1 : 1 : 1 : 1)$, the fibres of π_Q are curves.

Classification of the fibres of π_Q

1. The case where the coordinates a, b, c, and d are distinct:
   In this case, the fibres $\pi_Q^{-1}(a : b : c : d)$ are smooth elliptic curves by the proof of Proposition 2.
2. The case where only two of the coordinates a, b, c, and d are equal:
   We can assume that $a \neq b \neq c = d \neq a$ without loss of generality.

   Proposition 7 *If $a \neq b \neq c = d \neq a$, the fibre is a cubic curve with a double point, which is the intersection of the fibre $\pi_Q^{-1}(a : b : c : d)$ and the singular set $P = 0$, $Z + W = 0$, $X = Y = 0$, $c = d$ in* (9).

3. The case where two of the coordinates a, b, c, and d are equal and that the other two are also equal without further coincidence:
   We can assume that $a = b \neq c = d$ without loss of generality.

   Proposition 8 *If $a = b \neq c = d$, the fibre $\pi_Q^{-1}(a : b : c : d)$ is the union of two smooth rational curves intersecting at two points, which are the intersection of the fibre and the singular sets $P = 0$, $X + Y = 0$, $Z = W = 0$, $a = b$ and $P = 0$, $Z + W = 0$, $X = Y = 0$, $c = d$ in* (9), *respectively.*

4. The case where three of the coordinates are equal without further coincidence:
   We can assume that $a \neq b = c = d$ without loss of generality.

   Proposition 9 *If $a \neq b = c = d$, the fibre is a smooth rational curve, which intersects with the singular sets $P = 0$, $Y + Z = 0$, $X = W = 0$, $b = c$; $P = 0$, $Y + W = 0$, $X = Z = 0$, $b = d$; $P = 0$, $Z + W = 0$, $X = Y = 0$, $c = d$, at three distinct points.*

5. The case where $a = b = c = d$:

In this case, the fibre is a surface with six singular points $(P : X : Y : Z : W) = (0 : 1 : -1 : 0 : 0), (0 : 1 : 0 : -1 : 0), (0 : 1 : 0 : 0 : -1), (0 : 0 : 1 : -1 : 0), (0 : 0 : 1 : 0 : -1), (0 : 0 : 0 : 1 : -1)$, all of which are A_1-singular points. Note that these six singularities are the intersections of the fibre $\pi_Q^{-1}(1 : 1 : 1 : 1)$ and the six singular sets in (9).

Desingularizing the variety Q by blowing-ups, we can obtain a smooth elliptic fibration $\widehat{Q}$ over $P_3(\mathbb{C})$, which is bimeromorphic to Q, in the sense that there is a bimeromorphic mapping between the Zariski open subsets of the elliptic fibrations consisting of the regular fibres. In fact, the smooth variety $\widehat{Q}$ can be realized as the blowing-up of Q separately along the disjoint six surfaces (9). We mention the types of the singular fibres of the elliptic fibration $\pi_{\widehat{Q}} : \widehat{Q} \to P_3(\mathbb{C})$, which can be determined essentially by those of π_Q.

Classification of the fibres of $\pi_{\widehat{Q}}$

1. The case where the coordinates a, b, c, and d are distinct:
   In this case, the fibres $\pi_{\widehat{Q}}^{-1}(a : b : c : d)$ are smooth rational curves as in the classifications of the fibres of π_Q.
2. The case where only two of the coordinates a, b, c, and d are equal:
   In this case, the fibres $\pi_{\widehat{Q}}^{-1}(a : b : c : d)$ consist of two rational curves intersecting at two points. These are singular fibres of type I_2 in Kodaira's notation.
3. The case where two of the coordinates a, b, c, and d are equal and that the other two are also equal without further coincidence:
   In this case, the fibres $\pi_{\widehat{Q}}^{-1}(a : b : c : d)$ consist of four smooth rational curves intersecting cyclically. These are singular fibres of type I_4 in Kodaira's notation.
4. The case where three of the coordinates are equal without further coincidence:
   In this case, the fibres $\pi_{\widehat{Q}}^{-1}(a : b : c : d)$ consist of four rational curves intersecting as in Figure 1. These singular fibres do not belong to the list of singular fibres of elliptic surfaces.

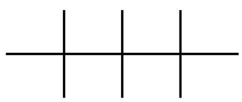

Figure 1.

5. The case where all the coordinates are equal:
   In this case, the fibre is the desingularization of the surface $\pi_Q^{-1}(1:1:1:1)$ along the six A_1-singularities. The resulting smooth surface is a del Pezzo surface of degree two, which can be obtained as a double covering of $P_2(\mathbb{C})$ branched over four lines in general position.

The elliptic fibration $\widehat{Q}$ is smooth, but is non-flat. From the viewpoint of complex algebraic geometry, we aim to construct an elliptic fibration $\widehat{W}$ which is bimeromorphic to Q and to $\widehat{Q}$, *i.e.* there are biholomorphic mappings respecting the fibrations between the Zariski open subsets of Q, $\widehat{Q}$, and $\widehat{W}$ consisting of the regular fibres, and which satisfies the following conditions:

(A) $\widehat{W}$ is smooth and is a flat elliptic fibration.
(B) The singular fibres of $\widehat{W}$ are of types contained in Kodaira's classification list of the singular fibres of elliptic surfaces (*cf.* [12] or [5]).

In order to construct the desired elliptic fibration $\widehat{W}$, it is needed to modify the base space $P_3(\mathbb{C})$ of π_Q and $\pi_{\widehat{Q}}$. In fact, it is shown that $\widehat{W}$ can be obtained from the elliptic fibration W in Weierstraß normal form, which is constructed in the next section.

As was mentioned at the last of the previous section, the quotient elliptic fibration has global meromorphic sections.

Proposition 10 *The quotient elliptic fibration $\pi_Q : Q \to P_3(\mathbb{C})$ has meromorphic sections.*

Proof. It can easily be checked that the mappings $(a:b:c:d) \mapsto (P:X:Y:Z:W) = (0:b-c:c-a:a-b:0), (0:b-d:d-a:0:a-b), (0:c-d:0:d-a:a-c), (0:0:c-d:d-b:b-c)$ give rise to global meromorphic sections of π_Q. □

This fact assures the existence of an elliptic fibration in Weierstraß normal form within its bimeromorphic equivalent class, but we construct the Weierstraß normal form in a concrete way as in the next section.

5. Weierstraß Normal Form

In order to construct the desired elliptic fibration stated as in the end of the previous section, we utilize the notion of Weierstraß normal form. Some of its generalities are mentioned in Subsection 5.1. See [8], [16], [17], [18], for detailed general discussion. In Subsection 5.2, we give the Weierstraß normal form $\pi_W : W \to P_3(\mathbb{C})$ for the naive elliptic fibration π_F. Since the Weierstraß normal form W has singularities on its total space, we desingularize it in Sub-subsection 5.3.1. After taking the blowing-up $\Phi_B : B \to P_3(\mathbb{C})$ of the base space of π_W with the centre at $(a : b : c : d) = (1 : 1 : 1 : 1)$, we introduce an elliptic fibration $\pi_T : T \to E$ in Weierstraß normal form over the exceptional set E of the blowing-up Φ_B, which will be related to π_W in Proposition 13. Because the total space T of π_T is still singular, we give its desingularization $\widehat{T}$. In fact, taking an appropriate blowing-up $\widehat{E}$ of the exceptional set E, the desingularization $\widehat{T}$ can be shown to be an elliptic fibation over $\widehat{E}$ which satisfies the conditions (A) and (B) stated at the last of the previous section. By means of this fibration and Proposition 13, the desired elliptic fibration $\pi_{\widehat{W}} : \widehat{W} \to \widehat{B}$ can be given as a fibraiton over an appropriate blowing-up $\widehat{B}$ of B. Its singular fibres are listed in Theorem 2. Finally, we give a bimeromorphic mapping between Q and W in Sub-subsection 5.3.2. Here, we add diagrams which explain the relation of the fibrations appearing in Subsection 5.3 as in Figure 2 and Figure 3:

$$\begin{tikzcd}
W \arrow[d, "\pi_W"'] & \Phi_B^* W \cong \tau_E^* T \arrow[l] \arrow[r] \arrow[d] & T \arrow[d, "\pi_T"] \\
P_3(\mathbb{C}) & B \arrow[l, "\Phi_B"] \arrow[r, "\tau_B"'] & E
\end{tikzcd}$$

Figure 2.

$$\begin{tikzcd}
\widehat{W} \arrow[rrr] \arrow[dr, "\pi_{\widehat{W}}"'] \arrow[dd] & & & \widehat{T} \arrow[dl, "\pi_{\widehat{T}}"] \arrow[dd] \\
& \widehat{B} \arrow[r] \arrow[dd] & \widehat{E} \arrow[dd] & \\
\Phi_B^* W \arrow[dr] & & & T \arrow[dl, "\pi_T"] \\
& B \arrow[r, "\tau_B"] & E &
\end{tikzcd}$$

Figure 3.

5.1. Some Generalities

Let S be a complex manifold and $L \to S$ a holomorphic line bundle over it. We choose three holomorphic sections $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3 \in H^0(S, L^{\otimes 2})$, such that $\mathrm{e}_1 + \mathrm{e}_2 + \mathrm{e}_3 = 0$. Let $(\mathsf{x} : \mathsf{y} : \mathsf{z})$ be the homogeneous fibre coordinates of the $P_2(\mathbb{C})$-bundle $P(L^{\otimes 2} \oplus L^{\otimes 3} \oplus \mathcal{O}_S)$ over S, where $\mathcal{O}_S$ denotes the structure sheaf of S, which we identify with the trivial line bundle over S. Consider the hypersurface Σ of the total space P of $P_2(\mathbb{C})$-bundle $P(L^{\otimes 2} \oplus L^{\otimes 3} \oplus \mathcal{O}_S)$ defined through the equation

$$\mathsf{y}^2\mathsf{z} = 4(\mathsf{x} - \mathrm{e}_1\mathsf{z})(\mathsf{x} - \mathrm{e}_2\mathsf{z})(\mathsf{x} - \mathrm{e}_3\mathsf{z}), \tag{10}$$

or, equivalently,

$$\mathsf{y}^2\mathsf{z} = 4\mathsf{x}^3 - g_2\mathsf{x}\mathsf{z}^2 - g_3\mathsf{z}^3, \tag{11}$$

where $g_2 = -4(\mathrm{e}_1\mathrm{e}_2 + \mathrm{e}_2\mathrm{e}_3 + \mathrm{e}_3\mathrm{e}_1)$, and $g_3 = 4\mathrm{e}_1\mathrm{e}_2\mathrm{e}_3$ are holomorphic sections of $L^{\otimes 4}$ and $L^{\otimes 6}$, respectively. The discriminant Δ and the functional invariant J are given by

$$\Delta = g_2^3 - 27g_3^2 = 16(\mathrm{e}_1 - \mathrm{e}_2)^2(\mathrm{e}_2 - \mathrm{e}_3)^2(\mathrm{e}_3 - \mathrm{e}_1)^2,$$

$$J = \frac{g_2^3}{\Delta} = -\frac{4(\mathrm{e}_1\mathrm{e}_2 + \mathrm{e}_2\mathrm{e}_3 + \mathrm{e}_3\mathrm{e}_1)^3}{(\mathrm{e}_1 - \mathrm{e}_2)^2(\mathrm{e}_2 - \mathrm{e}_3)^2(\mathrm{e}_3 - \mathrm{e}_1)^2}.$$

Restricting the natural projection $\mathsf{P} \to S$ to Σ, we have an elliptic fibration $\pi_\Sigma : \Sigma \to S$. This elliptic fibration is called in Weierstraß normal form. As is easily observed, this elliptic fibration is flat, but has singularities on its total space. These singularities can be determined as follows:

Proposition 11 ([16]) *Let G_2, G_3, and D be the divisors on S defined by $g_2 = 0$, $g_3 = 0$, and $\Delta = 0$, respectively. We write a point of P in the form $(\mathsf{x} : \mathsf{y} : \mathsf{z}; \sigma)$ with σ being its projection onto S. Then, we have the following statement.*

1. *Σ is smooth when $\mathsf{z} = 0$ and the set of $\{(\mathsf{x} : \mathsf{y} : \mathsf{z}) = (1 : 0 : 0)\}$ of the total space P gives a holomorphic section of the fibration π_Σ.*
2. *If Σ is singular at $(\mathsf{x} : \mathsf{y} : \mathsf{z}; \sigma)$, then we have $\mathsf{y} = 0$ and $\mathsf{z} \neq 0$ necessarily.*
3. *For Σ to be singular at $(0 : 0 : \mathsf{z}; \sigma)$, it is necessary and sufficient that both G_2 and G_3 contain σ and that G_3 is singular at σ.*
4. *For Σ to be singular at $(\mathsf{x} : 0 : \mathsf{z}; \sigma)$ with $\mathsf{x} \neq 0$, it is necessary and sufficient that neither G_2 nor G_3 contains σ, but D contains σ and that it is singular at σ. In this case, we have $(\mathsf{x} : 0 : \mathsf{z}) = (-3g_3 : 0 : 2g_2)$.*

From this proposition, it can be concluded that the singular fibres of the elliptic fibration π_Σ lies over the support of the divisor D on the base space S. As is discussed in the next subsection, we give the holomorphic sections e_1, e_2, and e_3 in association with the naive elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$, and construct an elliptic fibration in Weierstraß normal form.

Remark 3 In the general theory of elliptic fibrations, it is often required that an elliptic fibration in Weierstraß normal form should be minimal, in the sense that there is no prime divisor Γ on S with which both $G_2 \geq 4\Gamma$ and $G_3 \geq 6\Gamma$ are satisfied. In fact, the elliptic fibration in Weierstraß normal form which we construct in the next subsection is minimal.

5.2. Weierstraß Normal Form for F

In this subsection, we define the holomorphic line bundle L over $P_3(\mathbb{C})$ and the holomorphic sections e_1, e_2, and e_3 of $L^{\otimes 2}$, or equivalently g_2 of $L^{\otimes 4}$ and g_3 of $L^{\otimes 6}$, so that we obtain the Weierstraß normal form for the naive elliptic fibration π_F. Following Proposition 2, the regular fibres of π_F are the double covering of $P_1(\mathbb{C})$ over the four points a, b, c, and d. These four points are transformed into $\kappa := \frac{(a-c)(b-d)}{(a-d)(b-c)}$, 1, 0, and ∞ by the linear fraction $\frac{(z-c)(b-d)}{(z-d)(b-c)}$, and, subsequently, into $\frac{2\kappa-1}{3}$, $\frac{2-\kappa}{3}$, $-\frac{\kappa+1}{3}$, and ∞ by subtracting the mean of the first three. The triple $\frac{2\kappa-1}{3}$, $\frac{2-\kappa}{3}$, $-\frac{\kappa+1}{3}$ essentially determine the sections e_1, e_2, e_3, but there still remains the ambiguity of scale changes. For this reason, we are freed from the artificial denominator $(a-d)(b-c)$ and we can replace it by the square of any nontrivial section s of $\mathcal{O}_{P_3(\mathbb{C})}(1)$, so that we have on the open set U_s where $s \neq 0$:

$$
\begin{aligned}
\mathrm{e}_1 &= \frac{1}{3s^2}\{(a-b)(c-d)+(a-c)(b-d)\},\\
\mathrm{e}_2 &= \frac{1}{3s^2}\{-2(a-b)(c-d)+(a-c)(b-d)\},\\
\mathrm{e}_3 &= \frac{1}{3s^2}\{(a-b)(c-d)-2(a-c)(b-d)\}.
\end{aligned}
\tag{12}
$$

We can, then, regard them as the local expression on U_s of the sections e_1, e_2, e_3 of $\mathcal{O}_{P_3(\mathbb{C})}(2)$. Thus, we should take $L = \mathcal{O}_{P_3(\mathbb{C})}(1)$.

Further parameters can be calculated as follows:

$$
\begin{aligned}
g_2 = \frac{4}{3s^4}&\{(a-b)(c-d)+\omega(a-c)(b-d)\}\\
&\times\{(a-b)(c-d)+\omega^2(a-c)(b-d)\},\\
g_3 = \frac{4}{27s^6}&\{(a-b)(c-d)+(a-c)(b-d)\}\\
&\times\{-2(a-b)(c-d)+(a-c)(b-d)\}\\
&\times\{(a-b)(c-d)-2(a-c)(b-d)\},
\end{aligned}
$$

$$\Delta = \frac{16}{s^{12}}(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2,$$

$$J = \frac{\{(a-b)^2(c-d)^2 + (a-c)^2(b-d)^2 + (a-d)^2(b-c)^2\}^3}{54(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2}.$$

Here, $\omega = \frac{-1+\sqrt{-3}}{2}$ is a cubic root of the unit. We denote the elliptic fibration over $P_3(\mathbb{C})$ in Weierstraß normal form defined through Eq. (10) or Eq. (11) with the above parameters by $\pi_W : W \to P_3(\mathbb{C})$. (Here, W should not be confused with one of the coordinates in the weighted projective space $P_4(\mathbb{C}; 2:1:1:1:1)$.)

The three divisors G_2, G_3, and D can be described as follows:

- G_2 is the sum of the two quadric surfaces

$$(a-b)(c-d) + \omega(a-c)(b-d) = 0,$$
$$(a-b)(c-d) + \omega^2(a-c)(b-d) = 0.$$

- G_3 is the sum of the three quadric surfaces

$$(a-b)(c-d) + (a-c)(b-d) = 0,$$
$$-2(a-b)(c-d) + (a-c)(b-d) = 0,$$
$$(a-b)(c-d) - 2(a-c)(b-d) = 0.$$

- D is the sum of the six planes

$$a = b, \quad a = c, \quad a = d, \quad b = c, \quad b = d, \quad c = d,$$

  with multiplicities two.

Note that Eqs. (12) define the minimal Weierstraß normal form.

By means of Proposition 11, the singular set of W can be determined as the union of the six surfaces whose local expressions on the open set $U_s \times \{\mathsf{z} \neq 0\}$ are as follows:

$$(\mathsf{x} : \mathsf{y} : \mathsf{z}) = \left(\frac{(a-c)(a-d)}{3s^2} : 0 : 1\right), \quad a = b;$$

$$(\mathsf{x} : \mathsf{y} : \mathsf{z}) = \left(\frac{(a-b)(a-d)}{3s^2} : 0 : 1\right), \quad a = c;$$

$$
\begin{aligned}
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= \left(\frac{(a-b)(a-c)}{3s^2}:0:1\right), \quad a=d;\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= \left(\frac{(b-d)(b-a)}{3s^2}:0:1\right), \quad b=c;\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= \left(\frac{(b-c)(b-a)}{3s^2}:0:1\right), \quad b=d;\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= \left(\frac{(c-b)(c-a)}{3s^2}:0:1\right), \quad c=d.
\end{aligned} \tag{13}
$$

There are three curves in W, where two of the six surfaces in (13) intersect:

$$
\begin{aligned}
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= (0:0:1), \quad a=b, \quad c=d;\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= (0:0:1), \quad a=c, \quad b=d;\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= (0:0:1), \quad a=d, \quad b=c.
\end{aligned}
$$

There are four curves in W, where three of the six surfaces in (13) intersect:

$$
\begin{aligned}
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= (0:0:1), \quad a=b=c;\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= (0:0:1), \quad b=c=d;\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= (0:0:1), \quad c=d=a;\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= (0:0:1), \quad d=a=b.
\end{aligned}
$$

Note that all the above six surfaces in (13) intersect each other only at the point $((\mathsf{x}:\mathsf{y}:\mathsf{z}),(a:b:c:d)) = ((0:0:1),(1:1:1:1))$. We concentrate our attention on this point and construct the desingularization of W as in the next subsection.

5.3. Construction of Smooth Elliptic Fibration Bimeromorphic to Q

In this subsection, we construct an elliptic fibration which is bimeromorphic to Q and to $\widehat{Q}$ and which satisfies the conditions (A) and (B) at the last of Section 4. Firstly, we desingularize the elliptic fibration W in Weierstraß normal form. Then, we show that there is a bimeromorphic mapping between Q and W, which is biholomorphic on the Zariski open subsets of both elliptic fibrations consisting of the regular fibres.

5.3.1 Desingularization of W

In order to observe the fibration over a neighbourhood of the base point $(a : b : c : d) = (1 : 1 : 1 : 1)$, we blow up the base space $P_3(\mathbb{C})$ with $(a : b : c : d) = (1 : 1 : 1 : 1)$ as the centre. We denote the modified space by B, the canonical surjection by $\Phi_B : B \to P_3(\mathbb{C})$, and the exceptional set, isomorphic to $P_2(\mathbb{C})$, by E. Write $B^* = B \setminus E$ and $P_3(\mathbb{C})^* = P_3(\mathbb{C}) \setminus \{(1 : 1 : 1 : 1)\}$, then $B^* \cong P_3(\mathbb{C})^*$. We will regard E as the totality $\mathcal{L}$ of the lines in $P_3(\mathbb{C})$ passing through the point $(a : b : c : d) = (1 : 1 : 1 : 1)$, and the blowing-up B itself as the disjoint union of such lines. From this point of view, we naturally obtain the projection $\tau_B : B \to E$. Further, one can obtain an isomorphism from E to any plane E' of $P_3(\mathbb{C})$ not passing through $(a : b : c : d) = (1 : 1 : 1 : 1)$, letting each line of $\mathcal{L}$ corresponding to its intersection with the plane E'. We take the line bundle $L_E = \mathcal{O}_{P_2(\mathbb{C})}(1)$ over $E \cong P_2(\mathbb{C})$.

Lemma 2 *We have the following isomorphism of vector bundles*:

$$(\tau_B^*(L_E))|_{B^*} \cong (\Phi_B^*(L))|_{B^*}.$$

Proof. We can consider the line bundle $(\Phi_B^*(L))|_{B^*} \cong L|_{P_3(\mathbb{C})^*}$ to be constructed by the transition functions, e.g., $\frac{b-a}{c-a}$, $\frac{c-a}{d-a}$, and $\frac{d-a}{b-a}$, which are defined on the intersections of two of the open sets $b - a \neq 0$, $c - a \neq 0$, and $d - a \neq 0$. Note that these transition functions give rise to the line bundle L_E, when they restricted on $E' \cong E$. Since the transition functions are constant along each line $l \in \mathcal{L}$, the lemma follows. $\square$

It is to be noted that, for any positive integer k, the holomorphic sections in $H^0(P_3(\mathbb{C}), L^{\otimes k})$, which is isomorphic to $H^0(P_3(\mathbb{C}), \mathcal{O}_{P_3(\mathbb{C})}(k))$, can be identified with the homogeneous polynomials of degree k with a, b, c, d as the variables. We write the space of these homogeneous polynomials of degree k generated by $b - a, c - a, d - a$ as H'_k. Through the above identification, we can consider H'_k to be a subspace of $H^0(P_3(\mathbb{C}), L^{\otimes k})$.

Proposition 12 *We have the following identification of the holomorphic sections*:

$$H'_k \cong H^0\big(E, L_E^{\otimes k}\big).$$

Proof. Since all elements of H'_k are constant along each line $l \in \mathcal{L}$, we have $H'_k \subset H^0(E, L_E^{\otimes k})$. Then, the proposition follows by counting the dimensions of these vector spaces. $\square$

Note that the holomorphic sections $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3 \in H^0(P_3(\mathbb{C}), L^{\otimes 2})$ actually lie in the subspaces H'_2, whence $g_2 \in H^0(P_3(\mathbb{C}), L^{\otimes 4})$, $g_3 \in H^0(P_3(\mathbb{C}), L^{\otimes 6})$, and $\Delta \in H^0(P_3(\mathbb{C}), L^{\otimes 12})$ lie in H'_4, H'_6, and H'_{12}, respectively. Therefore, Proposition 12 implies that e_1, e_2, and e_3 also define a Weierstraß normal form over $E \cong P_2(\mathbb{C})$, denoted by $\pi_T : T \to E$, and that $W|_{B^*}$ coincides with the lifting of T by the projection $\tau_B|_{B^*}$ through the isomorphism of the Lemma 2. To sum up, we have the following proposition.

Proposition 13 *We have the following biholomorphic isomorphism which respects the elliptic fibrations*:

$$\tau_B^*(T)|_{B^*} \cong \Phi_B^*(W)|_{B^*}.$$

In particular, $\tau_B^*(T)$, W and $\Phi_B^*(W)$ are bimeromorphically equivalent.

We describe these holomorphic sections, by choosing the plane $a+b+c+d=0$ as the above plane E' without breaking the symmetry. For this choice, the blowing-up $\Phi_B : B \to P_3(\mathbb{C})$ can be performed in the following manner. Set $m = \frac{a+b+c+d}{4}$ and choose the point $(t_0 : t_1 : t_2 : t_3) \in P_3(\mathbb{C})$ which satisfies

$$t_0 + t_1 + t_2 + t_3 = 0,$$

and

$$\mathrm{rank}\begin{pmatrix} a-m & b-m & c-m & d-m \\ t_0 & t_1 & t_2 & t_3 \end{pmatrix} < 2.$$

Then, $(t_0 : t_1 : t_2 : t_3)$ with $t_0 + t_1 + t_2 + t_3 = 0$ serves as the blowing-up coordinates. The corresponding holomorphic sections can be described as follows:

$$\mathrm{e}_1 = \frac{1}{3s^2}\{(t_0 - t_1)(t_2 - t_3) + (t_0 - t_2)(t_1 - t_3)\},$$

$$\mathrm{e}_2 = \frac{1}{3s^2}\{-2(t_0 - t_1)(t_2 - t_3) + (t_0 - t_2)(t_1 - t_3)\},$$

$$
\begin{aligned}
\mathrm{e}_3 &= \frac{1}{3s^2}\{(t_0-t_1)(t_2-t_3)-2(t_0-t_2)(t_1-t_3)\},\\
g_2 &= \frac{4}{3s^4}\{(t_0-t_1)(t_2-t_3)+\omega(t_0-t_2)(t_1-t_3)\}\\
&\quad\times\{(t_0-t_1)(t_2-t_3)+\omega^2(t_0-t_2)(t_1-t_3)\},\\
g_3 &= \frac{4}{27s^6}\{(t_0-t_1)(t_2-t_3)+(t_0-t_2)(t_1-t_3)\}\\
&\quad\times\{-2(t_0-t_1)(t_2-t_3)+(t_0-t_2)(t_1-t_3)\}\\
&\quad\times\{(t_0-t_1)(t_2-t_3)-2(t_0-t_2)(t_1-t_3)\},\\
\Delta &= \frac{16}{s^{12}}(t_0-t_1)^2(t_0-t_2)^2(t_0-t_3)^2(t_1-t_2)^2(t_1-t_3)^2(t_2-t_3)^2, \qquad (14)
\end{aligned}
$$

where the linear polynomial s is chosen as in Subsection 5.2, so that these sections are given local representations on the open patch $s \neq 0$.

In order to construct the desingularization of W, we first desingularize T and construct the elliptic fibration $\widehat{T}$, bimeromorphic to T, and satisfying the conditions (A) and (B). To this aim, we have to modify the base plane E of π_T through the blowing-up by taking the four points $t_0 = t_1 = t_2$, $t_1 = t_2 = t_3$, $t_2 = t_3 = t_0$, and $t_3 = t_0 = t_1$ as its separate centres. After this, we blow up B along the inverse images of these four points of E through $\tau_B : B \to E$. Pulling back the elliptic fibration $\widehat{T}$ onto the modified space of B, the desired elliptic fibration, bimeromorphic to W and satisfying the conditions (A) and (B), can be obtained.

The divisor D for the elliptic three-fold $\pi_T : T \to E$ is the sum of the six lines

$$
\begin{aligned}
&l_{01} : t_0 = t_1, \quad l_{02} : t_0 = t_2, \quad l_{03} : t_0 = t_3,\\
&l_{12} : t_1 = t_2, \quad l_{13} : t_1 = t_3, \quad l_{23} : t_2 = t_3,
\end{aligned}
$$

with multiplicity two. The configuration of these six lines can be drawn as in Figure 4.

For the convenience in what follows, we introduce the homogeneous coordinates $(T_0 : T_1 : T_2) = \left(\frac{(t_0+t_1)-(t_2+t_3)}{2} : \frac{(t_0+t_2)-(t_1+t_3)}{2} : \frac{(t_1+t_2)-(t_0+t_3)}{2}\right)$ of the base plane E. Note that

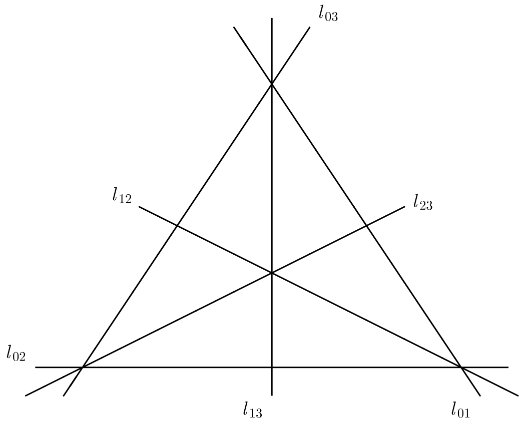

Figure 4.

$$\mathrm{e}_1 = \frac{1}{3s^2}\big(T_0^2 + T_1^2 - 2T_2^2\big),$$
$$\mathrm{e}_2 = \frac{1}{3s^2}\big(T_0^2 - 2T_1^2 + T_2^2\big),$$
$$\mathrm{e}_3 = \frac{1}{3s^2}\big(-2T_0^2 + T_1^2 + T_2^2\big),$$

in this coordinate system. The irreducible components of the discriminant divisor D can be described as

$$l_{01} : T_1 = T_2, \quad l_{02} : T_2 = T_0, \quad l_{03} : T_0 = -T_1,$$
$$l_{12} : T_0 = T_1, \quad l_{13} : T_2 = -T_0, \quad l_{23} : T_1 = -T_2.$$

By Proposition 11, the singularities of the three-fold T can be given as the union of the curves

$$(\mathsf{x} : \mathsf{y} : \mathsf{z}) = \left(\frac{T_0^2 - T_1^2}{3s^2} : 0 : 1\right), \quad T_1 = T_2;$$
$$(\mathsf{x} : \mathsf{y} : \mathsf{z}) = \left(\frac{T_1^2 - T_2^2}{3s^2} : 0 : 1\right), \quad T_2 = T_0;$$
$$(\mathsf{x} : \mathsf{y} : \mathsf{z}) = \left(\frac{T_2^2 - T_0^2}{3s^2} : 0 : 1\right), \quad T_0 = -T_1;$$

$$
\begin{aligned}
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= \left(\frac{T_2^2-T_0^2}{3s^2}:0:1\right), \quad T_0=T_1;\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= \left(\frac{T_1^2-T_2^2}{3s^2}:0:1\right), \quad T_2=-T_0;\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= \left(\frac{T_0^2-T_1^2}{3s^2}:0:1\right), \quad T_1=-T_2, \qquad (15)
\end{aligned}
$$

where $(\mathsf{x}:\mathsf{y}:\mathsf{z})$ denotes the homogeneous fibre coordinates of the $P_2(\mathbb{C})$-bundle $P(\mathcal{O}_E(2)\oplus\mathcal{O}_E(3)\oplus\mathcal{O}_E)$ over E.

Desingularization of T

(i) First, we resolve the singularities near the four points

$$
\begin{aligned}
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= (0:0:1), \quad P_0{:}(T_0:T_1:T_2)=(1:1:1);\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= (0:0:1), \quad P_1{:}(T_0:T_1:T_2)=(1:1:-1);\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= (0:0:1), \quad P_2{:}(T_0:T_1:T_2)=(1:-1:1);\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= (0:0:1), \quad P_3{:}(T_0:T_1:T_2)=(-1:1:1), \qquad (16)
\end{aligned}
$$

each of which is the intersection of three of the six singular curves in (15). For this purpose, we use the method of toroidal embeddings [10] in what follows. Note that the four points $P_0,\ P_1,\ P_2,\ P_3\in E$ can also be given by $T_0^2=T_1^2=T_2^2$. Near these four points, we have $\mathsf{z}\neq 0$, so that we can put $\mathsf{z}=1$ and regard (x,y) as the inhomogeneous fibre coordinates of the $P_2(\mathbb{C})$-bundle $P(\mathcal{O}_E(2)\oplus\mathcal{O}_E(3)\oplus\mathcal{O}_E)$. We set

$$\mathsf{p}=\mathsf{x}-\mathrm{e}_1,\quad \mathsf{q}=\mathsf{x}-\mathrm{e}_2,\quad \mathsf{r}=\mathsf{x}-\mathrm{e}_3,\quad \mathsf{y}'=\frac{\mathsf{y}}{2}.$$

Note that the quadratic function s in (14) is chosen on each neighbourhood of P_0, P_1, P_2, and P_3, such that $s\neq 0$. We can take, e.g., $s=T_0$ near these points. We can easily check that $(\mathsf{p},\mathsf{q},\mathsf{r},\mathsf{y}')$ give the local coordinates in $P(\mathcal{O}_E(2)\oplus\mathcal{O}_E(3)\oplus\mathcal{O}_E)$. Then, the equation of T is written as

$$\mathsf{y}'^2=\mathsf{pqr} \qquad (17)$$

around each of the four points (16). From now on, we operate in $\mathbb{C}^4{:}(\mathsf{p},\mathsf{q},\mathsf{r},\mathsf{y}')$ and observe the hypersurface defined by (17) in this affine space $\mathbb{C}^4$. The singular set (15) is the union of the three lines

$$\mathsf{y}' = \mathsf{p} = \mathsf{q} = 0, \quad \mathsf{y}' = \mathsf{q} = \mathsf{r} = 0, \quad \mathsf{y}' = \mathsf{r} = \mathsf{p} = 0,$$

which intersect only at a point $\mathsf{p} = \mathsf{q} = \mathsf{r} = \mathsf{y}' = 0$. Note that this point is nothing but the local expression of the above four points in (16). Regarding p, q, r, and y' as elements of $\mathbb{C}^*$, we have $\mathsf{r} = \mathsf{p}^{-1}\mathsf{q}^{-1}\mathsf{y}'^2$. For any monomial $\mathsf{p}^l\mathsf{q}^m\mathsf{y}'^n$ of p, q, y', we choose the point $(l, m, n) \in \mathbb{Z}^3$, so that any monomial corresponds to a point in $\mathbb{Z}^3$. Then, the specific monomials p, q, r, y' correspond to

$$(1, 0, 0),\ (0, 1, 0),\ (-1, -1, 2),\ (0, 0, 1),$$

respectively. The cone generated by these four vectors is described as

$$\begin{cases} 2l + n \geq 0, \\ 2m + n \geq 0, \\ n \geq 0, \end{cases}$$

where (l, m, n) are regarded as the coordinates of $\mathbb{R} \otimes \mathbb{Z}^3 = \mathbb{R}^3$, and its dual cone is generated by

$$(2, 0, 1),\ (0, 2, 1),\ (0, 0, 1),$$

which is not unimodular. To obtain a unimodular subdivision of this dual cone, we divide the triangle with the vertices $(2, 0, 1)$, $(0, 2, 1)$, and $(0, 0, 1)$, which is transverse to the dual cone, by adding the points $(1, 0, 1)$, $(1, 1, 1)$, and $(0, 1, 1)$ on the edges as in Figure 5.

The four cones

$$\begin{aligned} S_0 &: \text{generated by } (0, 1, 1),\ (1, 0, 1),\ (1, 1, 1), \\ S_1 &: \text{generated by } (2, 0, 1),\ (1, 1, 1),\ (1, 0, 1), \\ S_2 &: \text{generated by } (1, 1, 1),\ (0, 2, 1),\ (0, 1, 1), \\ S_3 &: \text{generated by } (1, 0, 1),\ (0, 1, 1),\ (0, 0, 1) \end{aligned}$$

are unimodular and give rise to the holomorphic mappings of $\mathbb{C}^4$:

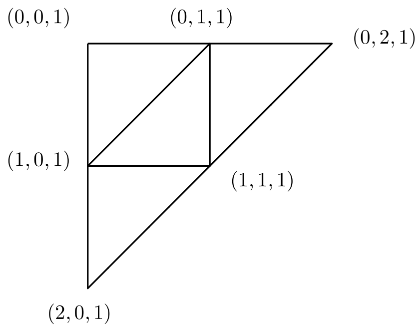

Figure 5.

$$\Omega_0 \ni (u_0, v_0, w_0) \mapsto (v_0w_0, w_0u_0, u_0v_0, u_0v_0w_0) \in \mathbb{C}^4{:}(\mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{y}'),$$

$$\Omega_1 \ni (u_1, v_1, w_1) \mapsto (u_1^2v_1w_1, v_1, w_1, u_1v_1w_1) \in \mathbb{C}^4{:}(\mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{y}'),$$

$$\Omega_2 \ni (u_2, v_2, w_2) \mapsto (u_2, u_2v_2^2w_2, w_2, u_2v_2w_2) \in \mathbb{C}^4{:}(\mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{y}'),$$

$$\Omega_3 \ni (u_3, v_3, w_3) \mapsto (u_3, v_3, u_3v_3w_3^2, u_3v_3w_3) \in \mathbb{C}^4{:}(\mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{y}').$$

Gluing these four affine spaces Ω_0, Ω_1, Ω_2, and Ω_3 in the natural way (cf. [10]), we obtain a smooth three-fold $\Omega_* = \bigcup_{i=0}^{3} \Omega_i$ and the natural holomorphic mapping $\varpi : \Omega_* \to \mathbb{C}^4$. The inverse images through ϖ of the points on $\mathsf{y}' = \mathsf{p} = \mathsf{q} = 0$, $\mathsf{y}' = \mathsf{q} = \mathsf{r} = 0$, and $\mathsf{y}' = \mathsf{r} = \mathsf{p} = 0$ except $\mathsf{p} = \mathsf{q} = \mathsf{r} = 0$ are smooth rational curves, while the inverse image of $\mathsf{p} = \mathsf{q} = \mathsf{r} = 0$ through ϖ is the union of three rational curves which intersect at a point. These three rational curves can be described as

$$\begin{aligned} v_0 = w_0 = 0, \\ w_0 = u_0 = 0, \\ u_0 = v_0 = 0, \end{aligned} \tag{18}$$

in the coordinate patch Ω_0. These three curves intersect each other only at the point $(u_0, v_0, w_0) = (0, 0, 0)$.

We look into the structure of the elliptic fibration $\pi_T \circ \varpi$ over a neighbourhood of one of the four points P_0, P_1, P_2, and P_3. The singular fibres lie over the lines l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, and l_{23}. It can immediately be checked that the singular fibres are of type I_2 in Kodaira's notation unless $T_0^2 = T_1^2 = T_2^2$, while the fibre over P_0, P_1, P_2, and P_3 is the union of the three rational curves in Eq. (18) and the one

$$u_0 = v_0 = w_0. \tag{19}$$

Although the singularity of T near the four points in (16) are resolved, the fibres are not of types in Kodaira's list over the points on P_0, P_1, P_2, and P_3.

In order to construct the desired elliptic fibration with conditions (A) and (B), we have to modify Ω_* as well as the base plane E as follows:

(a) Firstly, the total space Ω_* is blown up at the point $(u_0, v_0, w_0) = (0, 0, 0)$. The exceptional divisor $\mathcal{E}$ is isomorphic to $P_2(\mathbb{C})$. The proper transforms of the four lines in (18) and (19) intersect with $\mathcal{E}$ at distinct four points q_1, q_2, q_3, and q_4, respectively.

(b) Secondly, we blow up the base plane E at the four points P_0, P_1, P_2, and P_3, successively. We denote the resulting surface by $\widehat{E}$ and the canonical surjection by $\tau_{\widehat{E}} : \widehat{E} \to E$. Then, each of the four exceptional curves, in fact, parameterizes the pencil of conic curves in $\mathcal{E} \cong P_2(\mathbb{C})$ passing through q_1, q_2, q_3, and q_4.

(c) Finally, we blow up the modified space $\widehat{\Omega_*}$ of Ω_* along the proper transforms L_1, L_2, L_3, and L_4 of the four lines in (18) and (19), successively. Now, the fibres of the elliptic fibration become of type I_0^* or I_1^* over the exceptional rational curves of the blowing-up $\widehat{E}$. As will be seen, the singular fibres of type I_1^* correspond to the singular conics of the pencil in (b). It is to be noted that this third procedure is necessary for the condition (B) stated at the last of Section 4.

In what follows, we explain the above procedure of blowing-ups in a concrete manner. In each small neighbourhood of P_0, P_1, P_2, P_3, we can assume $T_0 \neq 0$. Thus, we take $s = T_0$ and use $(\widetilde{T_1}, \widetilde{T_2}) = \left(\left(\frac{T_1}{T_0}\right)^2 - 1, \left(\frac{T_2}{T_0}\right)^2 - 1\right)$ as the local coordinates centred at P_0, P_1, P_2, P_3, respectively. In these coordinates, we have $\mathrm{e}_1 = \frac{\widetilde{T_1} - 2\widetilde{T_2}}{3}$, $\mathrm{e}_2 = \frac{-2\widetilde{T_1} + \widetilde{T_2}}{3}$, $\mathrm{e}_3 = \frac{\widetilde{T_1} + \widetilde{T_2}}{3}$. Since $\mathrm{e}_1 = \frac{\mathsf{q}+\mathsf{r}-2\mathsf{p}}{3}$, $\mathrm{e}_2 = \frac{\mathsf{r}+\mathsf{p}-2\mathsf{q}}{3}$, the elliptic fibration of Ω_* over these neighbourhoods can be given by

$$\widetilde{T_1} = u_0(w_0 - v_0),$$

$$\widetilde{T_2} = v_0(w_0 - u_0),$$

in the patch Ω_0. The description on the other neighbourhoods can be performed similarly.

We pursue the blowing-ups step by step and determine the singular fibres of the elliptic fibration concretely.

(a) Blowing-up of Ω_* at $(u_0, v_0, w_0) = (0, 0, 0)$.
We blow up Ω_0 at $(u_0, v_0, w_0) = (0, 0, 0)$. The exceptional surface $\mathcal{E}$ is isomorphic to $P_2(\mathbb{C})$ and intersects at the four points q_1, q_2, q_3, and q_4 with the proper transforms L_1, L_2, L_3, and L_4 of the four lines in (18) and (19), respectively. Using the blowing-up coordinates $u = u_0$, $v = \frac{v_0}{u_0}$, and $w = \frac{w_0}{u_0}$ on the open set $u_0 \neq 0$, the proper transforms L_1 and L_4 of the two rational curves $v_0 = w_0 = 0$ in (18) and $u_0 = v_0 = w_0$ in (19) are given as $v = w = 0$ and $1 = v = w$, respectively, while the other proper transforms L_2 and L_3 are out of the coordinate neighbourhood $u_0 \neq 0$, which are described in the other open sets $v_0 \neq 0$ or $w_0 \neq 0$ in the same manner. Note that the proper transforms L_1, L_2, L_3, and L_4 are mutually disjoint. On the open set $u_0 \neq 0$, we have

$$\widetilde{T_1} = u^2(w - v),$$

$$\widetilde{T_2} = u^2 v(w - 1).$$

(b) Blowing-up of the base plane E.
On this step, we blow up E at P_0, P_1, P_2, and P_3. The canonical surjection from $\widehat{E}$ to E is denoted by $\tau_{\widehat{E}}$. Using the blowing-up coordinates $\overline{T_1} = \widetilde{T_1}$ and $\overline{T_2} = \frac{\widetilde{T_2}}{\widetilde{T_1}}$ in the open set $\overline{T_1} = \widetilde{T_1} \neq 0$, we have

$$\overline{T_2} = \frac{w - v}{v(w - 1)}. \tag{20}$$

On the exceptional rational curve $\overline{T_1} = 0$, the function $\overline{T_2}$ serves as the inhomogeneous coordinate. The homogeneous coordinates $(\widetilde{T_1} : \widetilde{T_2}) = (1 : \overline{T_2}) \in P_1(\mathbb{C})$ parameterize the pencil of the conic curves passing through the four points $q_1{:}v_0 = w_0 = 0$, $q_2{:}w_0 = u_0 = 0$, $q_3{:}u_0 = v_0 = 0$, and $q_4{:}u_0 = v_0 = w_0$ in the exceptional plane $\mathcal{E}$ by

$$\widetilde{T_1} v_0(w_0 - u_0) - \widetilde{T_2} u_0(w_0 - v_0) = 0,$$

where $(u_0 : v_0 : w_0)$ with $u_0 + v_0 + w_0 = 0$ is viewed as the homogeneous coordinates of $\mathcal{E} \cong P_2(\mathbb{C})$ in the similar manner as in p. 387. The points q_1, q_2, q_3, and q_4 correspond to the indeterminate points of the fraction (20), which are expressed as $v = w = 0$ and $v = w = 1$ on the open set $u = u_0 \neq 0$. Since (20) can be transformed into

$$w = \frac{(\overline{T_2} - 1)v}{\overline{T_2}(v-1)},$$

the singular conic curves correspond to the points $\overline{T_2} = 0, 1, \infty$ on the exceptional rational curve.

(c) Blowing-up of the modified space $\widehat{\Omega_*}$.
We blow up the modified space $\widehat{\Omega_*}$ obtained in (a) along the proper transforms L_1, L_2, L_3, and L_4 of the four rational curves in (18) and (19), successively. It suffices to consider the blowing-up along the rational curve $L_1 : v = w = 0$ by the symmetry. Using the blowing-up coordinates $\widetilde{u} = u$, $\widetilde{v} = v$, and $\widetilde{w} = \frac{w}{v}$ in the open subset $\widetilde{v} = v \neq 0$, we have

$$\overline{T_2} = \frac{\widetilde{w} - 1}{\widetilde{v}\widetilde{w} - 1}, \tag{21}$$

which is definite near $\widetilde{v} = \widetilde{w} = 0$. Over the points $\overline{T_2}$ on the exceptional rational curve in (b), the fibre of the elliptic fibration is the union of the conic curve given by Eq. (21) and the four rational curves $c_1(\overline{T_2})$, $c_2(\overline{T_2})$, $c_3(\overline{T_2})$, and $c_4(\overline{T_2})$ in the exceptional sets obtained by the blowing-up along L_1, L_2, L_3, and L_4, respectively. In the exceptional set $\widetilde{v} = 0$, corresponding to the proper transform $L_1 : v = w = 0$ as above, we have $\overline{T_2} = 1 - \widetilde{w}$, *i.e.* $\widetilde{w} = 1 - \overline{T_2}$, which is $c_1(\overline{T_2})$. Thus, we have the singular fibres of type I_0^* over the points on the exceptional curves in (b) with $\overline{T_2} \neq 0, 1, \infty$. Over the points $\overline{T_2} = 0, 1, \infty$, the conic curve defined by Eq. (21) is singular and consists of two rational curves intersecting at a point, one of which intersects two, e.g., $c_1(\overline{T_2})$ and $c_2(\overline{T_2})$ for $\overline{T_2} = 1$, of the above four rational curves $c_1(\overline{T_2})$, $c_2(\overline{T_2})$, $c_3(\overline{T_2})$, and $c_4(\overline{T_2})$, and the other of which intersects with the other two, e.g., $c_3(\overline{T_2})$, and $c_4(\overline{T_2})$ for $\overline{T_2} = 1$. We have the singular fibres of type I_1^* over the exceptional curves in (b) with $\overline{T_2} = 0, 1, \infty$.

(ii) Next, we consider the three singular points

$$
\begin{aligned}
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= \left(\frac{T_0^2}{3s^2}:0:1\right), \quad Q_0{:}(T_0:T_1:T_2) = (1:0:0);\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= \left(\frac{T_1^2}{3s^2}:0:1\right), \quad Q_1{:}(T_0:T_1:T_2) = (0:1:0);\\
(\mathsf{x}:\mathsf{y}:\mathsf{z}) &= \left(\frac{T_2^2}{3s^2}:0:1\right), \quad Q_2{:}(T_0:T_1:T_2) = (0:0:1). \qquad (22)
\end{aligned}
$$

By the symmetry, it suffices to analyse the fibre over Q_0. Using the coordinates

$$
\begin{aligned}
T_1'' &= \frac{T_1}{T_0} + \frac{T_2}{T_0},\\
T_2'' &= \frac{T_1}{T_0} - \frac{T_2}{T_0},\\
\mathsf{x}'' &= 2\left(\mathsf{x} - \frac{2T_0^2 - T_1^2 - T_2^2}{6T_0^2}\right) + \mathsf{y}\left(\mathsf{x} - \frac{-2T_0^2 + T_1^2 + T_2^2}{3T_0^2}\right)^{-1/2},\\
\mathsf{y}'' &= 2\left(\mathsf{x} - \frac{2T_0^2 - T_1^2 - T_2^2}{6T_0^2}\right) - \mathsf{y}\left(\mathsf{x} - \frac{-2T_0^2 + T_1^2 + T_2^2}{3T_0^2}\right)^{-1/2},
\end{aligned}
$$

and choosing $s = T_0$ as in p. 390, the equation for the Weierstraß normal form T can be written as

$$\mathsf{x}''\mathsf{y}'' = T_1''^2 T_2''^2. \qquad (23)$$

Note that the elliptic fibration can be realized by the mapping $(\mathsf{x}'', \mathsf{y}'', T_1'', T_2'') \mapsto (T_1'', T_2'')$ near Q_0. The singular fibres lie over the curves $T_1'' = 0$ and $T_2'' = 0$. The hypersurface (23) has singularity at $\mathsf{x}'' = \mathsf{y}'' = T_1'' = 0$ and $\mathsf{x}'' = \mathsf{y}'' = T_2'' = 0$. We desingularize this singularity by the method of toroidal embedding again. Regarding $\mathsf{x}'', \mathsf{y}'', T_1'', T_2''$ as elements in $\mathbb{C}^*$, we have $\mathsf{x}'' = \mathsf{y}''^{-1} T_1''^2 T_2''^2$, so that we can consider that Eq. (23) defines an Abelian Lie group isomorphic to $(\mathbb{C}^*)^3$ and y'', T_1'', T_2'' are the coordinates of this group. As in p. 391, we let $(l, m, n) \in \mathbb{Z}^3$ represent the character $\mathsf{y}''^l T_1''^m T_2''^n$, so that any monomial of y'', T_1'', T_2'' corresponds to a point in $\mathbb{Z}^3$. Then, the monomials x'', y'', T_1'', and T_2'' correspond to the elements

$$(-1,2,2),\ (1,0,0),\ (0,1,0),\ (0,0,1) \in \mathbb{Z}^3,$$

respectively. The cone generated by these four vectors is described as

$$\begin{cases} 2l+m \geq 0, \\ 2l+n \geq 0, \\ m \geq 0, \\ n \geq 0, \end{cases}$$

where $(l,m,n) \in \mathbb{R} \otimes \mathbb{Z}^3 = \mathbb{R}^3$. The dual cone is generated by

$$(2,1,0),\ (2,0,1),\ (0,1,0),\ (0,0,1). \tag{24}$$

To obtain the unimodular subdivision of this dual cone, we divide the rectangle with the vertices (24), which is transversal to the dual cone, by adding $(1,0,1)$ and $(1,1,0)$ as in Figure 6.

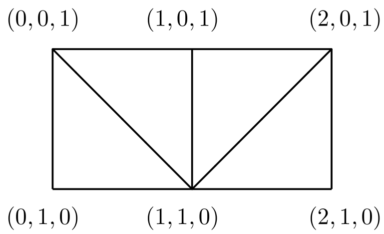

Figure 6.

The four cones

$$\begin{aligned} S_0' &: \text{generated by } \ (1,1,0),\ (0,1,0),\ (0,0,1), \\ S_1' &: \text{generated by } \ (1,0,1),\ (1,1,0),\ (0,0,1), \\ S_2' &: \text{generated by } \ (2,1,0),\ (1,1,0),\ (2,0,1), \\ S_3' &: \text{generated by } \ (2,0,1),\ (1,1,0),\ (1,0,1) \end{aligned}$$

are unimodular and the associated holomorphic mappings are

$$\begin{aligned}
\Omega_0' \ni (u_0, v_0, w_0) &\mapsto \quad \bigl(u_0 v_0^2 w_0^2, u_0, u_0 v_0, w_0\bigr) \in \mathbb{C}^4 \colon \bigl(\mathsf{x}'', \mathsf{y}'', T_1'', T_2''\bigr),\\
\Omega_1' \ni (u_1, v_1, w_1) &\mapsto \bigl(u_1 v_1 w_1^2, u_1 v_1, v_1, u_1 w_1\bigr) \in \mathbb{C}^4 \colon \bigl(\mathsf{x}'', \mathsf{y}'', T_1'', T_2''\bigr),\\
\Omega_2' \ni (u_2, v_2, w_2) &\mapsto \quad \bigl(v_2, u_2^2 v_2 w_2^2, u_2 v_2, w_2\bigr) \in \mathbb{C}^4 \colon \bigl(\mathsf{x}'', \mathsf{y}'', T_1'', T_2''\bigr),\\
\Omega_3' \ni (u_3, v_3, w_3) &\mapsto \bigl(v_3 w_3, u_3^2 v_3 w_3, v_3, u_3 w_3\bigr) \in \mathbb{C}^4 \colon \bigl(\mathsf{x}'', \mathsf{y}'', T_1'', T_2''\bigr),
\end{aligned}$$

respectively, where $\Omega_i' \cong \mathbb{C}^4$. Gluing the affine spaces Ω_i' $(i = 0, 1, 2, 3)$ naturally, we obtain a smooth three-fold $\Omega_*' := \bigcup_{i=0}^{3} \Omega_i'$ with the natural holomorphic mapping $\varpi' : \Omega_*' \to \mathbb{C}^4$. The exceptional set through ϖ' is a $P_1(\mathbb{C})$-fibration (with singular fibres) over the reducible curve $T_1'' T_2'' = 0$ in the base plane. As to the elliptic fibration, the fibre over (T_1'', T_2'') with $T_1'' = 0$ and $T_2'' \neq 0$ is described locally as

$$\begin{aligned}
u_0 = 0, w_0 = T_2''; \; v_0 = 0, &\quad\quad w_0 = T_2'' \text{ on } \Omega_0',\\
v_1 = 0, &\quad u_1 w_1 = T_2'' \text{ on } \Omega_1',\\
u_2 = 0, w_2 = T_2''; \; v_2 = 0, &\quad\quad w_2 = T_2'' \text{ on } \Omega_2',\\
v_3 = 0, &\quad u_3 w_3 = T_2'' \text{ on } \Omega_3'.
\end{aligned}$$

Four of them coincide, being extended to their closures ($\cong P_1(\mathbb{C})$). This rational curve and the remaining two curves intersect as is indicated in Figure 7. The horizontal curves in Figure 7 should be observed in a neighbourhood

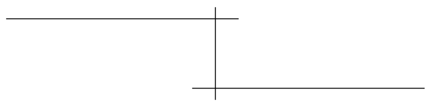

Figure 7.

of the central rational curve with respect to the original local situation. By completing the non-singular part of the fibre over (T_1'', T_2'') with $T_1'' = 0$ and $T_2'' \neq 0$, we obtain two rational curves intersecting at two points in the case where $T_2'' \neq 0$, so that this singular fibre is of type I_2. Note that $v_0 = 0, w_0 = T_2''$ on Ω_0' and $u_2 = 0, w_2 = T_2''$ on Ω_2' form the proper transforms of the two irreducible branches of the original fibre over (T_1'', T_2'') with $T_1'' = 0$ and $T_2'' \neq 0$, described by the equation $\mathsf{x}''\mathsf{y}'' = 0$. Near the exceptional set through ϖ', the proper transforms of the fibres are contained in two surfaces corresponding to the two local irreducible branches of each among them. By the similar argument, the singular fibres over (T_1'', T_2'') with $T_1'' \neq 0$ and $T_2'' = 0$ are shown to be of type I_2. On the other hand, the singular fibre over $(T_1'', T_2'') = (0, 0)$ is described as

$$u_0 = w_0 = 0;\ v_0 = w_0 = 0 \text{ on } \Omega_0',$$

$$u_1 = v_1 = 0;\ v_1 = w_1 = 0 \text{ on } \Omega_1',$$

$$u_2 = w_2 = 0;\ v_2 = w_2 = 0 \text{ on } \Omega_1',$$

$$u_3 = v_3 = 0;\ v_3 = w_3 = 0 \text{ on } \Omega_1'.$$

Each of the following three pairs of the closures ($\cong P_1(\mathbb{C})$) of these curves coincides:

$$\begin{array}{c} u_0 = w_0 = 0 \text{ on } \Omega_0'\ ;\ \ v_1 = w_1 = 0 \text{ on } \Omega_1' \\ \hline u_1 = v_1 = 0 \text{ on } \Omega_1'\ ;\ \ v_3 = w_3 = 0 \text{ on } \Omega_3' \\ \hline v_2 = w_2 = 0 \text{ on } \Omega_2'\ ;\ \ u_3 = v_3 = 0 \text{ on } \Omega_3' \end{array}$$

These three rational curves and the remaining two curves intersect as in Figure 8.

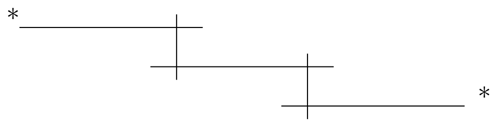

Figure 8.

The two curves indicated by $*$ in Figure 8 should be observed in a neighbourhood of the central three rational curves with respect to the original local situation. By completing the non-singular part of the fibre over $(T_1'', T_2'') = (0, 0)$, we obtain four rational curves intersecting cyclically, so that the singular fibres are of type I_4 over Q_0, Q_1, and Q_2.

Hence, we have the blowing-up of T at the three points. After blowing up T along the proper transforms of the six curves in (15) and gluing the above desingularizations, we have the elliptic fibration $\pi_{\widehat{T}} : \widehat{T} \to \widehat{E}$, bimeromorphic to W and satisfying the conditions (A) and (B). The singular fibres over the generic points on the proper transforms of the six lines l_{01}, l_{02}, l_{03}, l_{12}, l_{13}, and l_{23} are of type I_2 in Kodaira's notation.

(iii) We summarize the result from **(i)** and **(ii)** as follows:

Theorem 1 *The singular elliptic fibration $\pi_T : T \to E$ in Weierstraß normal form is bimeromorphic to the elliptic fibration*

$$\pi_{\widehat{T}} : \widehat{T} \to \widehat{E}$$

over the blowing-up $\widehat{E}$ of E at the four points $t_0 = t_1 = t_2$, $t_1 = t_2 = t_3$, $t_2 = t_3 = t_0$, and $t_3 = t_0 = t_1$, which satisfies the conditions (A) *and* (B)*. The singular fibres of $\pi_{\widehat{T}}$ can be determined as follows*:

- *The fibres over three intersection points of each of the four components of the exceptional set through $\tau_{\widehat{E}}$ and the proper transforms of the six lines $t_0 = t_1$, $t_0 = t_2$, $t_0 = t_3$, $t_1 = t_2$, $t_1 = t_3$, and $t_2 = t_3$ are of type* I_1^* *in Kodaira's notation.*
- *The fibres over the generic points on the exceptional set through $\tau_{\widehat{E}}$ are of type* I_0^*.
- *The fibres over the three points $t_0 = t_1,\ t_2 = t_3$; $t_0 = t_2,\ t_1 = t_3$; and $t_0 = t_3,\ t_1 = t_2$ are of type* I_4.
- *The fibres over the generic points of the six lines $t_0 = t_1$, $t_0 = t_2$, $t_0 = t_3$, $t_1 = t_2$, $t_1 = t_3$, and $t_2 = t_3$ are of type* I_2.

The other fibres are regular. Here, we use the coordinates $(t_0 : t_1 : t_2 : t_3)$ with the condition $t_0 + t_1 + t_2 + t_3 = 0$ as the points of E.

Remark 4 In [16], Miranda studies elliptic fibrations in Weierstraß normal form with some conditions and determines their singular fibres, using the double-covering technique. Although we use different method to desingularize the Weierstraß normal form, our result is completely consistent with his study.

Desingularization of *W*

By pulling back the elliptic fibration $\pi_{\widehat{T}} : \widehat{T} \to \widehat{E}$ through the projection $\tau_B : B \to E$, we can get the desingularization of W. More precisely, we have the following theorem. Let $\widehat{B}$ be the blowing-up of B along the proper transforms of the four lines $a = b = c$, $b = c = d$, $c = d = a$, and $d = a = b$ in $P_3(\mathbb{C})$. Denote the corresponding four exceptional divisors in $\widehat{B}$ by C_1, C_2, C_3, and C_4, respectively. Note that the natural projection $\tau_B : B \to E$ canonically induces the projection from $\widehat{B}$ onto $\widehat{E}$, which is denoted by $\tau_{\widehat{B}} : \widehat{B} \to \widehat{E}$. The pulling-back of $\pi_{\widehat{T}} : \widehat{T} \to \widehat{E}$ through $\tau_{\widehat{B}}$ is written as $\widehat{W}$ with the fibration $\pi_{\widehat{W}} : \widehat{W} \to \widehat{B}$.

Theorem 2 *The singular elliptic fibration $\pi_W : W \to P_3(\mathbb{C})$ in Weierstraß normal form is bimeromorphic to the elliptic fibration*

$$\pi_{\widehat{W}} : \widehat{W} \to \widehat{B},$$

which satisfies the conditions (A) *and* (B). *The singular fibres of $\pi_{\widehat{W}}$ can be described as follows*:

- *The fibres over the points on the intersection of the exceptional divisors C_1 (respectively C_2, C_3, and C_4) and the proper transforms of the three lines $a = b$, $b = c$, and $c = a$, (respectively $b = c$, $c = d$, and $d = b$; $c = d$, $d = a$, and $a = c$; $d = a$, $a = b$, and $b = d$) are of type I_1^* in Kodaira's notation.*
- *The fibres over the generic points on the exceptional divisors C_1, C_2, C_3, and C_4 are of type I_0^*.*
- *The fibres over the points on the proper transforms of the three lines $a = b$, $c = d$; $a = c$, $b = d$; $a = d$, $b = c$ are of type I_4.*
- *The fibres over the generic points on the proper transforms of the six planes $a = b$, $a = c$, $a = d$, $b = c$, $b = d$, and $c = d$ are of type I_2.*

5.3.2 Bimeromorphic Mapping between Q and W

We construct a bimeromorphic mapping from the quotient variety Q to W. Recall that Q is defined through Eqs. (8). We put

$$\begin{aligned} U &= a^2X + b^2Y + c^2Z + d^2W, \\ V &= a^3X + b^3Y + c^3Z + d^3W. \end{aligned} \tag{25}$$

(Here, we use W as one of the coordinates of the weighted projective space $P_5(\mathbb{C}; 2:1:1:1:1)$, while the Weierstraß normal form is denoted by the same symbol, which should not be confused.) Then, we have on the variety Q, if a, b, c, and d are distinct,

$$X = -\frac{(b+c+d)U + V}{(a-b)(a-c)(a-d)},$$

$$Y = -\frac{(c+d+a)U + V}{(b-a)(b-c)(b-d)},$$

$$Z = -\frac{(d+a+b)U+V}{(c-a)(c-b)(c-d)},$$
$$W = -\frac{(a+b+c)U+V}{(d-a)(d-b)(d-c)}, \tag{26}$$

so that

$$\begin{aligned} P^2 &= XYZW \\ &= \frac{1}{(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2} \\ &\quad \times \{(b+c+d)U+V\}\{(c+d+a)U+V\} \\ &\quad \times \{(d+a+b)U+V\}\{(a+b+c)U+V\}. \end{aligned}$$

If we put

$$\begin{aligned} \mathcal{X} &= \frac{1}{3(d-a)(d-b)(d-c)} \\ &\quad \times \big[\{(d-b)(d-c)(b+c+d) + (d-c)(d-a)(c+d+a) \\ &\qquad\qquad + (d-a)(d-b)(d+a+b)\}U \\ &\qquad + \{(d-b)(d-c) + (d-c)(d-a) + (d-a)(d-b)\}V\big], \\ \mathcal{Y} &= \frac{2(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)P}{\{(a+b+c)U+V\}}, \\ \mathcal{Z} &= -\frac{(a+b+c)U+V}{(d-a)(d-b)(d-c)}, \end{aligned} \tag{27}$$

we obtain the equation

$$\mathcal{Y}^2\mathcal{Z} = 4(\mathcal{X} - \widehat{\mathrm{e}_1}\mathcal{Z})(\mathcal{X} - \widehat{\mathrm{e}_2}\mathcal{Z})(\mathcal{X} - \widehat{\mathrm{e}_3}\mathcal{Z}), \tag{28}$$

where

$$\widehat{\mathrm{e}_1} = \frac{1}{3}\{(a-b)(c-d) + (a-c)(b-d)\},$$
$$\widehat{\mathrm{e}_2} = \frac{1}{3}\{-2(a-b)(c-d) + (a-c)(b-d)\},$$

$$\widehat{e_3} = \frac{1}{3}\{(a-b)(c-d) - 2(a-c)(b-d)\}. \tag{29}$$

Eq. (28) is equivalent to the equation of the elliptic fibration W in Weierstraß normal form defined in Subsection 5.2, if we divide the parameters by the suitable nontrivial holomorphic sections of $\mathcal{O}_{P_3(\mathbb{C})}(1)$. Since (25), (26), and (27), together with the substitution $P = xyzw$, $X = x^2$, $Y = y^2$, $Z = z^2$, $W = w^2$, are rational with respect to the coordinates $((a:b:c:d),(x:y:z:w))$, we conclude that there is a bimeromorphic mapping between Q and W, which is biholomorphic on the Zariski open subsets of the two elliptic fibrations consisting of the regular fibres.

Combining the previous result, we have the following theorem.

Theorem 3 *There is a four-to-one meromorphic mapping from the naive elliptic fibration $\pi_F : F \to P_3(\mathbb{C})$ onto the elliptic fibration $\pi_{\widehat{W}} : \widehat{W} \to \widehat{B}$ satisfying the conditions* (A) *and* (B)

$$g : F - \cdots \to \widehat{W}.$$

The mapping g is a four-to-one holomorphic mapping between the Zariski open subsets of F and $\widehat{W}$ consisting of regular fibres and induces the canonical four-to-one isogeny (*cf. the proof of Proposition* 2) *of the regular fibres of π_F onto those of $\pi_{\widehat{W}}$.*

6. Relation to the Family of the Spectral Curves

In this section, we use the notation in Section 2. The spectral curve is the completion of the affine curve defined through Eq. (2). This equation can be written as

$$(J_1^2\lambda - \mu)(J_2^2\lambda - \mu)(J_3^2\lambda - \mu) + 2h'\lambda - 2l\mu = 0, \tag{30}$$

where we set $h' = \frac{1}{2}(J_1^2p_1^2 + J_2^2p_2^2 + J_3^2p_3^2)$, which can be calculated as

$$h' = I_1I_2I_3h + \frac{(I_1+I_2+I_3)^2 - 4(I_1I_2+I_2I_3+I_3I_1)}{4}l.$$

Since the parameters J_1, J_2, and J_3 are determined by I_1, I_2, I_3, h, and l, Eq. (30) describes the family of spectral curves parameterized by I_1, I_2, I_3, h, and l. Furthermore, we can give its completion as follows: By putting

$$\lambda = \frac{\sqrt{-2l}}{2I_1I_2I_3}\left[\frac{\mathcal{X}}{\mathcal{Y}} - \frac{1}{3}\left\{\left(J_1^2 - \frac{h'}{l}\right)\left(J_2^2 - \frac{h'}{l}\right) + \left(J_2^2 - \frac{h'}{l}\right)\left(J_3^2 - \frac{h'}{l}\right)\right.\right.$$
$$\left.\left. + \left(J_3^2 - \frac{h'}{l}\right)\left(J_1^2 - \frac{h'}{l}\right)\right\}\frac{\mathcal{Z}}{\mathcal{Y}}\right],$$
$$\frac{h'}{l}\lambda - \mu = \left(J_1^2 - \frac{h'}{l}\right)\left(J_2^2 - \frac{h'}{l}\right)\left(J_3^2 - \frac{h'}{l}\right)\frac{\sqrt{-2l}}{2(I_1I_2I_3)^3}\frac{\mathcal{Z}}{\mathcal{Y}},$$

and through some lengthy calculation, we obtain

$$\mathcal{Y}^2\mathcal{Z} = 4(\mathcal{X} - \widehat{e_1}\mathcal{Z})(\mathcal{X} - \widehat{e_2}\mathcal{Z})(\mathcal{X} - \widehat{e_3}\mathcal{Z}),$$

where $\widehat{e_1}$, $\widehat{e_2}$, and $\widehat{e_3}$ are the same parameters as in Eqs. (29). Thus, the above family of the spectral curves is nothing but the elliptic fibration $\pi_W : W \to P_3(\mathbb{C})$ in Weierstraß normal form, or, in other words, is described by the Weierstraß normal form W, which is rather easier to deal with from the viewpoint of complex algebraic geometry. Furthermore, this would enrich the another significance of our study on the elliptic fibration W from the viewpoint of the theory of finite-dimensional integrable systems.

7. Concluding Remarks

Starting with the naive elliptic fibration which naturally appears from the integral curves of the Euler equation, we give the elliptic fibration $\pi_{\widehat{W}} : \widehat{W} \to \widehat{B}$ which satisfies the conditions (A) and (B) stated at the end of Section 4. In fact, we have determined the types of the singular fibres of $\pi_{\widehat{W}}$ in Theorem 2. Finally, we have shown that the elliptic fibration $\pi_{\widehat{W}}$ is bimeromorphic to the family of the spectral curves in Section 6. More precisely, we have shown that the Weierstraß normal form π_W is bimeromorphic to the family of the spectral curves. On the other hand, the Weierstraß normal form π_W has been naturally constructed from the naive fibration π_F and these two fibrations have been related to each other through a four-to-one meromorphic mapping. Thus, one can see that the Weierstraß normal form explains the relation between the naive elliptic fibration, which is nothing but the family of integral curves of the Euler equation, and the family of spectral curves associated with the Manakov equation.

As can be observed easily, the free rigid body dynamics is trivial in the case where $(a : b : c : d) = (1 : 1 : 1 : 1)$, since the Hamiltonian vector field is null when $(a : b : c : d) = (1 : 1 : 1 : 1)$. However, the asymptotic behavior of the dynamical system has a large diversity when the parameter $(a : b : c : d)$ approaches this critical point $(1 : 1 : 1 : 1)$. In fact, this diversity can be understood from the elliptic fibration π_T over the exceptional set E by blowing up $P_3(\mathbb{C})$ with the centre $(a : b : c : d) = (1 : 1 : 1 : 1)$. Moreover, the fibration π_T has essential importance, in the sense that the fibration $\pi_{\widehat{W}} : \widehat{W} \to \widehat{B}$ has been obtained from $\pi_T : T \to E$ through the projection $\tau_B : B \to E$. Thus, one can say that the flatness imposed in the condition (A) stated at the end of Section 4 is realized by the limit fibration on the space E parameterizing the asymptotic behavior.

The finally obtained fibration $\pi_{\widehat{W}} : \widehat{W} \to \widehat{B}$ admits only the singular fibres included in the Kodaira's list of singular fibres for elliptic surfaces. Such a condition is useful to determine the monodromy of the fibration. In fact, the generic monodromy matrices near the singular fibres of types I_2, I_4, and I_0^* are determined, up to conjugacy, to be

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

respectively. These monodromy matrices also reflect the bifurcation phenomena of the dynamical system of free rigid bodies around the critical points.

These geometric descriptions of the asymptotic behavior of the free rigid body dynamics will reveal deeper meaning of the branching phenomena of this dynamics, together with the explicit description of the solution to the dynamics, which should be left to further investigations.

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