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Representing and interpolating sequences on parabolic Bloch type spaces

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Abstract. Let H be the upper half-space of the Euclidean space. The α-parabolic Bloch type space $\mathcal{B}_\alpha(\sigma)$ on H is the set of all solutions u of the parabolic equation $(\partial/\partial t + (-\Delta_x)^\alpha)u = 0$ with $0 < \alpha \le 1$ which belong to $C^1(H)$ and have finite Bloch norm with weight t^σ. In this paper, we study representing and interpolating sequences on parabolic Bloch type spaces. In our previous paper [8], the reproducing formula on $\mathcal{B}_\alpha(\sigma)$ is given. A representing sequence gives a discrete version of the reproducing formula on $\mathcal{B}_\alpha(\sigma)$. Interpolating sequences are closely related to representing sequences, and such sequences are very interesting in their own right.

Key words: Bloch space, parabolic operator of fractional order, representing sequence, interpolating sequence.

1. Introduction

Let $n \ge 1$ and let H be the upper half-space of the $(n+1)$-dimensional Euclidean space, that is, $H = \{X = (x, t) \in \mathbb{R}^{n+1} : x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ t > 0\}$. For $0 < \alpha \le 1$, the parabolic operator $L^{(\alpha)}$ is defined by

$$L^{(\alpha)} := \partial_t + (-\Delta_x)^\alpha, \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $\partial_\ell = \partial/\partial x_\ell$, and $\Delta_x = \partial_1^2 + \cdots + \partial_n^2$. Let $C(H)$ be the set of all real-valued continuous functions on H, and let $C_0(H)$ be the set of all functions in $C(H)$ which vanish continuously at $\partial H \cup \{\infty\}$. For a positive integer k, $C^k(H)$ denotes the set of all k times continuously differentiable functions on H, and put $C^\infty(H) = \cap_k C^k(H)$. Furthermore, let $C_c^\infty(H)$ be the set of all functions in $C^\infty(H)$ with compact support. A function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if $L^{(\alpha)}u = 0$ in the sense of distributions (for details, see Section 2). Put $m(\alpha) = \min\{1, 1/(2\alpha)\}$. For a real number $\sigma > -m(\alpha)$, let $\mathcal{B}_\alpha(\sigma)$ be the set of all $L^{(\alpha)}$-harmonic functions $u \in C^1(H)$ with the norm

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$$\|u\|_{\mathcal{B}_\alpha(\sigma)} := |u(0,1)| + \sup_{(x,t)\in H} t^\sigma \big\{ t^{1/(2\alpha)} |\nabla_x u(x,t)| + t|\partial_t u(x,t)| \big\} < \infty, \tag{1.2}$$

where $\nabla_x = (\partial_1, \ldots, \partial_n)$. We call $\mathcal{B}_\alpha(\sigma)$ the α-parabolic Bloch type space. Since $\mathcal{B}_\alpha(\sigma)$ contains constant functions, we may identify $\mathcal{B}_\alpha(\sigma)/\mathbb{R} \cong \widetilde{\mathcal{B}}_\alpha(\sigma)$, where

$$\widetilde{\mathcal{B}}_\alpha(\sigma) := \big\{ u \in \widetilde{\mathcal{B}}_\alpha(\sigma) : u(0,1) = 0 \big\}.$$

The α-parabolic Bloch type space $\mathcal{B}_\alpha(\sigma)$ is introduced and studied in our previous paper [8]. The authors mainly studied fundamental properties and reproducing formulae for functions of $\mathcal{B}_\alpha(\sigma)$ in [8]. We remark that $\mathcal{B}_\alpha(\sigma)$ and $\widetilde{\mathcal{B}}_\alpha(\sigma)$ are Banach spaces with the norm (1.2) (see [8, Theorem 3.2]). It is also shown that when $\alpha = 1/2$, every $u \in \mathcal{B}_{1/2}(\sigma)$ is harmonic on H (see [8, Remark 3.3]). Thus, $\mathcal{B}_{1/2}(\sigma)$ coincides with the harmonic Bloch type space.

In this paper, we study representing and interpolating sequences on parabolic Bloch type spaces. First, we describe the definition of $\widetilde{\mathcal{B}}_\alpha(\sigma)$-representing sequences. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $k \in \mathbb{N}_0$, a function ω_α^k on $H \times H$ is defined by

$$\omega_\alpha^k(X;Y) = \omega_\alpha^k(x,t;y,s) := \mathcal{D}_t^k W^{(\alpha)}(x-y, t+s) - \mathcal{D}_t^k W^{(\alpha)}(-y, 1+s) \tag{1.3}$$

for all $X = (x,t), Y = (y,s) \in H$, where $\mathcal{D}_t = -\partial_t$ and $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$ (see Section 2 for definition). Let ℓ^∞ be the Banach space of all bounded sequences. Furthermore, let $\mathbb{X} = \{X_j\} = \{(x_j, t_j)\}$ be a sequence in H. For $\{\lambda_j\} \in \ell^\infty$, let

$$U_{\sigma,\mathbb{X}}^k\{\lambda_j\}(X) := \sum_j \lambda_j t_j^{n/2\alpha + k - \sigma} \omega_\alpha^k(X; X_j) \tag{1.4}$$

for all $X \in H$. We say that $\{X_j\}$ is a $\widetilde{\mathcal{B}}_\alpha(\sigma)$-representing sequence of order k if $U_{\sigma,\mathbb{X}}^k\{\lambda_j\} \in \widetilde{\mathcal{B}}_\alpha(\sigma)$ for all $\{\lambda_j\} \in \ell^\infty$ and the operator $U_{\sigma,\mathbb{X}}^k : \ell^\infty \to \widetilde{\mathcal{B}}_\alpha(\sigma)$ is bounded and onto.

Next, we describe definition of $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating sequences. Let $k \in \mathbb{N}$. For $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$, we define a sequence of real numbers $T_{\sigma,\mathbb{X}}^k u$ by

$$T^k_{\sigma,\mathbb{X}} u := \{ t_j^{k+\sigma} \partial_t^k u(X_j) \}. \tag{1.5}$$

We say that $\{X_j\}$ is a $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating sequence of order k if the operator $T^k_{\sigma,\mathbb{X}} : \widetilde{\mathcal{B}}_\alpha(\sigma) \to \ell^\infty$ is bounded and onto.

It is known that for every $k \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$t^{k+\sigma} \left| \partial_t^k u(x,t) \right| \leq C \|u\|_{\mathcal{B}_\alpha(\sigma)}$$

for all $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$ and $(x,t) \in H$ (see [8, Theorem 3.2 (4)]). Thus, $T^k_{\sigma,\mathbb{X}} : \widetilde{\mathcal{B}}_\alpha(\sigma) \to \ell^\infty$ is always bounded, and this is the reason why we consider a weight $t_j^{k+\sigma}$ in definition of the operator $T^k_{\sigma,\mathbb{X}}$. We note that our definitions and investigations for such sequences are more general, that is, we shall study properties of operators $U^k_{\sigma,\mathbb{X}}$ and $T^k_{\sigma,\mathbb{X}}$ when k is a fractional order.

Representation theorems for holomorphic and harmonic functions in L^p were studied in [3]. Also, interpolating sequences for the classical Hardy space H^∞ were studied by L. Carleson [1], and many investigations on various settings are well known. In [8], the authors give reproducing formulae on the function space $\widetilde{\mathcal{B}}_\alpha(\sigma)$. A representing sequence gives the discrete version of the reproducing formula on the function space $\widetilde{\mathcal{B}}_\alpha(\sigma)$. We study a sufficient condition for a sequence in H to be the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-representing sequence. The interpolating sequences are closely related to representing sequences, and such sequences are interesting in their own right. In this paper, we also study $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating sequences.

We describe the construction of this paper. In Section 2, we present preliminary results of parabolic Bloch type spaces. In particular, we recall definitions of $L^{(\alpha)}$-harmonic functions and the fundamental solution of $L^{(\alpha)}$. In Section 3, we study a necessary and sufficient condition for a sequence $\mathbb{X} \subset H$ which ensures that the operator $U^k_{\sigma,\mathbb{X}} : \ell^\infty \to \widetilde{\mathcal{B}}_\alpha(\sigma)$ is bounded. In Section 4, we study properties of the operator $T^k_{\sigma,\mathbb{X}}$. As mentioned above, $T^k_{\sigma,\mathbb{X}} : \widetilde{\mathcal{B}}_\alpha(\sigma) \to \ell^\infty$ is always bounded. Therefore, we study boundedness of $T^k_{\sigma,\mathbb{X}}$ on a subspace of $\widetilde{\mathcal{B}}_\alpha(\sigma)$. In Section 5, we give our representing theorem, that is, we give a sufficient condition for a sequence $\mathbb{X} \subset H$ to be the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-representing sequence. In Section 6, we give our interpolating theorem, that is, we give a sufficient condition for a sequence $\mathbb{X} \subset H$ to be the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating sequence.

Throughout this paper, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line.

2. Preliminaries

In this section, we recall some basic properties. We begin with describing the operator $(-\Delta_x)^\alpha$ and the $L^{(\alpha)}$-harmonic functions. Since the case $\alpha = 1$ is trivial, we only describe the case $0 < \alpha < 1$. For $0 < \alpha < 1$, $(-\Delta_x)^\alpha$ is the convolution operator defined by

$$(-\Delta_x)^\alpha \psi(x,t) := -C_{n,\alpha} \lim_{\delta \downarrow 0} \int_{|y|>\delta} \big(\psi(x+y,t) - \psi(x,t)\big)|y|^{-n-2\alpha} dy \quad (2.1)$$

for all $\psi \in C_c^\infty(H)$ and $(x,t) \in H$, where $C_{n,\alpha} = -4^\alpha \pi^{-n/2}\Gamma((n+2\alpha)/2)/\Gamma(-\alpha) > 0$. Let $\widetilde{L}^{(\alpha)} := -\partial_t + (-\Delta_x)^\alpha$ be the adjoint operator of $L^{(\alpha)}$. Then, a function $u \in C(H)$ is said to be $L^{(\alpha)}$-harmonic if u satisfies $L^{(\alpha)}u = 0$ in the sense of distributions, that is, $\int_H |u\, \widetilde{L}^{(\alpha)}\psi| dV < \infty$ and $\int_H u\, \widetilde{L}^{(\alpha)}\psi dV = 0$ for all $\psi \in C_c^\infty(H)$, where dV is the Lebesgue measure on H. We describe the fundamental solution of $L^{(\alpha)}$. For $x \in \mathbb{R}^n$, let

$$W^{(\alpha)}(x,t) := \begin{cases} \dfrac{1}{(2\pi)^n} \displaystyle\int_{\mathbb{R}^n} \exp(-t|\xi|^{2\alpha} + i\, x\cdot\xi)\, d\xi & (t > 0) \\ 0 & (t \le 0), \end{cases}$$

where $x\cdot\xi$ denotes the inner product on $\mathbb{R}^n$ and $|\xi| = (\xi\cdot\xi)^{1/2}$. The function $W^{(\alpha)}$ is the fundamental solution of $L^{(\alpha)}$ and it is $L^{(\alpha)}$-harmonic on H. We note that

$$W^{(\alpha)} > 0 \text{ on } H \quad \text{and} \quad \int_{\mathbb{R}^n} W^{(\alpha)}(x,t)dx = 1 \text{ for all } 0 < t < \infty. \quad (2.2)$$

Furthermore, $W^{(\alpha)} \in C^\infty(H)$.

Since we treat fractional calculus in our investigations, we recall definitions of the fractional integral and differential operators for functions on $\mathbb{R}_+ = (0,\infty)$ (for details, see [4]). For a real number $\kappa > 0$, let

$$\mathcal{FC}^{-\kappa} := \big\{\varphi \in C(\mathbb{R}_+) : \varphi(t) = O(t^{-\kappa'}) \ (t \to \infty) \text{ for some } \kappa' > \kappa\big\}. \quad (2.3)$$

For a function $\varphi \in \mathcal{FC}^{-\kappa}$, we can define the fractional integral $\mathcal{D}_t^{-\kappa}\varphi$ of φ by

$$\mathcal{D}_t^{-\kappa}\varphi(t) := \frac{1}{\Gamma(\kappa)}\int_0^\infty \tau^{\kappa-1}\varphi(\tau+t)d\tau, \quad t \in \mathbb{R}_+. \tag{2.4}$$

We put $\mathcal{FC}^0 := C(\mathbb{R}_+)$ and $\mathcal{D}_t^0\varphi := \varphi$. Moreover, let

$$\mathcal{FC}^\kappa := \left\{\varphi; \partial_t^{\lceil\kappa\rceil}\varphi \in \mathcal{FC}^{-(\lceil\kappa\rceil-\kappa)}\right\}, \tag{2.5}$$

where $\lceil\kappa\rceil$ is the smallest integer greater than or equal to κ. Then, we can also define the fractional derivative $\mathcal{D}_t^\kappa\varphi$ of $\varphi \in \mathcal{FC}^\kappa$ by

$$\mathcal{D}_t^\kappa\varphi(t) := \mathcal{D}_t^{-(\lceil\kappa\rceil-\kappa)}\big((-\partial_t)^{\lceil\kappa\rceil}\varphi\big)(t), \quad t \in \mathbb{R}_+. \tag{2.6}$$

Clearly, when $\kappa \in \mathbb{N}_0$, the operator $\mathcal{D}_t^\kappa$ coincides with the ordinary differential operator $(-\partial_t)^\kappa$. For a multi-index $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}_0^n$, let $\partial_x^\gamma := \partial_1^{\gamma_1} \cdots \partial_n^{\gamma_n}$. We present some properties of fractional derivatives of the fundamental solution $W^{(\alpha)}$.

Lemma 2.1 ([4, Theorem 3.1]) *Let $0 < \alpha \le 1$ and let ν be a real number such that $\nu > -\frac{n}{2\alpha}$. Let $\gamma \in \mathbb{N}_0^n$ be a multi-index. Then, the following statements hold.*

(1) *The derivatives $\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t)$ and $\mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t)$ can be defined, and the equation $\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t) = \mathcal{D}_t^\nu \partial_x^\gamma W^{(\alpha)}(x,t)$ holds. Furthermore, there exists a constant $C = C(n,\alpha,\gamma,\nu) > 0$ such that*

$$\left|\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t)\right| \le C(t+|x|^{2\alpha})^{-\left(\frac{n+|\gamma|}{2\alpha}+\nu\right)}$$

for all $(x,t) \in H$.

(2) *If a real number κ satisfies the condition $\kappa+\nu > -\frac{n}{2\alpha}$, then the derivative $\mathcal{D}_t^\kappa \partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t)$ is well defined, and*

$$\mathcal{D}_t^\kappa \partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t) = \partial_x^\gamma \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(x,t).$$

(3) *The derivative $\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t)$ is $L^{(\alpha)}$-harmonic on H.*

(4) *The derivative $\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t)$ satisfies the homogeneous property, that is,*

$$\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t) = t^{-\left(\frac{n+|\gamma|}{2\alpha}+\nu\right)}\left(\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}\right)(t^{-\frac{1}{2\alpha}}x,1)$$

for all $(x,t) \in H$.

We note that $\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(-x,t) = (-1)^{|\gamma|}\partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x,t)$ by the definition of $W^{(\alpha)}$. We also describe basic properties of fractional derivatives of functions in $\mathcal{B}_\alpha(\sigma)$.

Lemma 2.2 ([8, Proposition 5.4]) *Let* $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, *and let* κ *be a real number such that* $\kappa = 0$ *or* $\kappa > \max\{0, -\sigma\}$. *Let* $\gamma \in \mathbb{N}_0^n$ *be a multi-index. If* $u \in \mathcal{B}_\alpha(\sigma)$, *then the following statements hold.*

(1) *The derivatives* $\partial_x^\gamma \mathcal{D}_t^\kappa u(x,t)$ *and* $\mathcal{D}_t^\kappa \partial_x^\gamma u(x,t)$ *can be defined, and the equation* $\partial_x^\gamma \mathcal{D}_t^\kappa u(x,t) = \mathcal{D}_t^\kappa \partial_x^\gamma u(x,t)$ *holds. Furthermore, if* $(\gamma,\kappa) \neq (0,0)$, *then there exists a constant* $C = C(n,\alpha,\sigma,\gamma,\kappa) > 0$ *such that*

$$\left|\partial_x^\gamma \mathcal{D}_t^\kappa u(x,t)\right| \le C t^{-\left(\frac{|\gamma|}{2\alpha}+\kappa+\sigma\right)}\|u\|_{\mathcal{B}_\alpha(\sigma)}$$

for all $(x,t) \in H$.

(2) *If* $\nu = 0$ *or* $\nu > \max\{0,-\sigma\}$, *then*

$$\mathcal{D}_t^\nu \partial_x^\gamma \mathcal{D}_t^\kappa u(x,t) = \partial_x^\gamma \mathcal{D}_t^{\nu+\kappa} u(x,t) \tag{2.7}$$

Furthermore, if $\nu < 0$, *then* (2.7) *also holds when* $\nu < \sigma$ *and* $\nu + \kappa > \max\{0, -\sigma\}$.

(3) *The derivative* $\partial_x^\gamma \mathcal{D}_t^\kappa u$ *is* $L^{(\alpha)}$*-harmonic on* H.

We give the definition of the kernel function, which is generalization of (1.3). Let $I_{\alpha,n}$ be an interval $(-\frac{n}{2\alpha}, \infty)$. Then, for $(\gamma,\kappa) \in \mathbb{N}_0^n \times I_{\alpha,n}$, in view of Lemma 2.1, we define a function $\omega_\alpha^{\gamma,\nu}$ on $H \times H$ by

$$\begin{aligned}\omega_\alpha^{\gamma,\nu}(X;Y) &= \omega_\alpha^{\gamma,\nu}(x,t;y,s)\\ &:= \partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(x-y,t+s) - \partial_x^\gamma \mathcal{D}_t^\nu W^{(\alpha)}(-y,1+s)\end{aligned} \tag{2.8}$$

for all $X = (x,t), Y = (y,s) \in H$. We may write $\omega_\alpha^\nu = \omega_\alpha^{0,\nu}$. We describe the following lemma. In particular, Lemma 2.3 (1) is [5, Proposition 3.1 (1)]. The result Lemma 2.3 (2) is an immediate consequence of Lemma 2.3 (1).

Lemma 2.3 *Let $0 < \alpha \le 1$ and $(\gamma, \kappa) \in \mathbb{N}_0^n \times I_{\alpha,n}$. Then, the following statements hold.*

(1) *For any compact set $E \subset \mathbb{R}^n$ and any real number $T > 1$, there exist constants $C_1, C_2 > 0$ such that*

$$\left|\omega_\alpha^{\gamma,\kappa}(x,t;y,s)\right| \le \frac{C_1|x|}{(1+s+|y|^{2\alpha})^{\frac{n+|\gamma|+1}{2\alpha}+\kappa}} + \frac{C_2|t-1|}{(1+s+|y|^{2\alpha})^{\frac{n+|\gamma|}{2\alpha}+\kappa+1}}$$

for all $(x,t) \in E \times [T^{-1}, T]$ and $(y,s) \in H$.

(2) *For any compact set $K \subset H$, there exists a constant $C > 0$ such that*

$$\left|\omega_\alpha^{\gamma,\kappa}(x,t;y,s)\right| \le \frac{C}{(1+s+|y|^{2\alpha})^{\frac{n+|\gamma|}{2\alpha}+\kappa+m(\alpha)}}$$

for all $(x,t) \in K$ and $(y,s) \in H$.

We give definitions of some function spaces, which are closely related to parabolic Bloch type spaces. For $1 \le p < \infty$ and $\lambda > -1$, the Lebesgue space $L^p(\lambda) := L^p(H, t^\lambda dV)$ is defined to be the Banach space of all Lebesgue measurable functions u on H with

$$\|u\|_{L^p(\lambda)} := \left(\int_H |u(x,t)|^p t^\lambda dV(x,t)\right)^{1/p} < \infty.$$

The α-parabolic Bergman space $\boldsymbol{b}_\alpha^p(\lambda)$ is the set of all $L^{(\alpha)}$-harmonic functions u on H with $u \in L^p(\lambda)$. Furthermore, $L^\infty := L^\infty(H, dV)$ is defined to be the Banach space of all Lebesgue measurable functions u on H with

$$\|u\|_{L^\infty} := \text{ess sup}\{|u(x,t)|; (x,t) \in H\} < \infty,$$

and let $\boldsymbol{b}_\alpha^\infty$ be the set of all $L^{(\alpha)}$-harmonic functions u on H with $u \in L^\infty$. We also consider the subspace of $\mathcal{B}_\alpha(\sigma)$. The α-parabolic little Bloch type space $\mathcal{B}_{\alpha,0}(\sigma)$ is the set of all functions $u \in \mathcal{B}_\alpha(\sigma)$ with

$$\lim_{(x,t)\to\partial H\cup\{\infty\}} t^\sigma\left\{t^{1/(2\alpha)}|\nabla_x u(x,t)| + t|\partial_t u(x,t)|\right\} = 0. \tag{2.9}$$

Furthermore, let $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ be the set of all functions $u \in \mathcal{B}_{\alpha,0}(\sigma)$ with

$u(0,1)=0$. Clearly, $\mathcal{B}_{\alpha,0}(\sigma)$ and $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ are, respectively, the closed subspaces of $\mathcal{B}_\alpha(\sigma)$ and $\widetilde{\mathcal{B}}_\alpha(\sigma)$ by definition. We describe reproducing formulae by fractional derivatives on $\boldsymbol{b}^p_\alpha(\lambda)$ and $\mathcal{B}_\alpha(\sigma)$. We note that Lemma 2.4 (1) is [4, Theorem 5.2] and Lemma 2.4 (2) is [8, Theorem 5.7], respectively.

Lemma 2.4 *Let $0<\alpha\le 1$. Then, the following statements hold.*

(1) *Let $1\le p<\infty$ and $\lambda>-1$. If real numbers κ and ν satisfy $\kappa>-\frac{\lambda+1}{p}$ and $\nu>\frac{\lambda+1}{p}$, then*

$$u(x,t)=\frac{2^{\kappa+\nu}}{\Gamma(\kappa+\nu)}\int_H \mathcal{D}_t^\kappa u(y,s)\mathcal{D}_t^\nu W^{(\alpha)}(x-y,t+s)s^{\kappa+\nu-1}dV(y,s) \tag{2.10}$$

for all $u\in\boldsymbol{b}^p_\alpha(\lambda)$ and $(x,t)\in H$. Furthermore, (2.10) *also holds for $\nu=\lambda+1$ when $p=1$.*

(2) *Let $\sigma>-m(\alpha)$. If real numbers $\kappa\in\mathbb{R}_+$ and $\nu\in\mathbb{R}$ satisfy $\kappa>-\sigma$ and $\nu>\sigma$, then*

$$u(x,t)-u(0,1)=\frac{2^{\kappa+\nu}}{\Gamma(\kappa+\nu)}\int_H \mathcal{D}_t^\kappa u(y,s)\omega_\alpha^\nu(x,t;y,s)s^{\kappa+\nu-1}dV(y,s) \tag{2.11}$$

for all $u\in\mathcal{B}_\alpha(\sigma)$ and $(x,t)\in H$. Furthermore, (2.11) *also holds for $\nu>\max\{0,\sigma\}$ when $\kappa=0$.*

We also describe the following duality theorems. In the following lemma, Lemma 2.5 (1) is [8, Theorem 3] and Lemma 2.5 (2) is [8, Theorem 4], respectively.

Lemma 2.5 *Let $0<\alpha\le 1$, $\sigma>-m(\alpha)$, and $\lambda>-1$. Then, the following statements hold.*

(1) *The duality $(\boldsymbol{b}^1_\alpha(\lambda))^*\cong\widetilde{\mathcal{B}}_\alpha(\sigma)$ holds under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda,\sigma}$, where*

$$\langle u,v\rangle_{\lambda,\sigma}:=\frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)}\int_H u(y,s)\mathcal{D}_t v(y,s)s^{\lambda+\sigma+1}dV(y,s),$$
$$u\in\boldsymbol{b}^1_\alpha(\lambda),\ v\in\widetilde{\mathcal{B}}_\alpha(\sigma). \tag{2.12}$$

(2) *The duality $\boldsymbol{b}^1_\alpha(\lambda)\cong(\widetilde{\mathcal{B}}_{\alpha,0}(\sigma))^*$ holds under the pairing* (2.12)*, that is,*

$\langle u, v\rangle_{\lambda,\sigma}$ *with* $u \in \boldsymbol{b}^1_\alpha(\lambda)$ *and* $v \in \widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$.

Lemma 2.6 ([11, Lemma 5]) *Let* $\theta, c \in \mathbb{R}$. *If* $\theta > -1$ *and* $\theta - c + \frac{n}{2\alpha} + 1 < 0$, *then there exists a constant* $C = C(n, \alpha, \theta, c) > 0$ *such that*

$$\int_H \frac{s^\theta}{(t + s + |x-y|^{2\alpha})^c} dV(y,s) = C t^{\theta - c + \frac{n}{2\alpha} + 1}$$

for all $(x,t) \in H$.

We also need the following lemma.

Lemma 2.7 ([7, Theorem 3.1]) *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, *and* $\lambda \in \mathbb{R}$. *Suppose that a multi-index* $\gamma \in \mathbb{N}_0^n$, *and real numbers* κ, $\rho \in \mathbb{R}$ *with* $\kappa > -\frac{n}{2\alpha}$ *satisfy*

$$\lambda - \rho p < p - 1 < \left(\frac{|\gamma|}{2\alpha} + \kappa\right) p + \lambda - \rho p.$$

Then, for every $f \in L^p(\lambda)$,

$$v(x,t) := \int_H f(y,s) \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x-y, t+s) s^\rho dV(y,s)$$

is well defined for every $(x,t) \in H$. *Furthermore, let* $\beta \in \mathbb{N}_0^n$ *be a multi-index. If a real number* $\nu \in \mathbb{R}$ *satisfies*

$$\nu + \kappa > -\frac{n}{2\alpha} \text{ and } p - 1 < \left(\frac{|\gamma|}{2\alpha} + \nu + \kappa\right) p + \lambda - \rho p,$$

then

$$\partial_x^\beta \mathcal{D}_t^\nu v(x,t) = \int_H f(y,s) \partial_x^{\beta+\gamma} \mathcal{D}_t^{\nu+\kappa} W^{(\alpha)}(x-y, t+s) s^\rho dV(y,s).$$

Now, we recall the definition of α-parabolic cylinders, which are introduced in [12]. The α-parabolic cylinders will be used to define separated sequences below. For $Y = (y,s) \in H$ and $0 < \delta < 1$, an α-parabolic cylinder $S_\delta^{(\alpha)}(Y) = S_\delta^{(\alpha)}(y,s)$ is defined by

$$S_\delta^{(\alpha)}(y,s) := \left\{(x,t)\in H; |x-y| < \left(\frac{2\delta}{1-\delta^2}s\right)^{1/2\alpha},\ \frac{1-\delta}{1+\delta}s < t < \frac{1+\delta}{1-\delta}s\right\}.$$

Clearly, $\lim_{\delta\to 1} S_\delta^{(\alpha)}(Y) = H$ and $S_\delta^{(\alpha)}(y,s) = \Phi_Y^{(\alpha)}(S_\delta^{(\alpha)}(0,1))$ for each $Y \in H$, where $\Phi_Y^{(\alpha)}(X)$ is the function defined by

$$\Phi_Y^{(\alpha)}(X) := \bigl(s^{1/2\alpha}x + y, st\bigr), \quad X = (x,t) \in H.$$

Also, $V(S_\delta^{(\alpha)}(y,s)) = 2B_n(2\delta s/(1-\delta^2))^{n/(2\alpha)+1}$, where B_n is the volume of the unit ball in $\mathbb{R}^n$. For $0<\delta<1$, we say that a sequence $\{X_j\} \subset H$ is δ-separated in the α-parabolic sense if α-parabolic cylinders $S_\delta^{(\alpha)}(X_j)$ are pairwise disjoint. We also need the following lemma.

Lemma 2.8 ([6, Lemma 4.2]) *Let $0<\alpha\le 1$. For every $\theta > -1$ and $c>0$, there exists a constant $C>0$ such that*

$$\frac{s^\theta}{(t+s+|x-y|^{2\alpha})^c} \le C\frac{F(\delta)}{s^{n/(2\alpha)+1}}\int_{S_\delta^{(\alpha)}(y,s)} \frac{r^\theta}{(t+r+|x-z|^{2\alpha})^c}dV(z,r)$$

for all $0<\delta<1$ and (x,t), $(y,s)\in H$, where

$$F(\delta) = \frac{(1-\delta^2)^{n/(2\alpha)+\theta+1-c}}{\delta^{n/(2\alpha)}\{(1+\delta)^{2(\theta+1)} - (1-\delta)^{2(\theta+1)}\}}.$$

We describe representing and interpolating operators, which are studied in [6]. Let $\mathbb{X} = \{X_j\} = \{(x_j,t_j)\}$ be a sequence in H. First, we give the definition of the representing operators. Let $(\gamma,\kappa) \in \mathbb{N}_0^n \times I_{\alpha,n}$. For $\{\lambda_j\} \in \ell^p$, let

$$U_{p,\lambda,\mathbb{X}}^{\gamma,\kappa}\{\lambda_j\}(X) := \sum_j \lambda_j t_j^{\frac{n+|\gamma|}{2\alpha}+\kappa-\left(\frac{n}{2\alpha}+1+\lambda\right)\frac{1}{p}}\partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x-x_j, t+t_j) \tag{2.13}$$

for all $X=(x,t)\in H$. We call $U_{p,\lambda,\mathbb{X}}^{\gamma,\kappa}$ the representing operator of order (γ,κ). The following result is also given in [6].

Lemma 2.9 ([6, Theorem 4.3]) *Let $0<\alpha\le 1$, $1<p<\infty$, $\lambda>-1$, and let κ be a real number such that $\kappa > \frac{\lambda+1}{p}$. Let $\gamma\in\mathbb{N}_0^n$ be a multi-*

index. Furthermore, let $\mathbb{X} = \{X_j\} = \{(x_j, t_j)\}$ *be a sequence in* H. *Then,* $U^{\gamma,\kappa}_{p,\lambda,\mathbb{X}} : \ell^p \to \boldsymbol{b}^p_\alpha(\lambda)$ *is bounded if and only if for any* $0 < \delta < 1$, *there exists* $M \in \mathbb{N}$ *such that* $\mathbb{X} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_M$ *and each sequence* $\mathbb{X}_i$ *is* δ*-separated in the* α*-parabolic sense. When* $p = 1$, *the "if" part also holds.*

Next, we give the definition of the interpolating operators. Let $\gamma \in \mathbb{N}^n_0$ and let κ be a real number such that $\kappa > -\left(\frac{n}{2\alpha}+1+\lambda\right)$. Then, for $u \in \boldsymbol{b}^p_\alpha(\lambda)$, we define a sequence of real numbers $T^{\gamma,\kappa}_{p,\lambda,\mathbb{X}}u$ by

$$T^{\gamma,\kappa}_{p,\lambda,\mathbb{X}}u := \left\{ t_j^{\left(\frac{n}{2\alpha}+1+\lambda\right)\frac{1}{p}+\frac{|\gamma|}{2\alpha}+\kappa} \partial^\gamma_x \mathcal{D}^\kappa_t u(X_j) \right\}. \tag{2.14}$$

We call $T^{\gamma,\kappa}_{p,\lambda,\mathbb{X}}$ the interpolating operator of order (γ, κ). The boundedness of the operator $T^{\gamma,\kappa}_{p,\lambda,\mathbb{X}} : \boldsymbol{b}^p_\alpha(\lambda) \to \ell^p$ is characterized by the following lemma.

Lemma 2.10 ([6, Lemma 4.1]) *Let* $0 < \alpha \le 1$, $1 \le p < \infty$, $\lambda > -1$, *and* κ *be a real number such that* $\kappa > -\frac{\lambda+1}{p}$. *Let* $\gamma \in \mathbb{N}^n_0$ *be a multi-index. Furthermore, let* $\mathbb{X} = \{X_j\} = \{(x_j, t_j)\}$ *be a sequence in* H. *Then,* $T^{\gamma,\kappa}_{p,\lambda,\mathbb{X}} : \boldsymbol{b}^p_\alpha(\lambda) \to \ell^p$ *is bounded if and only if for any* $0 < \delta < 1$, *there exists* $M \in \mathbb{N}$ *such that* $\mathbb{X} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_M$ *and each sequence* $\mathbb{X}_i$ *is* δ*-separated in the* α*-parabolic sense.*

3. The $\widetilde{\mathcal{B}}_\alpha(\sigma)$-representing operator

In this section, we define the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-representing operators, and study their properties. First, we give the definition of the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-representing operators. Let $\sigma > -m(\alpha)$ and $\mathbb{X} = \{X_j\} = \{(x_j, t_j)\}$ be a sequence in H. Furthermore, let $(\gamma, \kappa) \in \mathbb{N}^n_0 \times I_{\alpha,n}$. For $\{\lambda_j\} \in \ell^\infty$, put

$$U^{\gamma,\kappa}_{\sigma,\mathbb{X}}\{\lambda_j\}(X) := \sum_j \lambda_j t_j^{\frac{n+|\gamma|}{2\alpha}+\kappa-\sigma} \omega^{\gamma,\kappa}_\alpha(X; X_j), \quad X \in H. \tag{3.1}$$

We call $U^{\gamma,\kappa}_{\sigma,\mathbb{X}}$ the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-representing operator of order (γ, κ). Let c_0 be the totality of sequences convergent to 0, which is a closed subspace of ℓ^∞, and we may regard a finite sequence as an element of c_0. Now, we give a necessary and sufficient condition for a sequence $\{X_j\}$ which ensures that $U^{\gamma,\kappa}_{\sigma,\mathbb{X}} : \ell^\infty \to \widetilde{\mathcal{B}}_\alpha(\sigma)$ is bounded and also that $U^{\gamma,\kappa}_{\sigma,\mathbb{X}}$ maps c_0 into $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$,

Theorem 3.1 *Let* $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, *and let* κ *be a real number* *such that $\kappa > \sigma$. Let $\gamma \in \mathbb{N}_0^n$ be a multi-index. Furthermore, let $\mathbb{X} = \{X_j\} = \{(x_j, t_j)\}$ be a sequence in H. Then, $U_{\sigma,\mathbb{X}}^{\gamma,\kappa} : \ell^\infty \to \widetilde{\mathcal{B}}_\alpha(\sigma)$ is bounded and $U_{\sigma,\mathbb{X}}^{\gamma,\kappa}$ maps c_0 into $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ if and only if for any $0 < \delta < 1$, there exists $M \in \mathbb{N}$ such that $\mathbb{X} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_M$ and each sequence $\mathbb{X}_i$ is δ-separated in the α-parabolic sense.*

Proof. First, suppose that $U_{\sigma,\mathbb{X}}^{\gamma,\kappa} : \ell^\infty \to \widetilde{\mathcal{B}}_\alpha(\sigma)$ is bounded and $U_{\sigma,\mathbb{X}}^{\gamma,\kappa}$ maps c_0 into $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$. Then, the restriction operator $S := U_{\sigma,\mathbb{X}}^{\gamma,\kappa}|c_0 : c_0 \to \widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ is bounded. Therefore, there exists the adjoint operator S^* of S such that $S^* : (\widetilde{\mathcal{B}}_{\alpha,0}(\sigma))^* \to (c_0)^*$ is bounded. Let $\lambda > -1$. Then, Lemma 2.5 (2) implies that $S^* : \boldsymbol{b}_\alpha^1(\lambda) \to \ell^1$ is bounded. Let $(\cdot\,,\cdot)$ be the usual pairing of ℓ^1 and ℓ^∞, and recall that $\langle\ \cdot\ ,\ \cdot\ \rangle_{\lambda,\sigma}$ is the pairing of $\boldsymbol{b}_\alpha^1(\lambda)$ and $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ described in Lemma 2.5. Furthermore, let $\{e_j\}$ be the standard basis of ℓ^∞. (We note that $e_j \in c_0$.) Then, for $u \in \boldsymbol{b}_\alpha^1(\lambda)$, we have

$$
\begin{aligned}
(S^*u, e_j) &= \langle u, Se_j\rangle_{\lambda,\sigma} = \langle u, U_{\sigma,\mathbb{X}}^{\gamma,\kappa} e_j\rangle_{\lambda,\sigma} \\
&= t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma} \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \\
&\quad \times \int_H u(y,s) \mathcal{D}_t \omega_\alpha^{\gamma,\kappa}(y,s;x_j,t_j) s^{\lambda+\sigma+1} dV(y,s) \\
&= t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma} \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \\
&\quad \times \int_H u(y,s) \partial_x^\gamma \mathcal{D}_t^{\kappa+1} W^{(\alpha)}(y-x_j, s+t_j) s^{\lambda+\sigma+1} dV(y,s). \quad (3.2)
\end{aligned}
$$

Making a change of variable $y = 2x_j - z$, we find that the right-hand side of (3.2) is equal to

$$
\begin{aligned}
&t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma} \frac{2^{\lambda+\sigma+2}}{\Gamma(\lambda+\sigma+2)} \\
&\quad \times \int_H v(z,s) \partial_x^\gamma \mathcal{D}_t^{\kappa+1} W^{(\alpha)}(x_j - z, t_j + s) s^{\lambda+\sigma+1} dV(z,s),
\end{aligned}
$$

where $v(z,s) = u(2x_j - z, s)$. Furthermore, Lemma 2.7 and Lemma 2.4 (1) imply that

$$
\begin{aligned}
&\int_H v(z,s)\partial_x^\gamma \mathcal{D}_t^{\kappa+1}\big(W^{(\alpha)}(x-z,t+s)\big)\Big|_{(x,t)=(x_j,t_j)} s^{\lambda+\sigma+1}dV(z,s)\\
&\quad= \partial_x^\gamma \mathcal{D}_t^{\kappa-(\lambda+\sigma+1)}\bigg(\int_H v(z,s)\\
&\qquad\qquad \times \mathcal{D}_t^{\lambda+\sigma+2}W^{(\alpha)}(x-z,t+s)s^{\lambda+\sigma+1}dV(z,s)\bigg)\bigg|_{(x,t)=(x_j,t_j)}\\
&\quad= \frac{\Gamma(\lambda+\sigma+2)}{2^{\lambda+\sigma+2}}\partial_x^\gamma \mathcal{D}_t^{\kappa-(\lambda+\sigma+1)} v(x,t)\Big|_{(x,t)=(x_j,t_j)}\\
&\quad= (-1)^{|\gamma|}\frac{\Gamma(\lambda+\sigma+2)}{2^{\lambda+\sigma+2}}\partial_x^\gamma \mathcal{D}_t^{\kappa-(\lambda+\sigma+1)} u(x_j,t_j).
\end{aligned}
$$

Hence, we obtain

$$
(S^*u, e_j) = (-1)^{|\gamma|} t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\partial_x^\gamma \mathcal{D}_t^{\kappa-(\lambda+\sigma+1)} u(x_j,t_j),
$$

that is,

$$
\begin{aligned}
S^*u &= (-1)^{|\gamma|}\big\{t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\partial_x^\gamma \mathcal{D}_t^{\kappa-(\lambda+\sigma+1)} u(X_j)\big\}\\
&= (-1)^{|\gamma|} T_{1,\lambda,\mathbb{X}}^{\gamma,\kappa-(\lambda+\sigma+1)} u.
\end{aligned}
$$

Since S^* is bounded, the operator $T_{1,\lambda,\mathbb{X}}^{\gamma,\kappa-(\lambda+\sigma+1)}$ is also bounded. Therefore, by Lemma 2.10, for any $0<\delta<1$, there exists $M\in\mathbb{N}$ such that $\mathbb{X} = \mathbb{X}_1\cup\cdots\cup\mathbb{X}_M$ and each sequence $\mathbb{X}_i$ is δ-separated in the α-parabolic sense.

Next, we show the "only if" part. It is sufficient to prove that if $\mathbb{X}$ is δ-separated in the α-parabolic sense for some $0<\delta<1$ then $U_{\sigma,\mathbb{X}}^{\gamma,\kappa} : \ell^\infty \to \widetilde{\mathcal{B}}_\alpha(\sigma)$ is bounded and $U_{\sigma,\mathbb{X}}^{\gamma,\kappa}$ maps c_0 into $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$. Thus, we suppose that $\mathbb{X} = \{X_j\} = \{(x_j,t_j)\}$ is δ-separated in the α-parabolic sense. Let $\{\lambda_j\}\in \ell^\infty$. We begin with showing that the series in (3.1) converges uniformly on compact subsets of H (we only use the pointwise convergence of this series later). Let K be a compact subset of H. Then, Lemma 2.3 (2) and Lemma 2.8 imply that there exists a constant $C = C(K) > 0$ such that

$$\begin{aligned}
&\bigl|\lambda_j t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\omega_\alpha^{\gamma,\kappa}(x,t;x_j,t_j)\bigr| \\
&\quad\le C\|\{\lambda_j\}\|_\infty \frac{t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}}{(1+t_j+|x_j|^{2\alpha})^{(n+|\gamma|)/(2\alpha)+\kappa+m(\alpha)}} \\
&\quad\le CF(\delta)\|\{\lambda_j\}\|_\infty \int_{S_\delta^{(\alpha)}(X_j)} \frac{r^{|\gamma|/(2\alpha)+\kappa-\sigma-1}}{(1+r+|z|^{2\alpha})^{(n+|\gamma|)/(2\alpha)+\kappa+m(\alpha)}}dV(z,r)
\end{aligned}$$

for all $0<\delta<1$, j, and $(x,t)\in K$, where $F(\delta)$ is the function defined in Lemma 2.8. Therefore, Lemma 2.6 shows that

$$\begin{aligned}
&\sum_j \bigl|\lambda_j t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\omega_\alpha^{\gamma,\kappa}(x,t;x_j,t_j)\bigr| \\
&\quad\le CF(\delta)\|\{\lambda_j\}\|_\infty \sum_j \int_{S_\delta^{(\alpha)}(X_j)} \frac{r^{|\gamma|/(2\alpha)+\kappa-\sigma-1}}{(1+r+|z|^{2\alpha})^{(n+|\gamma|)/(2\alpha)+\kappa+m(\alpha)}}dV(z,r) \\
&\quad\le CF(\delta)\|\{\lambda_j\}\|_\infty \int_H \frac{r^{|\gamma|/(2\alpha)+\kappa-\sigma-1}}{(1+r+|z|^{2\alpha})^{(n+|\gamma|)/(2\alpha)+\kappa+m(\alpha)}}dV(z,r) \\
&\quad\le CF(\delta)\|\{\lambda_j\}\|_\infty
\end{aligned}$$

for all $(x,t)\in K$, that is, the series in (3.1) converges uniformly on K. Put

$$u_N(x,t)=\sum_{j=1}^N \lambda_j t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\omega_\alpha^{\gamma,\kappa}(x,t;x_j,t_j), \quad (x,t)\in H.$$

Then, we claim that $\{u_N\}$ is bounded in $\widetilde{\mathcal{B}}_\alpha(\sigma)$. In fact, for each $(\beta,m)\in \mathbb{N}_0^n\times\mathbb{N}_0\backslash\{(0,0)\}$, Lemma 2.1 (1) and Lemma 2.8 imply that

$$\begin{aligned}
&\sum_{j=1}^N |\lambda_j| t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\bigl|\partial_x^\beta \mathcal{D}_t^m \omega_\alpha^{\gamma,\kappa}(x,t;x_j,t_j)\bigr| \\
&\quad=\sum_{j=1}^N |\lambda_j| t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\bigl|\partial_x^{\beta+\gamma}\mathcal{D}_t^{m+\kappa}W^{(\alpha)}(x-x_j,t+t_j)\bigr| \\
&\quad\le C\Bigl(\sup_{1\le j\le N}|\lambda_j|\Bigr)\sum_{j=1}^N \frac{t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}}{(t+t_j+|x-x_j|^{2\alpha})^{(n+|\beta|+|\gamma|)/(2\alpha)+m+\kappa}}
\end{aligned}$$

$$\leq CF(\delta)\Big(\sup_{1\leq j\leq N}|\lambda_j|\Big)$$
$$\times\sum_{j=1}^{N}\int_{S_\delta^{(\alpha)}(X_j)}\frac{r^{|\gamma|/(2\alpha)+\kappa-\sigma-1}}{(t+r+|x-z|^{2\alpha})^{(n+|\beta|+|\gamma|)/(2\alpha)+m+\kappa}}dV(z,r) \quad (3.3)$$

for all $X=(x,t)\in H$. Therefore, (3.3) and Lemma 2.6 also imply that

$$\sum_{j=1}^{N}|\lambda_j|t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\big|\partial_{x_\ell}\omega_\alpha^{\gamma,\kappa}(X;X_j)\big| \leq Ct^{-\sigma-1/(2\alpha)}\Big(\sup_{1\leq j\leq N}|\lambda_j|\Big) \quad (3.4)$$

and

$$\sum_{j=1}^{N}|\lambda_j|t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\big|\partial_t\omega_\alpha^{\gamma,\kappa}(X;X_j)\big|\leq Ct^{-\sigma-1}\Big(\sup_{1\leq j\leq N}|\lambda_j|\Big) \quad (3.5)$$

for all $X=(x,t)\in H$. Thus, (3.4) and (3.5) show $\|u_N\|_{\mathcal{B}_\alpha(\sigma)}\leq C\|\{\lambda_j\}\|_\infty$ for all $N\in\mathbb{N}$. Let $\lambda>-1$, and we recall the fact $(\boldsymbol{b}_\alpha^1(\lambda))^*\cong\widetilde{\mathcal{B}}_\alpha(\sigma)$ under the pairing $\langle\ \cdot\ ,\ \cdot\ \rangle_{\lambda,\sigma}$ defined in Lemma 2.5. Furthermore, since $L^1(\lambda)$ is separable, the subspace $\boldsymbol{b}_\alpha^1(\lambda)$ of $L^1(\lambda)$ is also separable. Therefore, the Banach-Alaoglu theorem implies that there exist a subsequence $\{u_{N_i}\}\subset\{u_N\}$ and a function $u\in\widetilde{\mathcal{B}}_\alpha(\sigma)$ such that $\{u_{N_i}\}$ converges to u in the w*-topology. By Lemma 2.3 (2) and Lemma 2.6, we have $\omega_\alpha^{\lambda+\sigma+1}(X;\cdot\ )=\omega_\alpha^{0,\lambda+\sigma+1}(X;\cdot\ )\in\boldsymbol{b}_\alpha^1(\lambda)$ for each $X\in H$. Hence, Lemma 2.4 (2) with $\kappa=1$ shows that

$$\begin{aligned}u(X)&=\big\langle\omega_\alpha^{\lambda+\sigma+1}(X;\cdot\ ),u\big\rangle_{\lambda,\sigma}\\&=\lim_i\big\langle\omega_\alpha^{\lambda+\sigma+1}(X;\cdot\ ),u_{N_i}\big\rangle_{\lambda,\sigma}=\lim_i u_{N_i}(X)=U_{\sigma,\mathbb{X}}^{\gamma,\kappa}\{\lambda_j\}(X).\end{aligned}$$

This implies $U_{\sigma,\mathbb{X}}^{\gamma,\kappa}\{\lambda_j\}\in\widetilde{\mathcal{B}}_\alpha(\sigma)$ and $\|U_{\sigma,\mathbb{X}}^{\gamma,\kappa}\{\lambda_j\}\|_{\mathcal{B}_\alpha(\sigma)}\leq\liminf_i\|u_{N_i}\|_{\mathcal{B}_\alpha(\sigma)}$ $\leq C\|\{\lambda_j\}\|_\infty$, that is, the operator $U_{\sigma,\mathbb{X}}^{\gamma,\kappa}:\ell^\infty\to\widetilde{\mathcal{B}}_\alpha(\sigma)$ is bounded. Next, let $\{\eta_j\}\in c_0$, and put

$$v_N(X) = \sum_{j=1}^{N} \eta_j t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma} \omega_\alpha^{\gamma,\kappa}(X; X_j), \quad X \in H.$$

Then, by (3.3), we have $v_N \in \widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$. Furthermore, (3.4) and (3.5) show that

$$\|v_M - v_N\|_{\mathcal{B}_\alpha(\sigma)} \le C\left(\sup_{N+1\le j\le M} |\eta_j|\right) \to 0 \quad (M > N \to \infty).$$

Hence, there exists a function $v \in \widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ such that $\{v_N\}$ converges to v in $\widetilde{\mathcal{B}}_\alpha(\sigma)$. Thus, $\{v_N\}$ also converges to v in the w*-topology. Therefore, Lemma 2.4 (2) with $\kappa = 1$ also implies that

$$\begin{aligned} v(X) &= \left\langle \omega_\alpha^{\lambda+\sigma+1}(X; \cdot\,), v\right\rangle_{\lambda,\sigma} \\ &= \lim_N \left\langle \omega_\alpha^{\lambda+\sigma+1}(X; \cdot\,), v_N\right\rangle_{\lambda,\sigma} = \lim_N v_N(X) = U_{\sigma,\mathbb{X}}^{\gamma,\kappa}\{\eta_j\}(X). \end{aligned}$$

It follows that $U_{\sigma,\mathbb{X}}^{\gamma,\kappa}$ maps c_0 into $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$. □

4. The $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating operator

In this section, we define $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating operators, and study their properties. First, we give the definition of the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating operators. Let $\sigma > -m(\alpha)$ and put $\Sigma_\sigma := \{0\} \cup \{\kappa \in \mathbb{R} : \kappa > \max\{0, -\sigma\}\}$. Furthermore, let $\mathbb{X} = \{X_j\} = \{(x_j, t_j)\}$ be a sequence in H, and let $(\gamma, \kappa) \in (\mathbb{N}_0^n \times \Sigma_\sigma)\backslash\{(0, 0)\}$. Then, for $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$, we define a sequence of real numbers $T_{\sigma,\mathbb{X}}^{\gamma,\kappa} u$ by

$$T_{\sigma,\mathbb{X}}^{\gamma,\kappa} u := \left\{ t_j^{|\gamma|/(2\alpha)+\kappa+\sigma} \partial_x^\gamma \mathcal{D}_t^\kappa u(X_j) \right\}. \tag{4.1}$$

By Lemma 2.2 (1), the linear operator $T_{\sigma,\mathbb{X}}^{\gamma,\kappa} : \widetilde{\mathcal{B}}_\alpha(\sigma) \to \ell^\infty$ is always bounded, and we call $T_{\sigma,\mathbb{X}}^{\gamma,\kappa}$ the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating operator of order (γ, κ). We also consider the operator $T_{\sigma,\mathbb{X}}^{\gamma,\kappa}$ on the subspace $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ of $\widetilde{\mathcal{B}}_\alpha(\sigma)$. We give sufficient conditions for a sequence $\{X_j\}$ which ensures that $T_{\sigma,\mathbb{X}}^{\gamma,\kappa}$ maps $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ into c_0. We give the following theorem.

Theorem 4.1 *Let $0 < \alpha \leq 1$, $\sigma > -m(\alpha)$, and $(\gamma, \kappa) \in (\mathbb{N}_0^n \times \Sigma_\sigma)\setminus \{(0,0)\}$. Then, the following statements hold.*

(1) *If $u \in \widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$, then $\lim_{(x,t)\to\partial H\cup\{\infty\}} t^{|\gamma|/(2\alpha)+\kappa+\sigma}\partial_x^\gamma \mathcal{D}_t^\kappa u(x,t) = 0$.*
(2) *If a sequence $\mathbb{X} = \{X_j\} \subset H$ satisfies $X_j \to \partial H \cup \{\infty\}$ $(j \to \infty)$, then $T_{\sigma,\mathbb{X}}^{\gamma,\kappa}$ maps $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ into c_0.*
(3) *If for any $0 < \delta < 1$, there exists $M \in \mathbb{N}$ such that $\mathbb{X} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_M$ and each sequence $\mathbb{X}_i$ is δ-separated in the α-parabolic sense, then $T_{\sigma,\mathbb{X}}^{\gamma,\kappa}$ maps $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ into c_0.*

Proof. (1) Let $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$. Then, by Lemma 2.4 (2) with $\kappa = 1$ and $\nu = \sigma + 1$, we have

$$u(x,t) = \frac{2^{\sigma+2}}{\Gamma(\sigma+2)} \int_H \mathcal{D}_t u(y,s)\omega_\alpha^{\sigma+1}(x,t;y,s)s^{\sigma+1}dV(y,s) \tag{4.2}$$

for all $(x,t) \in H$. Let $(\gamma,\kappa) \in (\mathbb{N}_0^n \times \Sigma_\sigma)\setminus\{(0,0)\}$. If $\kappa \notin \mathbb{N}_0$, then differentiating through the integral (4.2), we obtain

$$\begin{aligned}&\partial_x^\gamma \mathcal{D}_t^{\lceil\kappa\rceil} u(x,t)\\ &\quad= \frac{2^{\sigma+2}}{\Gamma(\sigma+2)} \int_H \mathcal{D}_t u(y,s)\partial_x^\gamma \mathcal{D}_t^{\lceil\kappa\rceil+\sigma+1} W^{(\alpha)}(x-y,t+s)s^{\sigma+1}dV(y,s).\end{aligned}$$

Thus, we have

$$\begin{aligned}\partial_x^\gamma \mathcal{D}_t^\kappa u(x,t) = \frac{2^{\sigma+2}}{\Gamma(\sigma+2)} \frac{1}{\Gamma(\lceil\kappa\rceil-\kappa)} \int_0^\infty \tau^{\lceil\kappa\rceil-\kappa-1} \int_H \mathcal{D}_t u(y,s)\\ \times \partial_x^\gamma \mathcal{D}_t^{\lceil\kappa\rceil+\sigma+1} W^{(\alpha)}(x-y,t+s+\tau)s^{\sigma+1}dV(y,s)d\tau.\end{aligned}$$

Here, Lemma 2.1 (1) and Lemma 2.6 imply that

$$\begin{aligned}&\int_0^\infty \tau^{\lceil\kappa\rceil-\kappa-1}\\ &\times \int_H \left|\mathcal{D}_t u(y,s)\partial_x^\gamma \mathcal{D}_t^{\lceil\kappa\rceil+\sigma+1} W^{(\alpha)}(x-y,t+s+\tau)\right|s^{\sigma+1}dV(y,s)d\tau\end{aligned}$$

$$
\begin{aligned}
&\le C\|u\|_{\mathcal{B}_\alpha(\sigma)} \int_0^\infty \tau^{\lceil\kappa\rceil-\kappa-1} \\
&\quad \times \int_H \frac{1}{(t+s+\tau+|x-y|^{2\alpha})^{(n+|\gamma|)/(2\alpha)+\lceil\kappa\rceil+\sigma+1}} dV(y,s)d\tau \\
&= C\|u\|_{\mathcal{B}_\alpha(\sigma)} \int_0^\infty \frac{\tau^{\lceil\kappa\rceil-\kappa-1}}{(t+\tau)^{|\gamma|/(2\alpha)+\lceil\kappa\rceil+\sigma}} d\tau < \infty,
\end{aligned}
$$

because $|\gamma|/(2\alpha)+\kappa+\sigma>0$. Therefore, the Fubini theorem shows

$$
\begin{aligned}
&\partial_x^\gamma \mathcal{D}_t^\kappa u(x,t) \\
&= \frac{2^{\sigma+2}}{\Gamma(\sigma+2)} \int_H \mathcal{D}_t u(y,s) \partial_x^\gamma \mathcal{D}_t^{\kappa+\sigma+1} W^{(\alpha)}(x-y,t+s) s^{\sigma+1} dV(y,s). \quad (4.3)
\end{aligned}
$$

If $\kappa \in \mathbb{N}_0$, then clearly we also obtain (4.3). Hence, we conclude that Equation (4.3) holds for every $(\gamma,\kappa) \in (\mathbb{N}_0^n \times \Sigma_\sigma)\setminus\{(0,0)\}$. Let $u \in \widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ and let $\eta>0$ be a real number such that $|\gamma|/(2\alpha)+\kappa+\sigma>\eta$. Then, given $\varepsilon>0$, there exists a compact set $K \subset H$ such that $s^{\sigma+1}|\mathcal{D}_t u(y,s)|<\varepsilon$ for all $(y,s) \in K^c$, because $u \in \widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$. Hence, Lemma 2.1 (1) and Lemma 2.6 again imply that

$$
\begin{aligned}
&t^{|\gamma|/(2\alpha)+\kappa+\sigma} |\partial_x^\gamma \mathcal{D}_t^\kappa u(x,t)| \\
&\le C t^{|\gamma|/(2\alpha)+\kappa+\sigma} \int_H \frac{s^{\sigma+1}|\mathcal{D}_t u(y,s)|}{(t+s+|x-y|^{2\alpha})^{(n+|\gamma|)/(2\alpha)+\kappa+\sigma+1}} dV(y,s) \\
&\le C t^\eta \int_H \frac{s^{\sigma+1}|\mathcal{D}_t u(y,s)|}{(t+s+|x-y|^{2\alpha})^{n/(2\alpha)+\eta+1}} dV(y,s) \\
&\le C\varepsilon t^\eta \int_{K^c} \frac{1}{(t+s+|x-y|^{2\alpha})^{n/(2\alpha)+\eta+1}} dV(y,s) \\
&\quad + C\|u\|_{\mathcal{B}_\alpha(\sigma)} t^\eta \int_K \frac{1}{(t+s+|x-y|^{2\alpha})^{n/(2\alpha)+\eta+1}} dV(y,s) \\
&\le C\varepsilon + C\|u\|_{\mathcal{B}_\alpha(\sigma)} \frac{t^\eta}{(1+t+|x|^{2\alpha})^{n/(2\alpha)+\eta+1}} \\
&\le C\varepsilon + C\|u\|_{\mathcal{B}_\alpha(\sigma)} \frac{1}{(1+t+|x|^{2\alpha})^{n/(2\alpha)+1}}
\end{aligned}
$$

for all $(x,t) \in H$. Thus, we obtain

$$\lim_{(x,t)\to\partial H\cup\{\infty\}} t^{|\gamma|/(2\alpha)+\kappa+\sigma}|\partial_x^\gamma \mathcal{D}_t^\kappa u(x,t)| \leq C\varepsilon.$$

(2) The desired result immediately follows from Theorem 4.1 (1).

(3) Let $\mathbb{X} = \{X_j\}$ and $0 < \delta < 1$. Suppose that there exists $M \in \mathbb{N}$ such that $\mathbb{X} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_M$ and each sequence $\mathbb{X}_i$ is δ-separated in the α-parabolic sense. Then clearly, for any compact set $K \subset H$, there exists $j_0 \in \mathbb{N}$ such that $X_j \in K^c$ for all $j \geq j_0$, that is, $X_j \to \partial H \cup \{\infty\}$ $(j \to \infty)$. □

5. The $\widetilde{\boldsymbol{\mathcal{B}}}_{\boldsymbol{\alpha}}(\boldsymbol{\sigma})$-representing theorem

In this section, we give a representing theorem for $\widetilde{\mathcal{B}}_\alpha(\sigma)$. Let $\sigma > -m(\alpha)$ and $\mathbb{X} = \{X_j\} = \{(x_j,t_j)\}$ be a sequence in H. Furthermore, let $(\gamma,\kappa) \in \mathbb{N}_0^n \times I_{\alpha,n}$. For $\{\lambda_j\} \in \ell^\infty$, we recall the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-representing operator

$$U_{\sigma,\mathbb{X}}^{\gamma,\kappa}\{\lambda_j\}(X) = \sum_j \lambda_j t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma} \omega_\alpha^{\gamma,\kappa}(X;X_j), \quad X \in H. \tag{5.1}$$

We say that $\{X_j\}$ is a $\widetilde{\mathcal{B}}_\alpha(\sigma)$-representing sequence of order (γ,κ) if $U_{\sigma,\mathbb{X}}^{\gamma,\kappa}\{\lambda_j\} \in \widetilde{\mathcal{B}}_\alpha(\sigma)$ for all $\{\lambda_j\} \in \ell^\infty$ and the operator $U_{\sigma,\mathbb{X}}^{\gamma,\kappa} : \ell^\infty \to \widetilde{\mathcal{B}}_\alpha(\sigma)$ is bounded and onto. We also say that $\{X_j\}$ is a $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$-representing sequence of order (γ,κ) if $U_{\sigma,\mathbb{X}}^{\gamma,\kappa}\{\lambda_j\} \in \widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ for all $\{\lambda_j\} \in c_0$ and the operator $U_{\sigma,\mathbb{X}}^{\gamma,\kappa} : c_0 \to \widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ is bounded and onto. In this section, we give a representing theorem for $\widetilde{\mathcal{B}}_\alpha(\sigma)$ and $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$, that is, we give a sufficient condition for a sequence $\{X_j\}$ to be the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-representing and $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$-representing sequence. We need the following lemma.

Lemma 5.1 ([6, Lemma 5.2]) *Let $0 < \alpha \leq 1$, $\gamma \in \mathbb{N}_0^n$, $\kappa > -n/(2\alpha)$, and $\theta \in \mathbb{R}$. Then, there exists a constant $C = C(n,\alpha,\gamma,\kappa,\theta) > 0$ such that*

$$\begin{aligned}&\left|s^\theta \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x-y,t+s) - r^\theta \partial_x^\gamma \mathcal{D}_t^\kappa W^{(\alpha)}(x-z,t+r)\right| \\ &\leq C\frac{(\delta+\delta^{1/(2\alpha)})r^\theta}{(t+r+|x-z|^{2\alpha})^{(n+|\gamma|)/(2\alpha)+\kappa}}\end{aligned}$$

for all (x,t), $(y,s) \in H$, $(z,r) \in S_\delta^{(\alpha)}(y,s)$, and $0 < \delta \leq 1/3$.

We also give the Lipschitz type estimates of functions in $\widetilde{\mathcal{B}}_\alpha(\sigma)$.

Proposition 5.2 *Let $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, and $(\gamma, \kappa) \in (\mathbb{N}_0^n \times \Sigma_\sigma)\backslash\{(0,0)\}$. Then, there exists a constant $C = C(n, \alpha, \sigma, \gamma, \kappa) > 0$ such that*

$$\left|\partial_x^\gamma \mathcal{D}_t^\kappa u(y,s) - \partial_x^\gamma \mathcal{D}_t^\kappa u(x,t)\right| \le C\big(\delta + \delta^{\frac{1}{2\alpha}}\big) s^{-(|\gamma|/(2\alpha)+\kappa+\sigma)} \|u\|_{\mathcal{B}_\alpha(\sigma)} \quad (5.2)$$

for all $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$, $(x,t) \in H$, $(y,s) \in S_\delta^{(\alpha)}(x,t)$, and $0 < \delta \le 1/3$.

Proof. Let $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$, $(x,t) \in H$, $(y,s) \in S_\delta^{(\alpha)}(x,t)$, and $0 < \delta \le 1/3$. Then, by (4.3) and Lemma 5.1, we have

$$\begin{aligned}
&\left|\partial_x^\gamma \mathcal{D}_t^\kappa u(y,s) - \partial_x^\gamma \mathcal{D}_t^\kappa u(x,t)\right| \\
&\quad \le C \int_H |\mathcal{D}_t u(z,r)| \big|\partial_x^\gamma \mathcal{D}_t^{\kappa+\sigma+1} W^{(\alpha)}(y-z, s+r) \\
&\qquad\qquad - \partial_x^\gamma \mathcal{D}_t^{\kappa+\sigma+1} W^{(\alpha)}(x-z, t+r)\big| r^{\sigma+1} dV(z,r) \\
&\quad \le C(\delta + \delta^{1/(2\alpha)}) \int_H \frac{|\mathcal{D}_t u(z,r)| r^{\sigma+1}}{(r+s+|z-y|^{2\alpha})^{(n+|\gamma|)/(2\alpha)+\kappa+\sigma+1}} dV(z,r) \\
&\quad \le C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)} \\
&\qquad \times \int_H \frac{1}{(r+s+|z-y|^{2\alpha})^{(n+|\gamma|)/(2\alpha)+\kappa+\sigma+1}} dV(z,r).
\end{aligned}$$

Hence, (5.2) follows from Lemma 2.6, where C is independent of δ. □

Given $0 < \delta < 1$, we say that a sequence $\{X_j\} \subset H$ is a δ-lattice in the α-parabolic sense if $H = \bigcup_j S_\delta^{(\alpha)}(X_j)$ and $\{X_j\}$ is ε-separated in the α-parabolic sense for some ε, $0 < \varepsilon < \delta$. The notion of the δ-lattice in the α-parabolic sense is introduced in [13] and an example of the δ-lattice is given in [13, Remark 4.3].

Let $0 < \delta \le 1/3$ and $\{X_j\}$ be a δ-lattice in the α-parabolic sense (ε-separated for some $0 < \varepsilon < \delta$). Then, we take a pairwise disjoint covering $\{S_j\}$ of H as follows:

$$S_1 = S_\delta^{(\alpha)}(X_1) \backslash \bigcup_{k \ge 2} S_\varepsilon^{(\alpha)}(X_k)$$

$$S_j = S_\delta^{(\alpha)}(X_j) \setminus \Big\{ \Big(\bigcup_{m \le j-1} S_m \Big) \bigcup \Big(\bigcup_{k \ge j+1} S_\varepsilon^{(\alpha)}(X_k) \Big) \Big\}, \quad (j \ge 2). \quad (5.3)$$

It is easy to see that $S_\varepsilon^{(\alpha)}(X_j) \subset S_j \subset S_\delta^{(\alpha)}(X_j) \subset S_{1/3}^{(\alpha)}(X_j)$, and there exists a constant $C > 0$ independent of δ such that $V(S_j) \le C t_j^{n/2\alpha+1}$ for all $j \ge 1$. We show the main theorem of this section.

Theorem 5.3 *Let* $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, *and* κ *be a real number such that* $\kappa > \sigma$. *Then, there exists* $0 < \delta_0 < 1$ *such that if a sequence* $\{X_j\}$ *in* H *is the* δ*-lattice in the* α*-parabolic sense with* $0 < \delta \le \delta_0$, *then* $\{X_j\}$ *is the* $\widetilde{\mathcal{B}}_\alpha(\sigma)$*-representing and* $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$*-representing sequence of order* $(0, \kappa)$.

Proof. Suppose that $0 < \delta \le 1/3$ and $\mathbb{X} = \{X_j\} = \{(x_j, t_j)\}$ is the δ-lattice in the α-parabolic sense (ε-separated for some $0 < \varepsilon < \delta$). Here constraints of δ will be imposed later. Theorem 3.1 implies that $U_{\sigma,\mathbb{X}}^{0,\kappa} : \ell^\infty \to \widetilde{\mathcal{B}}_\alpha(\sigma)$ is bounded and $U_{\sigma,\mathbb{X}}^{0,\kappa}$ maps c_0 into $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$. Let $\{S_j\}$ be a pairwise disjoint covering of H defined in (5.3). Then, we define an operator $B_{\sigma,\mathbb{X}}$ on $\widetilde{\mathcal{B}}_\alpha(\sigma)$ by

$$B_{\sigma,\mathbb{X}} u := \Big\{ t_j^{1+\sigma-(n/(2\alpha)+1)} \mathcal{D}_t u(X_j) V(S_j) \Big\} = \Big\{ t_j^{\sigma - n/(2\alpha)} \mathcal{D}_t u(X_j) V(S_j) \Big\}.$$

We note that $B_{\sigma,\mathbb{X}} : \widetilde{\mathcal{B}}_\alpha(\sigma) \to \ell^\infty$ is bounded and $B_{\sigma,\mathbb{X}}$ maps $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ into c_0, because $V(S_j) \le C t_j^{n/(2\alpha)+1}$ and $\{X_j\}$ is ε-separated for some $0 < \varepsilon < \delta$. Thus, we define an operator $A_{\sigma,\mathbb{X}}^\kappa$ on $\widetilde{\mathcal{B}}_\alpha(\sigma)$ by

$$\begin{aligned} A_{\sigma,\mathbb{X}}^\kappa u(x,t) &:= \frac{2^{\kappa+1}}{\Gamma(\kappa+1)} U_{\sigma,\mathbb{X}}^{0,\kappa} B_{\sigma,\mathbb{X}} u(x,t) \\ &= \frac{2^{\kappa+1}}{\Gamma(\kappa+1)} \sum_j t_j^\kappa \mathcal{D}_t u(x_j, t_j) \omega_\alpha^\kappa(x,t;x_j,t_j) V(S_j). \end{aligned}$$

Then, $A_{\sigma,\mathbb{X}}^\kappa : \widetilde{\mathcal{B}}_\alpha(\sigma) \to \widetilde{\mathcal{B}}_\alpha(\sigma)$ is bounded and $A_{\sigma,\mathbb{X}}^\kappa$ maps $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$ into itself. It suffices to show that $A_{\sigma,\mathbb{X}}^\kappa$ is invertible on $\widetilde{\mathcal{B}}_\alpha(\sigma)$ for all δ sufficiently small.

We shall show that $\|I - A_{\sigma,\mathbb{X}}^\kappa\| < 1$ for all δ sufficiently small, where I is the identity operator on $\widetilde{\mathcal{B}}_\alpha(\sigma)$. In fact, Lemma 2.4 (2) implies that for $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$ and $(x,t) \in H$,

$$\begin{aligned}u(x,t)&=\frac{2^{\kappa+1}}{\Gamma(\kappa+1)}\int_H \mathcal{D}_t u(y,s)\omega_\alpha^\kappa(x,t;y,s)s^\kappa dV(y,s)\\&=\frac{2^{\kappa+1}}{\Gamma(\kappa+1)}\sum_j\int_{S_j}\mathcal{D}_t u(y,s)\omega_\alpha^\kappa(x,t;y,s)s^\kappa dV(y,s).\end{aligned}$$

Hence, we obtain

$$\big(I-A^\kappa_{\sigma,\mathbb{X}}\big)u(x,t)=\frac{2^{\kappa+1}}{\Gamma(\kappa+1)}(\Pi_1(x,t)+\Pi_2(x,t)),$$

where

$$\Pi_1(x,t)=\sum_j\int_{S_j}\mathcal{D}_t u(y,s)\big(s^\kappa\omega_\alpha^\kappa(x,t;y,s)-t_j^\kappa\omega_\alpha^\kappa(x,t;x_j,t_j)\big)dV(y,s)$$

and

$$\Pi_2(x,t)=\sum_j\int_{S_j}\big(\mathcal{D}_t u(y,s)-\mathcal{D}_t u(x_j,t_j)\big)t_j^\kappa\omega_\alpha^\kappa(x,t;x_j,t_j))dV(y,s).$$

First, we shall show that there exists a constant $C>0$ independent of δ and u such that $\|\Pi_1\|_{\mathcal{B}_\alpha(\sigma)}\le C(\delta+\delta^{1/(2\alpha)})\|u\|_{\mathcal{B}_\alpha(\sigma)}$. By Lemmas 5.1 and 2.6, we have for each $1\le\ell\le n$,

$$\begin{aligned}&|\partial_{x_\ell}\Pi_1(x,t)|\\&\quad\le\sum_j\int_{S_j}|\mathcal{D}_t u(y,s)|\big|s^\kappa\partial_{x_\ell}\mathcal{D}_t^\kappa W^{(\alpha)}(x-y,t+s)\\&\qquad\qquad-t_j^\kappa\partial_{x_\ell}\mathcal{D}_t^\kappa W^{(\alpha)}(x-x_j,t+t_j)\big|dV(y,s)\\&\quad\le C(\delta+\delta^{1/(2\alpha)})\sum_j\int_{S_j}\frac{|\mathcal{D}_t u(y,s)|s^\kappa}{(t+s+|x-y|^{2\alpha})^{(n+1)/(2\alpha)+\kappa}}dV(y,s)\\&\quad\le C(\delta+\delta^{1/(2\alpha)})\|u\|_{\mathcal{B}_\alpha(\sigma)}\int_H\frac{s^{-1-\sigma+\kappa}}{(t+s+|x-y|^{2\alpha})^{(n+1)/(2\alpha)+\kappa}}dV(y,s)\\&\quad\le C(\delta+\delta^{1/(2\alpha)})\|u\|_{\mathcal{B}_\alpha(\sigma)}\cdot t^{-\sigma-1/(2\alpha)},\end{aligned}$$

and

$$
\begin{aligned}
&|\partial_t \Pi_1(x,t)| \\
&\quad \le \sum_j \int_{S_j} |\mathcal{D}_t u(y,s)| \big| s^\kappa \mathcal{D}_t^{\kappa+1} W^{(\alpha)}(x-y, t+s) \\
&\qquad\qquad\qquad - t_j^\kappa \mathcal{D}_t^{\kappa+1} W^{(\alpha)}(x-x_j, t+t_j)\big| dV(y,s) \\
&\quad \le C(\delta + \delta^{1/(2\alpha)}) \sum_j \int_{S_j} \frac{|\mathcal{D}_t u(y,s)| s^\kappa}{(t+s+|x-y|^{2\alpha})^{n/(2\alpha)+\kappa+1}} dV(y,s) \\
&\quad \le C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)} \int_H \frac{s^{-1-\sigma+\kappa}}{(t+s+|x-y|^{2\alpha})^{n/(2\alpha)+\kappa+1}} dV(y,s) \\
&\quad \le C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)} \cdot t^{-\sigma-1}.
\end{aligned}
$$

Therefore, we obtain $\|\Pi_1\|_{\mathcal{B}_\alpha(\sigma)} \le C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)}$, where the constant C is independent of δ and u.

Next, we shall show that there exists a constant $C > 0$ independent of δ and u such that $\|\Pi_2\|_{\mathcal{B}_\alpha(\sigma)} \le C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)}$. By Lemma 2.1 (1) and Proposition 5.2, we have for each $1 \le \ell \le n$,

$$
\begin{aligned}
&|\partial_{x_\ell} \Pi_2(x,t)| \\
&\quad \le \sum_j \int_{S_j} |\mathcal{D}_t u(y,s) - \mathcal{D}_t u(x_j,t_j)| t_j^\kappa \big| \partial_{x_\ell} \mathcal{D}_t^\kappa W^{(\alpha)}(x-x_j, t+t_j)\big| dV(y,s) \\
&\quad \le \sum_j \int_{S_j} \frac{|\mathcal{D}_t u(y,s) - \mathcal{D}_t u(x_j,t_j)| t_j^\kappa}{(t+t_j+|x-x_j|^{2\alpha})^{(n+1)/(2\alpha)+\kappa}} dV(y,s) \\
&\quad \le C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)} \\
&\qquad \times \sum_j \int_{S_j} \frac{s^{-1-\sigma} t_j^\kappa}{(t+t_j+|x-x_j|^{2\alpha})^{(n+1)/(2\alpha)+\kappa}} dV(y,s),
\end{aligned}
$$

and

$$
\begin{aligned}
&|\partial_t \Pi_2(x,t)| \\
&\quad \le \sum_j \int_{S_j} |\mathcal{D}_t u(y,s) - \mathcal{D}_t u(x_j,t_j)| t_j^\kappa \big| \mathcal{D}_t^{\kappa+1} W^{(\alpha)}(x-x_j, t+t_j)\big| dV(y,s)
\end{aligned}
$$

$$\leq \sum_j \int_{S_j} \frac{|\mathcal{D}_t u(y,s) - \mathcal{D}_t u(x_j,t_j)| t_j^{\kappa}}{(t+t_j+|x-x_j|^{2\alpha})^{n/(2\alpha)+\kappa+1}} dV(y,s)$$

$$\leq C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)} \sum_j \int_{S_j} \frac{s^{-1-\sigma} t_j^{\kappa}}{(t+t_j+|x-x_j|^{2\alpha})^{n/(2\alpha)+\kappa+1}} dV(y,s).$$

Since $S_j \subset S_\delta^{(\alpha)}(X_j) \subset S_{1/3}^{(\alpha)}(X_j)$, there exists a constant $C > 0$ independent of δ such that

$$C^{-1}s \leq t_j \leq Cs, \quad t+s+|x-y|^{2\alpha} \leq C(t+t_j+|x-x_j|^{2\alpha})$$

for all $(y,s) \in S_j$ and j. Therefore, Lemma 2.6 implies that there exists a constant $C > 0$ independent of δ such that for each $1 \leq \ell \leq n$,

$$\begin{aligned}
&|\partial_{x_\ell} \Pi_2(x,t)| \\
&\quad \leq C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)} \sum_j \int_{S_j} \frac{s^{-1-\sigma+\kappa}}{(t+s+|x-y|^{2\alpha})^{(n+1)/(2\alpha)+\kappa}} dV(y,s) \\
&\quad \leq C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)} \int_H \frac{s^{-1-\sigma+\kappa}}{(t+s+|x-y|^{2\alpha})^{(n+1)/(2\alpha)+\kappa}} dV(y,s) \\
&\quad \leq C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)} \cdot t^{-\sigma-1/(2\alpha)},
\end{aligned}$$

and

$$\begin{aligned}
&|\partial_t \Pi_2(x,t)| \\
&\quad \leq C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)} \sum_j \int_{S_j} \frac{s^{-1-\sigma+\kappa}}{(t+s+|x-y|^{2\alpha})^{n/(2\alpha)+\kappa+1}} dV(y,s) \\
&\quad \leq C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)} \int_H \frac{s^{-1-\sigma+\kappa}}{(t+s+|x-y|^{2\alpha})^{n/(2\alpha)+\kappa+1}} dV(y,s) \\
&\quad \leq C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)} \cdot t^{-\sigma-1}.
\end{aligned}$$

Hence, we obtain $\|\Pi_2\|_{\mathcal{B}_\alpha(\sigma)} \leq C(\delta + \delta^{1/(2\alpha)}) \|u\|_{\mathcal{B}_\alpha(\sigma)}$, where the constant C is independent of δ and u. $\square$

6. The $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating theorem

In this section, we give a interpolating theorem for the α-parabolic Bloch type spaces. Let $\sigma > -m(\alpha)$ and $\mathbb{X} = \{X_j\} = \{(x_j, t_j)\}$ be a sequence in H. Furthermore, let $(\gamma, \kappa) \in (\mathbb{N}_0^n \times \Sigma_\sigma)\backslash\{(0,0)\}$. For $u \in \widetilde{\mathcal{B}}_\alpha(\sigma)$, we recall the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating operator

$$T_{\sigma,\mathbb{X}}^{\gamma,\kappa} u := \Big\{ t_j^{|\gamma|/(2\alpha)+\kappa+\sigma} \partial_x^\gamma \mathcal{D}_t^\kappa u(X_j) \Big\}. \tag{6.1}$$

We say that $\{X_j\}$ is a $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating sequence of order (γ, κ) if the operator $T_{\sigma,\mathbb{X}}^{\gamma,\kappa} : \widetilde{\mathcal{B}}_\alpha(\sigma) \to \ell^\infty$ is bounded and onto. Again, we remark that $T_{\sigma,\mathbb{X}}^{\gamma,\kappa} : \widetilde{\mathcal{B}}_\alpha(\sigma) \to \ell^\infty$ is always bounded by Lemma 2.2 (1). We also say that $\{X_j\}$ is a $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$-interpolating sequence of order (γ, κ) if $T_{\sigma,\mathbb{X}}^{\gamma,\kappa} : \widetilde{\mathcal{B}}_{\alpha,0}(\sigma) \to c_0$ is bounded and onto. In this section, we give an interpolating theorem for $\widetilde{\mathcal{B}}_\alpha(\sigma)$ and $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$, that is, we give a sufficient condition for a sequence $\{X_j\}$ to be the $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating and $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$-interpolating sequence. We need the following lemma.

Lemma 6.1 *Let $0 < \alpha \le 1$, $\sigma > -m(\alpha)$, and κ be a real number such that $\kappa > \sigma$. Let $\gamma \in \mathbb{N}_0^n$ be a multi-index. Furthermore, let $\mathbb{X} = \{X_j\} = \{(x_j, t_j)\}$ be δ-separated in the α-parabolic sense. If $(\beta, \nu) \in \mathbb{N}_0^n \times \Sigma_\sigma\backslash\{(0,0)\}$ and $\{\lambda_j\} \in \ell^\infty$, then*

$$\begin{aligned}
&\partial_x^\beta \mathcal{D}_t^\nu \big(U_{\sigma,\mathbb{X}}^{\gamma,\kappa}\{\lambda_j\}\big)(x,t) \\
&\quad = \sum_{j=1}^\infty \lambda_j t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma} \partial_x^{\beta+\gamma} \mathcal{D}_t^{\nu+\kappa} W^{(\alpha)}(x - x_j, t + t_j)
\end{aligned} \tag{6.2}$$

for all $(x,t) \in H$.

Proof. Let $(\beta, \nu) \in \mathbb{N}_0^n \times \Sigma_\sigma\backslash\{(0,0)\}$.

Suppose $\nu \in \mathbb{N}_0$. Put

$$u_N(x,t) = \sum_{j=1}^N \lambda_j t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma} \omega_\alpha^{\gamma,\kappa}(x,t; x_j, t_j), \quad (x,t) \in H.$$

Then, $\{\partial_x^\beta \mathcal{D}_t^\nu u_N\}$ converges uniformly on $\mathbb{R}^n \times [\tau, \infty)$ for every $\tau > 0$. In fact, by (3.3), we have

$$\sum_{j=1}^{N}|\lambda_j|t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\left|\partial_x^\beta\mathcal{D}_t^\nu\omega_\alpha^{\gamma,\kappa}(x,t;x_j,t_j)\right|$$
$$\le CF(\delta)\|\{\lambda_j\}\|_\infty$$
$$\times\sum_{j=1}^{N}\int_{S_\delta^{(\alpha)}(X_j)}\frac{r^{|\gamma|/(2\alpha)+\kappa-\sigma-1}}{(t+r+|x-z|^{2\alpha})^{(n+|\beta|+|\gamma|)/(2\alpha)+\nu+\kappa}}dV(z,r)$$
$$\le CF(\delta)\|\{\lambda_j\}\|_\infty\int_H\frac{r^{|\gamma|/(2\alpha)+\kappa-\sigma-1}}{(t+r+|x-z|^{2\alpha})^{(n+|\beta|+|\gamma|)/(2\alpha)+\nu+\kappa}}dV(z,r)$$

for all $X\in H$. Since $(\beta,\nu)\in\mathbb{N}_0^n\times\mathbb{N}_0\backslash\{(0,0)\}$, Lemma 2.6 implies

$$\sum_{j=1}^{N}|\lambda_j|t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\left|\partial_x^\beta\mathcal{D}_t^\nu\omega_\alpha^{\gamma,\kappa}(X;X_j)\right|$$
$$\le CF(\delta)\|\{\lambda_j\}\|_\infty t^{-(|\beta|/(2\alpha)+\nu+\sigma)}.$$

Thus, we have $\{\partial_x^\beta\mathcal{D}_t^\nu u_N\}$ converges uniformly on $\mathbb{R}^n\times[\tau,\infty)$ for every $\tau>0$. It follows that we can differentiate term by term, so that (6.2) is obtained.

Suppose $\nu\notin\mathbb{N}_0$. Then, Lemma 2.6 also implies

$$\int_0^\infty\tau^{\lceil\nu\rceil-\nu-1}\sum_{j=1}^{\infty}|\lambda_j|t_j^{(n+|\gamma|)/(2\alpha)+\kappa-\sigma}\left|\partial_x^\beta\mathcal{D}_t^{\lceil\nu\rceil}\omega_\alpha^{\gamma,\kappa}(x,t+\tau;x_j,t_j)\right|d\tau$$
$$\le CF(\delta)\|\{\lambda_j\}\|_\infty\int_0^\infty\tau^{\lceil\nu\rceil-\nu-1}$$
$$\times\int_H\frac{r^{(|\gamma|)/(2\alpha)+\kappa-\sigma-1}}{(t+\tau+r+|x-z|^{2\alpha})^{(n+|\beta|+|\gamma|)/(2\alpha)+\lceil\nu\rceil+\kappa}}dV(z,r)d\tau$$
$$\le CF(\delta)\|\{\lambda_j\}\|_\infty\int_0^\infty\frac{\tau^{\lceil\nu\rceil-\nu-1}}{(t+\tau)^{|\beta|/(2\alpha)+\lceil\nu\rceil+\sigma}}d\tau<\infty,$$

because $\nu>\max\{0,-\sigma\}$. Hence, differentiating term by term, we obtain (6.2) from the Fubini theorem. □

We show the main theorem of this section.

Theorem 6.2 *Let $0 < \alpha \leq 1$, $\sigma > -m(\alpha)$, and $(\gamma, \kappa) \in (\mathbb{N}_0^n \times \Sigma_\sigma) \setminus \{(0,0)\}$. Then, there exists $0 < \delta_0 < 1$ such that if a sequence $\{X_j\}$ in H is δ-separated in the α-parabolic sense with $\delta_0 \leq \delta < 1$, then $\{X_j\}$ is a $\widetilde{\mathcal{B}}_\alpha(\sigma)$-interpolating and $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$-interpolating sequence of order (γ, κ).*

Proof. Let ν be a real number such that $\nu > \sigma$. We note that the function

$$s^{(n+2|\gamma|)/(2\alpha)+\kappa+\nu} \partial_x^{2\gamma} \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(0, 2s)$$

is constant on H. In fact, by Lemma 2.1 (4), we have

$$\begin{aligned}
&\partial_x^{2\gamma} \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(0, 2s) \\
&\quad = 2^{-((n+2|\gamma|)/(2\alpha)+\kappa+\nu)} s^{-((n+2|\gamma|)/(2\alpha)+\kappa+\nu)} \partial_x^{2\gamma} \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(0, 1).
\end{aligned}$$

Thus, $s^{(n+2|\gamma|)/(2\alpha)+\kappa+\nu} \partial_x^{2\gamma} \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(0, 2s)$ is constant on H. Put

$$\begin{aligned}
c_{\gamma,\kappa,\nu} &:= s^{(n+2|\gamma|)/(2\alpha)+\kappa+\nu} \partial_x^{2\gamma} \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(0, 2s) \\
&= 2^{-((n+2|\gamma|)/(2\alpha)+\kappa+\nu)} \partial_x^{2\gamma} \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(0, 1).
\end{aligned}$$

Then, as in the proof of [14, Proposition 1 (2)], it is easy to see that $\partial_x^{2\gamma} \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(0, 1) \neq 0$. Therefore, we obtain $c_{\gamma,\kappa,\nu} \neq 0$.

Suppose that $\mathbb{X} = \{X_j\} = \{(x_j, t_j)\}$ is δ-separated in the α-parabolic sense. Here constraints of δ will be imposed later. By Theorem 3.1, the operator $U_{\sigma,\mathbb{X}}^{\gamma,\nu} : \ell^\infty \to \widetilde{\mathcal{B}}_\alpha(\sigma)$ is bounded and $U_{\sigma,\mathbb{X}}^{\gamma,\nu}$ maps c_0 into $\widetilde{\mathcal{B}}_{\alpha,0}(\sigma)$. Therefore, $T_{\sigma,\mathbb{X}}^{\gamma,\kappa} U_{\sigma,\mathbb{X}}^{\gamma,\nu} : \ell^\infty \to \ell^\infty$ is bounded and $T_{\sigma,\mathbb{X}}^{\gamma,\kappa} U_{\sigma,\mathbb{X}}^{\gamma,\nu}$ maps c_0 into c_0 by Theorem 4.1 (3). As in the proof of Theorem 5.3, it suffices to show that there exists $0 < \delta_0 < 1$ such that if $\delta_0 \leq \delta < 1$ then $\|I - S_{\sigma,\mathbb{X}}^{\gamma,\kappa,\nu}\| < 1$, where I is the identity operator on ℓ^∞ and $S_{\sigma,\mathbb{X}}^{\gamma,\kappa,\nu} = c_{\gamma,\kappa,\nu}^{-1} T_{\sigma,\mathbb{X}}^{\gamma,\kappa} U_{\sigma,\mathbb{X}}^{\gamma,\nu}$. In fact, the operator $I - S_{\sigma,\mathbb{X}}^{\gamma,\kappa,\nu}$ maps a sequence $\{\lambda_j\}$ in ℓ^∞ to a sequence $\{\xi_m\}$ in ℓ^∞ given by

$$\xi_m = \lambda_m - c_{\gamma,\kappa,\nu}^{-1} t_m^{|\gamma|/(2\alpha)+\kappa+\sigma} \partial_x^\gamma \mathcal{D}_t^\kappa \big(U_{\sigma,\mathbb{X}}^{\gamma,\nu}\{\lambda_j\}\big)(X_m).$$

By Lemma 6.1, we have

$$\begin{aligned}\xi_m &= \lambda_m - c_{\gamma,\kappa,\nu}^{-1} t_m^{|\gamma|/(2\alpha)+\kappa+\sigma} \\ &\quad \times \sum_{j=1}^{\infty} \lambda_j t_j^{(n+|\gamma|)/(2\alpha)+\nu-\sigma} \partial_x^{2\gamma} \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(x_m - x_j, t_m + t_j) \\ &= c_{\gamma,\kappa,\nu}^{-1} t_m^{|\gamma|/(2\alpha)+\kappa+\sigma} \\ &\quad \times \sum_{j\neq m} \lambda_j t_j^{(n+|\gamma|)/(2\alpha)+\nu-\sigma} \partial_x^{2\gamma} \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(x_m - x_j, t_m + t_j).\end{aligned}$$

Thus, Lemma 2.1 (1) and Lemma 2.8 imply

$$\begin{aligned}|\xi_m| &\le \left|c_{\gamma,\kappa,\nu}^{-1}\right| t_m^{|\gamma|/(2\alpha)+\kappa+\sigma} \\ &\quad \times \sum_{j\neq m} |\lambda_j| t_j^{(n+|\gamma|)/(2\alpha)+\nu-\sigma} \left|\partial_x^{2\gamma} \mathcal{D}_t^{\kappa+\nu} W^{(\alpha)}(x_m - x_j, t_m + t_j)\right| \\ &\le C\|\{\lambda_j\}\|_\infty t_m^{|\gamma|/(2\alpha)+\kappa+\sigma} \sum_{j\neq m} \frac{t_j^{(n+|\gamma|)/(2\alpha)+\nu-\sigma}}{(t_m + t_j + |x_m - x_j|^{2\alpha})^{(n+2|\gamma|)/(2\alpha)+\kappa+\nu}} \\ &\le CF(\delta/2)\|\{\lambda_j\}\|_\infty t_m^{|\gamma|/(2\alpha)+\kappa+\sigma} \\ &\quad \times \sum_{j\neq m} \int_{S_{\delta/2}^{(\alpha)}(X_j)} \frac{r^{|\gamma|/(2\alpha)+\nu-\sigma-1}}{(t_m + r + |x_m - z|^{2\alpha})^{(n+2|\gamma|)/(2\alpha)+\kappa+\nu}} dV(z,r) \\ &\le CF(\delta/2)\|\{\lambda_j\}\|_\infty t_m^{|\gamma|/(2\alpha)+\kappa+\sigma} \\ &\quad \times \int_{H\setminus S_\delta^{(\alpha)}(X_m)} \frac{r^{|\gamma|/(2\alpha)+\nu-\sigma-1}}{(t_m + r + |x_m - z|^{2\alpha})^{(n+2|\gamma|)/(2\alpha)+\kappa+\nu}} dV(z,r) \\ &= CF(\delta/2)\|\{\lambda_j\}\|_\infty \int_{H\setminus S_\delta^{(\alpha)}(0,1)} \frac{t^{|\gamma|/(2\alpha)+\nu-\sigma-1}}{(1 + t + |x|^{2\alpha})^{(n+2|\gamma|)/(2\alpha)+\kappa+\nu}} dV(z,r),\end{aligned}$$

where C is independent of δ. Since $F(\delta/2)$ is bounded for all $1/2 \le \delta < 1$, Lemma 2.6 shows that there exists $0 < \delta_0 < 1$ such that if $\delta_0 \le \delta < 1$ then $\|I - S_{\sigma,\mathbb{X}}^{\gamma,\kappa,\nu}\| < 1$. □

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