

Equinormalizable theory for plane curve singularities with embedded points and the theory of equisingularity

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Abstract. In this paper we give some criteria for a family of generically reduced plane curve singularities to be equinormalizable. The first criterion is based on the δ -invariant of a (non-reduced) curve singularity which is introduced by Brücker-Greuel ([BG]). The second criterion is based on the I-equisingularity of a k -parametric family ($k \geq 1$) of generically reduced plane curve singularities, which is introduced by Nobile ([No]) for one-parametric families. The equivalence of some kinds of equisingularities of a family of generically reduced plane curve singularities is also studied.

Key words: Local deformations, equinormalizable, equisingularity, δ -invariant, plane curve singularities.

1. Introduction

The theory of equinormalizable deformations has been initiated by Teissier ([Tei1]) for deformations of reduced curve singularities over $(\mathbb{C}, 0)$. It is generalized to higher dimensional base spaces by Teissier himself and Raynaud in 1980 ([Tei2]). Recently, it is developed by Chiang-Hsieh and Lipman ([Ch-Li]) for projective deformations of reduced complex spaces over normal base spaces, and it is studied by Kollár for projective deformation of generically reduced algebraic schemes over semi-normal base spaces ([Ko]). The theory of equinormalizable deformations of not necessarily reduced curve singularity over $(\mathbb{C}, 0)$ is studied by Brücker and Greuel in 1990 ([BG]). Some generalizations of the results of Brücker and Greuel to deformations of (not necessarily reduced) curve singularities over normal base spaces are given by Greuel and the author in the forthcoming paper [GL]. In this paper we study the equinormalizable deformations of not necessarily reduced plane curve singularities over smooth base spaces $(\mathbb{C}^k, 0)$ ($k \geq 0$). We show in Theorem 4.1 that *the induced morphism on the pure-dimensional part of the total space is equinormalizable if and only if the given deformation is δ -constant*. This result is generalized in [GL] to deformations of not necessarily reduced curve singularities over normal spaces, however the technique

used to prove it in the context of plane curve singularities (in \mathbb{C}^2) is quite special, applying a consequence of the Hilbert-Burch Theorem (Lemma 4.1). This technique cannot be used for deformations of curve singularities which are not planar.

The theory of equisingularity for reduced plane curve singularities has been introduced by Zariski (1970, [Za]), Wahl (1974, [Wa]). In [No] Nobile defined three kinds of equisingularities for a one-parametric family of generically reduced plane curve singularities: I-, T- and C-equisingularity. He showed that I-equisingularity is equivalent to T-equisingularity, C-equisingularity implies I-equisingularity, and gave a criterion for a family to be C-equisingular. In this paper we generalize the results of Nobile to k -parametric families ($k \geq 1$) of generically reduced plane curve singularities, and based on these equisingularities we give a criterion for a deformation to be equinormalizable (Theorem 5.3).

2. δ -invariant of (not necessarily reduced) curve singularities

Following Greuel and Brücker ([BG], for curves), Greuel and the author ([GL], for arbitrary complex spaces), we recall in this section the definition of the δ -invariant of a curve which is not necessarily reduced, having an *isolated singularity*¹.

For a complex curve C , we denote by C^{red} its reduction and by $i : C^{red} \rightarrow C$ the inclusion. For $\nu^{red} : \bar{C} \rightarrow C^{red}$ we mean the normalization of the reduced curve C^{red} , and we call the composition $\nu := i \circ \nu^{red} : \bar{C} \rightarrow C$ the *normalization of the curve C* . Then we have the induced map on the structure sheaves

$$\nu^\# : \mathcal{O}_C \rightarrow \nu_* \mathcal{O}_{\bar{C}}.$$

We have $\text{Ker}(\nu^\#) = \text{Nil}(\mathcal{O}_C)$, the sheaf of nilpotent elements of \mathcal{O}_C , and $\text{Coker}(\nu^\#) = \nu_* \mathcal{O}_{\bar{C}} / \mathcal{O}_{C^{red}}$. Since the map ν is finite, these sheaves are coherent \mathcal{O}_C -modules, whose supports are $\text{NRed}(C)$ and $\text{Sing}(C)$, respectively. Thus, if $x \in C$ is an isolated non-normal point then $\text{Ker}(\nu^\#)$ and $\text{Coker}(\nu^\#)$

¹A point p in a curve C is said to be *non-reduced* (resp. *singular*) if the local ring at p , $\mathcal{O}_{C,p}$ is not reduced (resp. not regular). The set of all non-reduced (resp. singular) points in C is denoted by $\text{NRed}(C)$ (resp. $\text{Sing}(C)$), and called the *non-reduced locus* (*singular locus*) of C . If $p \in \text{NRed}(C)$ (resp. $\text{Sing}(C)$) is isolated then it is called an *isolated non-reduced point* (resp. *isolated singular point*) of C .

are finite dimensional \mathbb{C} -vector spaces, and we have $\text{Ker}(\nu^\#) = H_{\{x\}}^0(\mathcal{O}_C)$, the local cohomology.

Definition 2.1 Let C be a complex curve and $x \in C$ an isolated singular point. The number

$$\delta(C^{red}, x) := \dim_{\mathbb{C}}(\nu_*^{red} \mathcal{O}_{\overline{C}})_x / \mathcal{O}_{C^{red}, x}$$

is called the *delta-invariant of C^{red} at x* ,

$$\epsilon(C, x) := \dim_{\mathbb{C}} H_{\{x\}}^0(\mathcal{O}_C)$$

is called the *epsilon-invariant of C at x* , and

$$\delta(C, x) := \delta(C^{red}, x) - \epsilon(C, x)$$

is called the *delta-invariant of C at x* .

If C has only finitely many singular points then the number

$$\delta(C) := \sum_{x \in \text{Sing}(C)} \delta(C, x)$$

is called the *delta-invariant of C* .

It is easy to see that $\delta(C^{red}, x) \geq 0$, and $\delta(C^{red}, x) = 0$ if and only if x is an isolated point of C or the germ (C^{red}, x) is smooth. Hence, if $x \in C$ is an isolated point of C then $\delta(C, x) = -\dim_{\mathbb{C}} \mathcal{O}_{C, x}$.

Example 2.1 We compute the δ -invariant of the curve singularity $(X_0, 0) \subseteq (\mathbb{C}^2, 0)$ defined by the ideal

$$I_0 = \langle x^2 - y^3 \rangle \cap \langle y \rangle \cap \langle x, y^5 \rangle \subseteq \mathbb{C}\{x, y\}.$$

The curve singularity $(X_0, 0)$ is the union of a cusp C and a straight line L with an embedded non-reduced point at the origin. We have

$$\delta(X_0^{red}, 0) = \delta(C, 0) + \delta(L, 0) + i_{(0,0)}(C, L) = 1 + 0 + 2 = 3,$$

where $i_{(0,0)}(C, L)$ denotes the intersection multiplicity of C and L at the

origin. Note that $H_{\{0\}}^0(\mathcal{O}_{X_0}) = \text{Nil}(\mathcal{O}_{X_0,0})$ is the kernel of the surjection

$$p : \mathcal{O}_{X_0} \cong \mathbb{C}\{x, y\}/I_0 \rightarrow \mathcal{O}_{X_0^{red}} \cong \mathbb{C}\{x, y\}/\text{rad}(I_0).$$

Hence

$$\epsilon(X_0, 0) = \dim_{\mathbb{C}} \text{rad}(I_0)/I_0 = 1,$$

where $\text{rad}(I_0) = \langle x^2 - y^3 \rangle \cap \langle y \rangle$. Therefore

$$\delta(X_0) = \delta(X_0, 0) = \delta(X_0^{red}, 0) - \epsilon(X_0, 0) = 2.$$

3. Simultaneous normalizations and equinormalizable morphisms

Following Kollár ([Ko]) and Chiang-Hsieh-Lipman ([Ch-Li]), in [GL] we gave the following definition of a simultaneous normalization of a map between complex spaces, and we defined also equinormalizable maps between complex spaces (or germs).

Definition 3.1 Let $f : X \rightarrow S$ be a morphism of complex spaces. A *simultaneous normalization of f* is a morphism $n : \tilde{X} \rightarrow X$ such that

- (1) n is finite,
- (2) $\tilde{f} := f \circ n : \tilde{X} \rightarrow S$ is *normal*, i.e., for each $z \in \tilde{X}$, \tilde{f} is flat at z and the fiber $\tilde{X}_{\tilde{f}(z)} := \tilde{f}^{-1}(\tilde{f}(z))$ is normal,
- (3) the induced map $n_s : \tilde{X}_s := \tilde{f}^{-1}(s) \rightarrow X_s$ is *bimeromorphic*² for each $s \in f(X)$.

The morphism f is called *equinormalizable* if the normalization $\nu : \bar{X} \rightarrow X$ is a simultaneous normalization of f . It is called *equinormalizable at $x \in X$* if the restriction of f to some neighborhood of x is equinormalizable.

If $f : (X, x) \rightarrow (S, s)$ is a morphism of germs, then a *simultaneous normalization of f* is a morphism n from a multi-germ $(\tilde{X}, n^{-1}(x))$ to (X, x) such that some representative of n is a simultaneous normalization of a

²A map $f : X \rightarrow S$ is called *bimeromorphic* if there exists a nowhere dense analytic subset A of S such that $f^{-1}(A)$ is nowhere dense in X and the induced map $X \setminus f^{-1}(A) \rightarrow S \setminus A$ is an isomorphism.

representative of f . The germ f is *equinormalizable* if some representative of f is equinormalizable.

Remark 3.1 Our definition of simultaneous normalizations of a map does not require the flatness of the map. We do also not require the reducedness of the fibers (however, all non-empty fibers are generically reduced if the map admits a simultaneous normalization, see [GL]). The total space of the map is also not required to be pure-dimensional. One may see [Ch-Li], [Ko] and [GL] for more discussions about simultaneous normalizations and equinormalizability.

In this paper we consider flat deformations of plane curve singularities, find a criterion for such a deformation to be equinormalizable. First of all we give an example of a deformation of a plane curve singularity that is equinormalizable.

Example 3.1 We consider again the curve singularity $(X_0, 0) \subseteq (\mathbb{C}^2, 0)$ given in Example 2.1. Let

$$f : (X, 0) \longrightarrow (\mathbb{C}^2, 0), (x, y, u, v) \mapsto (u, v),$$

be the restriction to $(X, 0) \subseteq (\mathbb{C}^4, 0)$ of the projection $\pi : (\mathbb{C}^4, 0) \rightarrow (\mathbb{C}^2, 0)$, where $(X, 0)$ is defined by the ideal

$$I := \langle x^2 - y^3 + uy^2 \rangle \cap \langle y - u \rangle \cap \langle x - v, y \rangle \subseteq \mathbb{C}\{x, y, u, v\}.$$

The map f is flat since u, v is an $\mathcal{O}_{X,0}$ -regular sequence. Hence the map f is a deformation of $(X_0, 0)$ over $(\mathbb{C}^2, 0)$.

It is easy to see that the total space $(X, 0)$ of the deformation f is reduced with two 3-dimensional irreducible components and one 2-dimensional irreducible component. The normalization of these components are given by

$$\begin{aligned} \nu_1 : (\mathbb{C}^3, 0) &\longrightarrow (X, 0) \\ (T_1, T_2, T_3) &\mapsto (T_3^3 + T_1 T_3, T_3^2 + T_1, T_1, T_2), \\ \nu_2 : (\mathbb{C}^3, 0) &\longrightarrow (X, 0) \\ (T_1, T_2, T_3) &\mapsto (T_1, T_2, T_2, T_3), \end{aligned}$$

and

$$\begin{aligned} \nu_3 : (\mathbb{C}^2, 0) &\longrightarrow (X, 0) \\ (T_1, T_2) &\mapsto (T_2, 0, T_1, T_2). \end{aligned}$$

For each $i = 1, 2, 3$, the morphism $\bar{f}_i := f \circ \nu_i$ is given by the last two components of ν_i and all of them are flat. The special fibers of \bar{f}_1 and \bar{f}_2 are straight lines in \mathbb{C}^3 , while the special fiber of \bar{f}_3 is a (normal) point in \mathbb{C}^2 . It follows that the given deformation is equinormalizable.

4. Equinormalizability of deformations of curve singularities in the plane with embedded non-reduced points

In this section we give a criterion for a deformation of an isolated curve singularity in the plane with embedded non-reduced points over a smooth base space of dimension $k \geq 1$ to be equinormalizable. As a consequence of the Hilbert-Burch theorem (cf. [Bur, Theorem 5] or [BG, Satz 6.1]), each ideal defining a curve singularity in \mathbb{C}^n can be factorized as a product of an ideal defining a hypersurface singularity and an ideal defining a Cohen-Macaulay singularity of codimension 2 in \mathbb{C}^n . For $n = 2$, the hypersurface singularity is the pure 1-dimensional part of the curve singularity and the Cohen-Macaulay singularity becomes a (non-reduced) point.

The following result is one of the main ideas in the proof of the numerical criterion for the equinormalizability.

Lemma 4.1 (cf. [BG], Prop. 6.3) *Let $(X_0, 0) \subseteq (\mathbb{C}^n, 0)$ be a curve singularity defined by the ideal $\langle g_1, \dots, g_m \rangle = \langle g \rangle \cdot \langle p_1, \dots, p_m \rangle$. Let $f : (X, 0) \longrightarrow (\mathbb{C}^k, 0)$ be a deformation of $(X_0, 0)$, where $(X, 0)$ is given by the ideal $I(X, 0) = \langle g_i + \sum_{j=1}^k t_j \bar{g}_{ij} \rangle_{1 \leq i \leq m} \subseteq \mathcal{O}_{\mathbb{C}^n, 0} \{ \mathbf{t} \}$, $\mathbf{t} := (t_1, \dots, t_k)$. Then there exist functions \bar{g}_j and \bar{p}_{ij} in $\mathcal{O}_{\mathbb{C}^n, 0} \{ \mathbf{t} \}$ such that*

$$\left\langle gp_i + \sum_{j=1}^k t_j \bar{g}_{ij} \right\rangle_{1 \leq i \leq m} = \left\langle g + \sum_{j=1}^k t_j \bar{g}_j \right\rangle \cdot \left\langle p_i + \sum_{j=1}^k t_j \bar{p}_{ij} \right\rangle_{1 \leq i \leq m},$$

where $f_G : (G, 0) \longrightarrow (\mathbb{C}^k, 0)$ with $I(G, 0) = \langle g + \sum_{j=1}^k t_j \bar{g}_j \rangle$ is a deformation of $(G_0, 0)$ defined by $\langle g \rangle \subseteq \mathcal{O}_{\mathbb{C}^n, 0}$, and $f_P : (P, 0) \longrightarrow (\mathbb{C}^k, 0)$ with $I(P, 0) = \langle p_i + \sum_{j=1}^k t_j \bar{p}_{ij} \rangle_{1 \leq i \leq m}$ is a deformation of $(P_0, 0)$ defined

by $\langle p_1, \dots, p_m \rangle \subseteq \mathcal{O}_{\mathbb{C}^n, 0}$.

Let $f : X \rightarrow S$ be a morphism of complex spaces whose fibers have only finitely many non-normal points. It is called (locally) *delta-constant* if the function $s \mapsto \delta(X_s)$ is (locally) constant on S . A morphism of gemrs is *delta-constant* if some of its representatives is δ -constant.

The following theorem is the first main result of this paper.

Theorem 4.1 *Let $(X_0, 0) \subseteq (\mathbb{C}^2, 0)$ be an isolated (not necessarily reduced) curve singularity. Let $f : (X, 0) \rightarrow (\mathbb{C}^k, 0)$, $k \geq 1$, be a deformation of $(X_0, 0)$. Denote by $(X^u, 0)$ the unmixed subgerm³ of $(X, 0)$. Then the following holds:*

- (1) *The restriction $f^u : (X^u, 0) \rightarrow (\mathbb{C}^k, 0)$ is flat.*
- (2) *f is δ -constant if and only if $f^u : (X^u, 0) \rightarrow (\mathbb{C}^k, 0)$ is equinormalizable.*

Proof. (1) As a consequence of the Hilbert-Burch theorem, the germ $(X_0, 0)$ is a union of a hypersurface $(X_0^u, 0) \subseteq (\mathbb{C}^2, 0)$ and an embedded (non-reduced) point. More precisely, if $(X_0, 0)$ is defined by the ideal I_0 then it can be factorized as

$$I_0 = \langle g \rangle \cdot J_0,$$

where $\langle g \rangle \subseteq \mathcal{O}_{\mathbb{C}^2, 0}$ defines $(X_0^u, 0) \subseteq (\mathbb{C}^2, 0)$ and the ideal $J_0 \subseteq \mathcal{O}_{\mathbb{C}^2, 0}$ defines such a (non-reduced) point. By Lemma 4.1 we can write the ideal I defining $(X, 0) \subseteq (\mathbb{C}^2 \times \mathbb{C}^k, 0)$ as $I = \langle G \rangle \cdot J$, where $\langle G \rangle \subseteq \mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^k, 0}$ defines a deformation $(H, 0) \subseteq (\mathbb{C}^2 \times \mathbb{C}^k, 0)$ of the hypersurface $(X_0^u, 0)$ given by $\langle g \rangle$ and $J \subseteq \mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^k, 0}$ defines a deformation $(P, 0) \subseteq (\mathbb{C}^2 \times \mathbb{C}^k, 0)$ of the (non-reduced) point given by J_0 .

Note that $(H, 0)$ is reduced and pure $(k + 1)$ -dimensional, because it is the total space of a deformation of the reduced and pure 1-dimensional singu-

³Let R be a ring and $I \subseteq R$ be an ideal of dimension m . Assume that I has an irredundant primary decomposition $I = \bigcap_{i=1}^r Q_i$. For an integer $0 \leq k \leq m$, we define the *pure k -dimensional part* $I^{(k)}$ of the ideal I to be the intersection of all Q_i with $\dim Q_i = k$. The ideal $I^{(k)}$ is well-defined for each $0 \leq k \leq m$ because this part of the primary decomposition is uniquely determined. We define the *unmixed part* I^u of the ideal I to be the pure m -dimensional part of the radical \sqrt{I} . If the germ $(X, 0) \subseteq (\mathbb{C}^n, 0)$ is defined by an ideal $I \subseteq \mathcal{O}_{\mathbb{C}^n, 0}$, the *unmixed subgerm* $(X^u, 0)$ of $(X, 0)$ is the one defined by the unmixed part I^u of I .

larity $(X_0^u, 0) \subseteq (\mathbb{C}^2, 0)$ over $(\mathbb{C}^k, 0)$ which is reduced and pure k -dimensional (cf. [GLS, Theorem I. 1.85] and [GLS, Theorem I. 1.101]). Hence $(H, 0) \equiv (X^u, 0)$. Therefore the restriction map $f^u : (X^u, 0) \equiv (H, 0) \rightarrow (\mathbb{C}^k, 0)$ is flat and it is actually a deformation of $(X_0^u, 0)$ over $(\mathbb{C}^k, 0)$.

(2) **First we prove the “only if” part.** Let $f : X \rightarrow S$ be a sufficiently small representative of the given deformation such that $f^u : X^u \rightarrow S$ is equinormalizable. Since $(X_0, 0)$ has isolated singularities, it follows from the *generic principle* (cf. [BF, Theorem 2.2]) that there exists an open dense subset $U \subseteq S$ such that $(X^u)_s := (f^u)^{-1}(s)$ are reduced for all $s \in U$.

We first show that f is δ -constant on U , i.e., $\delta(X_s) = \delta(X_0)$ for any $s \in U$. In fact, for any $s \in U$, there exist an irreducible reduced curve singularity $C \subseteq S$ passing through s . Let $\alpha : T \rightarrow C \subseteq S$ be the normalization of this curve singularity such that $\alpha(T \setminus \{0\}) \subseteq U, \alpha(0) = s$, where $T \subseteq \mathbb{C}$ is a small disc with center at 0. Denote $X_T := X \times_S T, X_T^u := X^u \times_S T, \bar{X}_T := \bar{X} \times_S T$, where $\nu^u : \bar{X} \rightarrow X^u$ is the normalization of X^u . Thus we have the following Cartesian diagram:

$$(\Delta) \quad \begin{array}{ccc} \bar{X}_T & \longrightarrow & \bar{X} \\ \nu_T^u \downarrow & \square & \downarrow \nu^u \\ X_T^u & \longrightarrow & X^u \\ \downarrow i_T & \square & \downarrow i \\ f_T^u X_T & \longrightarrow & X \quad f^u \\ \downarrow f_T & \square & \downarrow f \\ T & \longrightarrow & S \end{array}$$

For any $t \in T, s = \alpha(t) \in S$, we have

$$\mathcal{O}_{(X_T)_t} := \mathcal{O}_{f_T^{-1}(t)} \cong \mathcal{O}_{X_s},$$

$$\mathcal{O}_{(X_T^u)_t} := \mathcal{O}_{f_T^{u-1}(t)} \cong \mathcal{O}_{(X^u)_s}, \quad \mathcal{O}_{(\bar{X}_T)_t} := \mathcal{O}_{\bar{f}_T^{-1}(t)} \cong \mathcal{O}_{\bar{X}_s}.$$

In the diagram (Δ) , f is flat by hypothesis, \bar{f} is flat since f^u is equinormalizable and f^u is flat by (1). It follows from the preservation of flatness under base change (cf. [GLS, Prop. I. 1.87]) that the induced morphisms f_T, \bar{f}_T and f_T^u are also flat over T . Moreover, for any $t \in T \setminus \{0\}$ we have

$s := \alpha(t) \in U$, hence $(X_T^u)_t \cong (X^u)_s$ is reduced. It follows from [BG, Prop. 3.1.1 (3)] that X_T^u is reduced. On the other hand, since X^u and \mathbb{C}^k are pure dimensional, the fiber of f^u is pure dimensional. It implies that the fiber of $f_T^u : X_T^u \rightarrow T$ is also pure dimensional. Hence X_T^u is pure dimensional. Thus X_T^u is the unmixed space $(X_T)^u$ of X_T .

Moreover, since $(\overline{X}_T)_t \cong \overline{X}_s$ for any $t \in T$, $s = \alpha(t)$, it implies that the special fiber $(\overline{X}_T)_0$ of \overline{f}_T is normal. Then \overline{f}_T is regular by the *regularity criterion for morphisms* (cf. [GLS, Theorem I. 1.117]). It implies that $\overline{X}_T \cong (\overline{X}_T)_0 \times T$ which is smooth, hence normal, and it is the normalization of $X_T^u \equiv (X_T)^u$. Consider the morphism $f_T^u : X_T^u \rightarrow T$ with X_T^u reduced and pure 2-dimensional, hence X_T^u is unmixed. Since $(\overline{X}_T^u)_0 \cong (\overline{X}_T)_0 \cong \overline{X}_0$ which is normal, it implies that the morphism f_T^u is equinormalizable. It follows from [BG, Korollar 2.3.5] that $f_T : X_T \rightarrow T$ is δ -constant, hence $f : X \rightarrow S$ is δ -constant on U .

Let us now take $s_0 \in S \setminus U$. Since U is dense in S , $s_0 \in S$, there exists always a point $s_1 \in U$ which is closed to s_0 . It follows from the semi-continuity of the δ -function (Lemma 4.2) that

$$\delta(X_0) \geq \delta(X_{s_0}) \geq \delta(X_{s_1}).$$

Moreover, $\delta(X_0) = \delta(X_{s_1})$ as above. It implies that $\delta(X_{s_0}) = \delta(X_0)$. Hence $f : X \rightarrow S$ is δ -constant.

Now we show the “if” part. Let $f : X \rightarrow S$ be a sufficiently small representative of the given deformation such that it is δ -constant. For each $s \in S$, denote $X_s := f^{-1}(s)$ and for each $x_s \in \text{Sing}(X_s)$, consider the family

$$\begin{array}{ccc} (X_s, x_s) & \hookrightarrow & (X, x_s) \\ \downarrow & & \downarrow \\ \{s\} & \hookrightarrow & (S, s) \end{array}$$

Each germ $(X_s, x_s) \subseteq (\mathbb{C}^2, 0)$ is defined by an ideal of the form $\langle g_s \rangle \cdot J_s \subseteq \mathcal{O}_{\mathbb{C}^2 \times \{s\}, x_s}$. We have

$$\delta(X_s, x_s) = \delta(X_s^u, x_s) - \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2 \times \{s\}, x_s} / J_s,$$

and

$$\begin{aligned} \delta(X_s) &:= \sum_{x_s \in \text{Sing}(X_s)} \delta(X_s, x_s) \\ &= \sum_{x_s \in \text{Sing}(X_s)} \delta(X_s^u, x_s) - \sum_{x_s \in \text{Sing}(X_s)} \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2 \times \{s\}, x_s} / J_s. \end{aligned} \tag{4.1}$$

Since $(X_0, 0)$ is isolated, it follows from the *local finiteness theorem* (cf. [GLS, Theorem I.1.66]) that the restriction $f : \text{Sing}(f) \rightarrow S$ is finite and $\text{Sing}(X_0) = \text{Sing}(f) \cap X_0 = \{0\}$. Then $f : P \rightarrow S$ is also finite. Obviously, it is flat. Therefore it follows from the *semi-continuity of fibre functions* (cf. [GLS, Theorem I.1.81]) that for all $s \in S$, we have

$$\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2 \times \{0\}, 0} / J_0 = \sum_{x_s \in \text{Sing}(X_s)} \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2 \times \{s\}, x_s} / J_s. \tag{4.2}$$

Moreover, f is δ -constant by assumption, that is,

$$\delta(X_0) = \delta(X_s) \text{ for all } s \in S. \tag{4.3}$$

It follows from (4.1), (4.2) and (4.3) that

$$\delta(X_0^u) = \delta(X_0^u, 0) = \sum_{x_s \in \text{Sing}(X_s)} \delta(X_s^u, x_s) = \delta(X_s^u) \text{ for all } s \in S,$$

i.e., $\delta(X_s^u)$ is constant. Therefore we have a *delta-constant* family of reduced curve singularities $f^u : X^u \cong H \rightarrow S$. Then it is equinormalizable by the criterion of Teissier, Raynaud, Chiang-Hsieh and Lipman (cf. [Ch-Li, Theorem 5.6]). □

In the proof of the theorem above we used the following semi-continuity of the delta-function.

Lemma 4.2 *Let $f : (X, 0) \rightarrow (S, 0)$ be a deformation of an isolated (not necessarily reduced) curve singularity $(X_0, 0) \subseteq (\mathbb{C}^n, 0)$. Then the δ -function, $s \mapsto \delta(X_s)$, is upper semi-continuous in the following sense: there exists a representative $f : X \rightarrow S$ of the given deformation such that $\delta(X_s) \leq \delta(X_0)$ for all $s \in S$.*

Proof. Let $f : X \rightarrow S$ be a sufficiently small representative of the given deformation. For any $s \in S$, there exists an irreducible reduced curve sin-

gularity $C \subseteq S$ passing through 0 and s and let $\alpha : T \rightarrow C \subseteq S$ be the normalization of this curve singularity, where $T \subseteq \mathbb{C}$ is a small disc. Denote $X_T := X \times_S T$. Let $f_T : X_T \rightarrow T$ be the morphism induced by f . The morphism f_T is flat by the preservation of flatness under base change (cf. [GLS, Prop. I.1.85]). Moreover, for any $t \in T, s := \alpha(t) \in S$, we have $\mathcal{O}_{(X_T)_t} := \mathcal{O}_{f_T^{-1}(t)} \cong \mathcal{O}_{f^{-1}(s)} =: \mathcal{O}_{X_s}$. It follows from [BG, Satz 3.1.2 (iii)] that, for $t \in T$ such that $\alpha(t) = s$, we have

$$\delta(X_0) - \delta(X_s) = \delta((X_T)_0) - \delta((X_T)_t) \geq 0. \quad \square$$

Example 4.1 Let us consider again the deformation $f : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ of the plane curve singularity $(X_0, 0)$ given in Example 3.1. As we have shown there, this deformation is equinormalizable. The δ -invariant of the special fiber $(X_0, 0)$ is equal to 2 (Example 2.1). Moreover, for each $u \neq 0, v \neq 0$ close to 0, the reduced fiber $X_{uv} = f^{-1}(u, v)$ consists of a cubic curve C , a straight line L and a reduced point $(v, 0)$. We can compute $\delta(X_{uv}) = 2$.

Furthermore, the fibers X_{0u} and X_{v0} are non-reduced and their delta-invariants are also 2. Hence the given deformation is δ -constant.

Remark 4.1 With the above notations, if the total space $(X, 0)$ of the deformation $f : (X, 0) \rightarrow (\mathbb{C}^k, 0)$ of the plane curve singularity $(X_0, 0)$ is reduced and pure $(k + 1)$ -dimensional then $(X_0, 0)$ is necessarily reduced. In fact, since $(X, 0)$ is reduced, the ideal I defining $(X, 0)$ is radical, i.e., $I = \sqrt{I}$. Moreover, since $(H, 0)$ is pure $(k + 1)$ -dimensional and $(P, 0)$ is pure k -dimensional, it follows that $(P, 0) \subseteq (H, 0)$. Then $(X, 0) = (H, 0) \cup (P, 0) = (H, 0)$, i.e., $V(I) = V(G)$. It follows from Hilbert-Rückert's Nullstellensatz (cf. [GLS, Theorem I.1.72]) that

$$\langle G \rangle \cdot J = I = \sqrt{I} = I(V(I)) = I(V(G)) = \sqrt{\langle G \rangle} = \langle G \rangle.$$

Hence $G \in \langle G \rangle \cdot J$. Then there exists $h \in J$ such that $G = Gh$, or $G(1 - h) = 0$. Since G is a non-zerodivisor of $\mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^k, 0}$ we get $h = 1$. Hence $1 \in J$ and we have $J = \langle 1 \rangle$. This implies $I = \langle G \rangle$ and hence $I_0 = \langle g \rangle$. It means that $(X_0, 0)$ is reduced.

Thus, if the plane curve singularity $(X_0, 0)$ is not reduced then the total spaces of deformations over smooth base spaces are either not reduced or not pure dimensional. However this fact is not true for deformations of non-plane curve singularities. We show in the following example that there

exists a deformation of a non-reduced curve singularity in a 4-dimensional complex space which has a reduced and pure dimensional total space.

Example 4.2 Let us consider the curve singularity $(X_0, 0) \subseteq (\mathbb{C}^4, 0)$ defined by the ideal

$$I_0 := \langle x^2 - y^3, z, w \rangle \cap \langle x, y, w \rangle \cap \langle x, y, z, w^2 \rangle \subseteq \mathbb{C}\{x, y, z, w\},$$

which was considered by Steenbrink ([St]). The curve singularity $(X_0, 0)$ is a union of a cusp C in the plane $z = w = 0$, a straight line $L = \{x = y = w = 0\}$ and an embedded non-reduced point $O = (0, 0, 0, 0)$. Now we consider the restriction $f : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ of the projection $\pi : (\mathbb{C}^6, 0) \rightarrow (\mathbb{C}^2, 0)$, $(x, y, z, w, u, v) \mapsto (u, v)$, to the complex space $(X, 0)$ which is defined by the ideal

$$I = \langle x^2 - y^3 + uy^2, z, w \rangle \cap \langle x, y, w - v \rangle \subseteq \mathbb{C}\{x, y, z, w, u, v\}.$$

It is easy to show that the total space $(X, 0)$ is reduced and pure 3-dimensional. Moreover, by a similar way to Example 3.1 and Example 4.1 we can show that this deformation is equinormalizable and delta-constant (with $\delta = 1$).

5. The theory of equisingularity

In this section we study the theory of equisingularity for plane curve singularities with embedded points which is introduced in [No] (for one-parametric family), where the author formulated and proved the equivalence between I-equisingularity and T-equisingularity, also the relation between I-equisingularity and C-equisingularity.

Definition 5.1 A k -parametric family ($k \in \mathbb{N}^*$) of generically reduced plane curve singularities is a diagram

$$\begin{array}{ccc} (X, 0) & \xrightarrow{i} & (\mathbb{C}^{k+2}, 0) \\ & \searrow f & \swarrow \pi \\ & & (\mathbb{C}^k, 0) \end{array}$$

where π is smooth and surjective, f is flat, $X_t := p^{-1}(t)$ is a generically plane curve singularity for each $t \in \mathbb{C}^k$ closed to 0. We denote this family

shortly by (X, f, π) . As we have seen in the previous sections, this family is a *deformation* of the generically reduced plane curve singularity $(X_0, 0)$, which is the *special fiber* of the deformation. Throughout this section we restrict our attention to the families whose restriction of f on its singular locus⁴ $\text{Sing}(f)$ is finite.

Assume that the germ $(X, 0)$ is defined by the ideal

$$I := I(X, 0) \subseteq \mathbb{C}\{x, y, u_1, \dots, u_k\},$$

where x, y (resp. u_1, \dots, u_k) are local coordinates in \mathbb{C}^2 (resp. \mathbb{C}^k). Then a k -parametric family of plane curve singularities induces a k -parametric flat family of plane ideals (I, π) (compare to [No, Definition 3.3]).

We associate to each ideal $J \subseteq \mathbb{C}\{x, y\}$ a *weighted directed tree* $\tau(J)$ as defined in [No, Section 1.2]. The induced family of plane ideals (I, π) mentioned above is called *equisingular* if $\tau(I(t)) \approx \tau(I(0))$ for all $t \in \mathbb{C}^k$ closed to 0, where $I(t) := I\mathcal{O}_{X_t, 0} \subseteq \mathbb{C}\{x, y\}$.

Definition 5.2 A k -parametric family of generically reduced plane curve singularities (X, f, π) is said to be *I-equisingular* if the induced family of plane ideals $(I := I(X), \pi)$ is equisingular in the sense of weighted directed tree.

We also associate to each generically reduced plane curve $C \subseteq \mathbb{C}^2$ a *bi-weighted directed tree* $T_2(C, \gamma)$ as defined in [No, Section 2.4], where γ is an ordering of the branches of the reduction C^{red} of C .

Definition 5.3 A k -parametric family of generically reduced plane curve singularities is said to be *T-equisingular* if for any pair of points t, t' in the same connected component of \mathbb{C}^k , closed to 0, we can choose suitable orderings γ, γ' on the reduction X_t^{red} and $X_{t'}^{red}$ of the fibers $X_t, X_{t'}$ respectively such that the corresponding bi-weighted directed trees $T_2(X_t, \gamma_t)$ and $T_2(X_{t'}, \gamma_{t'})$ are isomorphic.

Definition 5.4 A k -parametric family of generically reduced plane curve singularities is said to be *C-equisingular* if it is I-equisingular and if we

⁴The *singular locus* of a flat map $f : X \rightarrow S$ is the set of all points $x \in X$ such that the fiber $X_{f(x)} := f^{-1}(f(x))$ is singular. If S is regular and f is flat then $\text{Sing}(X) \subseteq \text{Sing}(f)$ ([GLS, Theorem I.1.117]).

denote by $\pi_i : Z_i \rightarrow Z_{i-1}$ ($i \geq 1$) the blowing up of Z_{i-1} with the center $\text{Sing}(X_i)$, where X_i denotes the proper transform of X under $\pi^{(i)} := \pi \circ \pi_1 \circ \dots \circ \pi_i : Z_i \rightarrow Z_0 = Z$, then the induced morphism $(X_i, p_i) \rightarrow (\mathbb{C}^k, 0)$ is flat, $p_i \in (\pi^{(i)})^{-1}(0)$.

The following theorem gives an equivalence of three kinds of equisingularity mentioned above for a k -parametric family of plane curve singularities. A similar result for one-parametric families is given by Nobile ([No, Theorem 5.5 and Prop. 5.8]).

Theorem 5.1 *Let (X, f, π) be a k -parametric family of generically reduced plane curve singularities. Then*

- (i) *I-equisingularity is equivalent to T-equisingularity.*
- (ii) *C-equisingularity implies I-equisingularity. Conversely, if the given family is I-equisingular and all the fibers $X_t, t \in \mathbb{C}^k$ close to 0, are smooth, then the family is C-equisingular.*

Proof. It suffices to prove the theorem for a sufficiently small representative

$$\begin{array}{ccc}
 X & \xrightarrow{i} & Z \\
 & \searrow f & \swarrow \pi \\
 & & S,
 \end{array}$$

of the given family, where X, Z and S are sufficiently small neighborhoods of 0.

For each $s \in S$, there exists an irreducible reduced curve singularity $C \subseteq S$ passing through s . Let $\alpha : T \rightarrow S$ be the normalization of this reduced curve singularity, where $T \subseteq \mathbb{C}$ is a small disc with center at $0 \in \mathbb{C}$. Denote by $X_T := X \times_S T$, the Castesian product of X and T over S , and by $f_T : X_T \rightarrow T$ the induced morphism of f . By the preservation of flatness under base change (cf. [GLS, Prop. I.1.87]), f_T is flat. Moreover, for each $t \in T, s = \alpha(t)$, we have

$$\mathcal{O}_{(X_T)_t} = \mathcal{O}_{X_T} \otimes_{\mathcal{O}_{T,t}} \mathbb{C} = (\mathcal{O}_X \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{T,t}) \otimes_{\mathcal{O}_{T,t}} \mathbb{C} \cong \mathcal{O}_X \otimes_{\mathcal{O}_{S,s}} \mathbb{C} \cong \mathcal{O}_{X_s}.$$

Hence the fiber of f over each $s \in S$ is isomorphic to the fiber of f_T over $t \in T, \alpha(t) = s$.

- (i) The family (X, f, π) is I-equisingular if and only if $\tau(I(s)) \approx \tau(I(0))$

for all $s \in S$. Hence the trees $\tau(I_T(t))$ and $\tau(I_T(0))$ are isomorphic for all $t \in T$, where $I_T := I_T(X_T) \subseteq \mathbb{C}\{x, y, t\}$ denotes the ideal defining X_T . Equivalently, the induced family (X_T, f_T, π_T) is I-equisingular (by a theorem of Risler, cf. [No, Theorem 3.7]). Thus the family (X_T, f_T, π_T) is T-equisingular. It means that the tree $T_2((X_T)_t, \gamma_t)$ is isomorphic to the tree $T_2((X_T)_{t'}, \gamma_{t'})$ for each pair t, t' in the same connected component of T . Hence, for each pair s, s' in the same connected component of S , $\alpha(t) = s$, $\alpha(t') = s'$, the trees $T_2(X_s, \gamma_s)$ and $T_2(X_{s'}, \gamma_{s'})$ are isomorphic. This is equivalent to the T-equisingularity of the family (X, f, π) .

(ii) It is clear that C-equisingularity implies I-equisingularity. Now we assume that the family is I-equisingular and all fibers $X_s, s \in S$, are smooth. Since $f : X \rightarrow S$ is flat, S is smooth, it follows that X is smooth (cf. [GLS, Theorem I.1.117]). Therefore, for each $i \in \mathbb{N}^*$, the proper transform X_i of X under $\pi^{(i)} := \pi \circ \pi_1 \circ \dots \circ \pi_i : Z_i \rightarrow Z_0 = Z$ is smooth, of pure $(k + 1)$ -dimensional. Moreover, each fiber $(X_i)_s := (\pi^{(i)}|_{X_i})^{-1}(s)$ is a proper transform of the smooth fiber X_s , hence $(X_i)_s$ is smooth of pure 1-dimensional. Thus we have the *dimension formula*

$$\dim(X_i, x) = \dim((X_i)_s, x) + \dim(S, s), \quad \pi^{(i)}(x) = s.$$

It follows that $\pi^{(i)}|_{X_i}$ is open (cf. [Fi, Section 3.10, Theorem, p.145]). Moreover, X_i is smooth, hence Cohen-Macaulay. It follows that $\pi^{(i)}|_{X_i}$ is flat ([Fi, Section 3.20, Proposition, p.158]). Hence the given family is C-equisingular. □

In the following we may induce an I-equisingular (hence T-equisingular) family of reduced plane curve singularities from a given I-equisingular family of generically reduced plane curve singularities.

Theorem 5.2 *Let (X, f, π) be a k -parametric family of generically reduced plane curve singularities which is I-equisingular. Then all the fibers of the restriction $f^u : (X^u, 0) \rightarrow (\mathbb{C}^k, 0)$ are reduced and the induced family*

$$\begin{array}{ccc} (X^u, 0) & \xrightarrow{i} & (\mathbb{C}^{k+2}, 0) \\ & \searrow f^u & \swarrow \pi \\ & (\mathbb{C}^k, 0) & \end{array}$$

is I-equisingular.

Proof. Let $f : X \rightarrow S$ be a sufficiently small representative of the germ $f : (X, 0) \rightarrow (\mathbb{C}^k, 0)$. Suppose f is a deformation of the generically reduced plane curve singularity $X_0 \subseteq \mathbb{C}^2$ which is defined by the ideal I_0 . As a consequence of the Hilbert-Burch theorem, $I_0 = \langle g \rangle \cdot J_0$, where $\langle g \rangle$ defines the unmixed part X_0^u of X_0 and J_0 defines the embedded non-reduced point $0 \in X_0$. It follows from Lemma 4.1 and the proof of Theorem 4.1 that the restriction $f^u : X^u \rightarrow S$ is a deformation of the reduction X_0^u , which is reduced and pure 1-dimensional. Hence the special fiber and the nearby fibers of f^u are all reduced.

Now we consider the induced family

$$\begin{array}{ccc} (X^u, 0) & \xrightarrow{i} & (\mathbb{C}^{k+2}, 0) \\ & \searrow f^u & \swarrow \pi \\ & & (\mathbb{C}^k, 0). \end{array}$$

Let $f^u : X^u \rightarrow S$ be a sufficiently small representative of the restriction map-germ f^u in the family. By the same notation as in the proof of Theorem 5.1 we have the following diagram

$$\begin{array}{ccc} (X^u)_T & \rightarrow & X^u \\ (f^u)_T \downarrow & & \downarrow f^u \\ T & \xrightarrow{\alpha} & S, \end{array}$$

where the induced map $f^u_T : (X^u)_T \rightarrow T$ is flat by the preservation of flatness under base change. We have already showed in the proof of Theorem 4.1 that $(X^u)_T = (X_T)^u$. Since the induced family (X_T, f_T, π_T) is I-equisingular, it follows from [No, Theorem 5.11] that the family $((X_T)^u = (X^u)_T, f_T, \pi_T)$ is I-equisingular with reduced fibers. On the other hand, the fibers of $f^u_T : (X^u)_T \rightarrow T$ and $f^u : X^u \rightarrow S$ are isomorphic. Therefore the family (X^u, f^u, π) is I-equisingular. \square

As a consequence we have the following equivalence of I-equisingularity and the delta-constancy of a family of generically reduced plane curve singularities. A family (X, f, π) is said to be δ -constant if the morphism $f : (X, 0) \rightarrow (\mathbb{C}^k, 0)$ is δ -constant.

Theorem 5.3 *Let (X, f, π) be a k -parametric family of generically reduced*

plane curve singularities. Then it is δ -constant if and only if the family (X^u, f^u, π) is I-equisingular.

Proof. The I-equisingularity of the family (X^u, f^u, π) implies that the induced family $((X^u)_T, f_T^u, \pi_T)$ is also I-equisingular with reduced fibers. It follows that the deformation $f_T^u : (X^u)_T \rightarrow T$ with reduced fibers is equinormalizable (cf. [Tei2, Theorem 5.3.1]). By a result of Teissier (cf. [Tei1, Corollary 1, p. 609]),

$$\delta(((X^u)_T)_t) = \delta(((X^u)_T)_0), \quad \forall t \in T \text{ close to } 0.$$

Since the fibers of $f^u : X^u \rightarrow S$ and $f_T^u : (X^u)_T \rightarrow T$ are isomorphic, it implies that the deformation $f^u : X^u \rightarrow S$ is δ -constant. Hence it is equinormalizable (cf. [Ch-Li, Theorem 5.6]). It follows again from Theorem 4.1 that $f : X \rightarrow S$ is δ -constant. The converse of the argument given above is also satisfied. This proves the theorem. \square

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