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Asymptotic behavior of solutions to a crystalline flow

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Abstract. The paper extends an author's result that, under certain technical assumptions, presents the self-similar solutions to a crystalline flow defined independently by M. Gurtin and J.E. Taylor as attractors for any other solutions. The existing result concerns the crystalline flow defined on the space of closed, convex polygons with respect to a reference convex body. On the same space of polygons, the present work shows that even for an arbitrary weight function γ defined on a certain set of normal directions, the self-similar solutions are attractors in the following sense: Let the system of equations defining the flow be

$$\frac{dh_i(t)}{dt}=\frac{\gamma_i}{l_i(t)}, \quad 1\leq i\leq n, \tag{$*$}$$

where $h_i(t)=h(\theta_i, t)$ is the distance from the origin to the i-th side of the evolving convex polygon, while $l_i(t)=l(\theta_i, t)$ is the length of the i-th side at the moment t, and $\gamma_i=\gamma(\theta_i)$ is a strictly positive function on the set of normal directions to the sides of the polygon.

Our main result says that if the family of convex polygonal curves which evolve by $(*)$ is normalized to enclose constant area, then for any sequence of times diverging to infinity, there is a convergent subsequence of polygons which converges to the shape of a self-similar solution. Moreover, for a π-periodic weight function, there is a unique self-similar solution of the flow which is a global attractor for the family of evolving polygons.

Key words: crystalline flows, self-similar solutions, anisotropic energy density.

1. Introduction

Motion by crystalline energy or crystalline curvature is viewed as a typical example of geometric evolution by nonsmooth interfacial energy. Analyzing the behavior of planar crystalline interfaces endowed with energy densities defined on a finite set of normal directions to the curve is a problem of interest in material sciences. The finite set of normal directions is the set of orientations that appear on the Wulff shape, which is the crystal of least total boundary energy at fixed enclosed area. Motion of planar curves by crystalline curvature is often considered for piecewise linear curves with the same ordered set of orientations as the Wulff shape. Such curves are called

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admissible. The evolution of admissible curves can be then described by a simple system of ordinary differential equations.

Angenent and Gurtin [AG] derived a model for the crystal evolution based on the balance of forces and the second law of thermodynamics. The planar case is governed by the following differential equations for the normal velocitites, V_i, of the i-th side of an admissible curve of normal directions $\{n_i\}_{1\leq i\leq n}$:

$$V_i(t) = \frac{1}{\beta(n_i)} \cdot \left(\tilde{l}_i(t)K_i(t) - U\right), \tag{1}$$

where $\tilde{l}_i$ denotes the length of the i-th side of the Wulff shape, $K_i = \frac{\chi_i}{l_i}$ is the crystalline curvature, and χ_i has the constant value -1, $+1$ or 0 depending whether the crystal is strictly convex, strictly concave, or neither near the i-th side, $\beta(n_i) > 0$ is the kinetic modulus, which depends only on the set of admissible directions, and U is the constant bulk energy.

Independently, Taylor ([T]) proposed an evolution for the planar crystalline motion under the assumptions: $U = 0$, $\beta(n_i) = \frac{1}{\tilde{h}_i}$ where $\tilde{h}_i$ is the distance from the origin to the support line of the i-th side of the Wulff shape.

The goal of this paper is to settle the asymptotic behavior of closed, convex, polygonal solutions of a version of the anisotropic flow proposed by Angenent and Gurtin for interfacial motion, with no driving term U:

$$V_i = -\gamma_i K_i, \qquad 1 \leq i \leq n, \tag{2}$$

where $\gamma_i = \gamma(n_i)$ is a strictly positive function on the set of normal directions to the sides of the polygon.

Our main result is:

Theorem 1.1 *If the family of closed, convex polygonal curves which evolve by* (2) *is normalized to enclose constant area, then for any sequence of times diverging to infinity, there is a convergent subsequence of polygons which converges in the Hausdorff metric to the shape of a self-similar solution.*

The theorem 1.1 implies also a couple of results concerning self-similar solutions, i.e. the solutions for which the family of evolving polygons are self-similar with the initial polygonal curve.

Corollary 1.1 (The existence of self-similar solutions)

Given any positive function γ defined on the set of admissible directions, there exists a self-similar solution to (2).

Corollary 1.2 (The decomposition of the weight function)

Every positive function γ defined on the set of admissible directions can be written as

$$\gamma_i = \tilde{h}_i \tilde{l}_i \qquad 1 \leq i \leq n,$$

where $\tilde{h}_i$ is the distance from the origin to the i-th side of a convex polygon and $\tilde{l}_i$ is the length of the i-th side of normal direction n_i.

These results are known for smooth energies ([DGM], [G1], [GL]). A direct proof for the decomposition of the weight function and the existence of the self-similar solutions which requires in fact only boundedness of the weight function is provided by Dohmen, Giga and Mizoguchi ([DGM]).

In many respects the crystalline evolution has a behavior typical of the parabolic partial differential equation satisfied by the curvature flows in the case of smooth planar curves. This is supported by a recent result by Giga and Gurtin ([GG]). They have established a comparison theorem for the planar crystalline evolution which sets the basis for extending the usual comparison principle from the smooth case to weak formulations of evolutions problems.

The system (2) represents a discrete version of the weighted curvature equation for convex, closed, smooth curves. In fact, Girão and Kohn ([GK], [G3]) use crystalline evolution as a method of approximating generalized curve-shortening equations.

The weighted curvature equation has been also studied in the context of geometric evolutions of curves represented by graphs including the case when the interface energy is not necessarily smooth ([FG]). In that particular paper the evolution law with $U \equiv 0$ for the crystalline interface energy is justified.

More recently, M.-H. Giga and Y. Giga extended the theory of generalized solutions for crystalline energy even if U is non-identically zero, ([GMHG]). The comparison principle by Giga and Gurtin ([GG]), which we have mentioned earlier, is fundamental in describing the large time behavior when the driving term comes in.

We would like to recall now an earlier uniqueness result of the author:

Theorem 1.2 ([S]) *Let $\tilde{K}$ be a convex polygonal body, symmetric with respect to the origin, whose distances from the origin to the sides are denoted by $\tilde{h}_i$ and the lengths of the sides are $\tilde{l}_i$. Then the flow defined by*

$$V_i = -\tilde{h}_i \tilde{l}_i K_i, \qquad 1 \leq i \leq n, \quad n > 4 \tag{3}$$

has a unique self-similar solution which is a homothety of $\tilde{K}$ and this is an attractor for any other solution to the flow.

In conjunction to the theorem 1.2, the corollary 1.2 leads to the following conclusion:

Corollary 1.3 *If γ is positive and π-periodic on the set of admissible directions of cardinal strictly greater than four, there is a unique self-similar solution to* (2) *and any evolving family of polygonal curves shrinks to a self-similar shaped point.*

The proof of our main result is modeled on the proof that M. Gage and Y. Li ([GL]) used for the existence of self-similar solutions to the anisotropic curve shortening equation.

I would like to thank Mike Gage for pointing out the possibility to solve this problem using similar techniques. In addition, I am grateful to the referee for his comments and suggestions.

2. Estimates

Let $K(t)$ be a family of convex, polygonal bodies whose boundaries are a solution to the crystalline flow (2) re-written:

$$\frac{dh_i(t)}{dt} = -\frac{\gamma_i}{l_i(t)}, \qquad 1 \leq i \leq n, \tag{2'}$$

where $h_i(t) = h(\theta_i, t)$ is the distance from the origin to the i-th side of the evolving convex polygon at the moment t, while $l_i(t) = l(\theta_i, t)$ is the length of the i-th side at time t. We identify the normal direction to the i-th side, $n_i = (\cos\theta_i, \sin\theta_i)$, by θ_i and the set of all such normal directions will be refered to as the set of admissible directions.

It is known that a convex polygon evolving by (2) stays convex until its area becomes zero in finite time ω, ([G3]). Moreover, it is not hard to see that at time ω all the sides of the evolving polygon become of zero length, ([T]). Thus also $\text{Area}(K(\omega)) =: A(\omega) = 0$.

In fact, during the evolution the area enclosed by $K(t)$ is decreasing at a constant rate:

Lemma 2.1

$$\frac{dA}{dt} = -\sum_{i=1}^{n} \gamma_i, \tag{4}$$

where $A(t) = \text{Area}(K(t))$.

Proof. From the geometry of the polygonal body K it follows that:

$$\begin{aligned} l_i(t) &= h_{i+1}(t) \csc \Delta\theta_{i+1} + h_{i-1}(t) \csc \Delta\theta_i \\ &\quad - h_i(t)(\cot \Delta\theta_{i+1} + \cot \Delta\theta_i), \end{aligned}$$

where $\Delta\theta_i$ is the angle of the i-th vertex of ∂K, made by the $i-1$-th and i-th side. More precisely, $\Delta\theta_i = \pi - (\theta_i - \theta_{i-1})$.

The lemma follows now from the evolution equations of the h_i's, reindexing two sums (The indices $n+1 = 1 \pmod n$ and $n = 0 \pmod n$.) $\square$

Remark 2.1. The area of $K(t)$ is given by the following expression:

$$A(t) = (\omega - t) \sum_{i=1}^{n} \gamma_i.$$

This will suggest to consider the normalized flow in which the figure is expanded to keep the enclosed area constant.

Another reason to do so is the lack of control over the blow up rate of $\frac{1}{l_i}$. The most we can say about the unnormalized quantity $\frac{1}{l_i}$ lies in the following two estimates:

Lemma 2.2 *There exist two strictly positive constants m and M depending on the initial conditions only such that*

$$\begin{aligned} &\left[\left(\min_i \left(\frac{\gamma_i}{l_i(0)}\right)\right)^{-2} - m \cdot \frac{2t}{\max_i \gamma_i}\right]^{-1/2} \\ &\qquad \le \min_i \frac{\gamma_i}{l_i} \le \left(\frac{m}{\max_i \gamma_i}\right)^{-1/2} \cdot (2\omega - 2t)^{-1/2} \end{aligned}$$

and

$$\left(\frac{M}{\min_i \gamma_i}\right)^{-1/2} \cdot (2\omega - 2t)^{-1/2} \leq \max_i \frac{\gamma_i}{l_i}$$
$$\leq \left[\left(\max_i \left(\frac{\gamma_i}{l_i(0)}\right)\right)^{-2} - M \cdot \frac{2t}{\min_i \gamma_i}\right]^{-1/2}$$

Remark 2.2. The first inequality will imply the existence of a positive constant depending on the initial conditions only which bounds from below each $\frac{1}{l_i}$, $1 \leq i \leq n$ during the entire evolution.

Proof of the Lemma 2.2 Consider the evolution equation for $\frac{\gamma_i}{l_i}$:

$$\left(\frac{\gamma_i}{l_i}\right)_t = -\frac{\gamma_i}{l_i{}^2} \cdot (l_i)_t = -\frac{\gamma_i}{l_i^2}\left(\frac{\gamma_i}{l_i}A_i - \frac{\gamma_{i+1}}{l_{i+1}}B_i - \frac{\gamma_{i-1}}{l_{i-1}}C_i\right)$$
$$= -\frac{1}{\gamma_i}\left[\left(\frac{\gamma_i}{l_i}\right)^3 A_i - \left(\frac{\gamma_i}{l_i}\right)^2 \frac{\gamma_{i+1}}{l_{i+1}}B_i - \left(\frac{\gamma_i}{l_i}\right)^2 \frac{\gamma_{i-1}}{l_{i-1}}C_i\right],$$

where, for simplicity, we made the following notations:

$$A_i := \cot \Delta\theta_{i+1} + \cot \Delta\theta_i$$

$$B_i := \csc \Delta\theta_{i+1}, \qquad C_i := \csc \Delta\theta_i = B_{i-1}$$

Then if $y = \frac{\gamma_i}{l_i} = \min_j \frac{\gamma_j}{l_j}$ we have $\frac{\gamma_j}{l_j} \geq y$ for all j's which implies together with $B_i, C_i > 0$ that:

$$y_t \geq \frac{y^3}{\gamma_i}(B_i + C_i - A_i),$$

and furthermore, labeling by m the following constant, depending on the set of admissible directions, defined a priori by the initial conditions:

$$m := \min_i (B_i + C_i - A_i),$$

we have

$$y_t \geq \frac{y^3}{\max_i \gamma_i} \cdot m.$$

The minimum of $\frac{\gamma_i}{l_i}$, as well as its maximum, is a continuous function, but it may not be differentiable. It is however Lipschitz. Therefore the notation y_t refers to $y_t = \liminf_{\epsilon \nearrow 0} \frac{y(t+\epsilon)-y(t)}{\epsilon}$.

From a comparison principle we can see that once y is bigger than a solution of $z' = \frac{z^3}{\max_i \gamma_i} \cdot m$, it must stay bigger. So if y is bigger than the solution $\left(m \cdot \frac{2\omega}{\max_i \gamma_i} - \delta - m \cdot \frac{2t}{\max_i \gamma_i}\right)^{-1/2}$ for any t and any positive δ, then we obtain a contradiction, since in this case the blow-up of $\min_i \frac{\gamma_i}{l_i}$ must occur at an earlier time than ω.

We conclude that $\min_i \frac{\gamma_i}{l_i} \leq \left(m \cdot \frac{2\omega}{\max_i \gamma_i} - \delta - m \cdot \frac{2t}{\max_i \gamma_i}\right)^{-1/2}$ for any t and any positive δ and this proves half of the first sequence of inequalities.

The left inequality of the sequence follows immediately from the comparison principle by comparing y with the solution of the equation $z' = \frac{z^3}{\max_i \gamma_i} \cdot m$ having the same initial condition, $z(0) = y(0) = \min_i \frac{\gamma_i}{l_i(0)}$.

The second sequence of inequalities follows in a similar way. $\square$

Remark 2.3. Note that on the other hand, if $y = \min_i \frac{\gamma_i}{l_i}$ is strictly less than the solution $\left(m \cdot \frac{2\omega}{\max_i \gamma_i} + \delta - m \cdot \frac{2t}{\max_i \gamma_i}\right)^{-1/2}$ for all t and any positive δ, then we obtain a contradiction, since in this case $\min_i \frac{\gamma_i}{l_i}$ remains finite at $t = \omega$ and the blow up must occur at a later time. But this only proves that for each δ there exists a time $t(\delta)$ such that $y \geq \left(m \cdot \frac{2\omega}{\max_i \gamma_i} + \delta - m \cdot \frac{2t}{\max_i \gamma_i}\right)^{-1/2}$ for all $t \geq t(\delta)$. Unfortunately, $t(\delta)$ increases as δ goes to zero. In conclusion, we cannot determine the exact blow up rate of the crystalline curvature.

Before we proceed to normalize the flow we consider the entropy functional whose study is essential for our work:

Definition 2.1 Let the entropy be:

$$\mathcal{E}(t) = \sum_{i=1}^{n} \gamma_i \log\left(\frac{\gamma_i}{l_i}\right).$$

Proposition 2.1

$$\frac{d\mathcal{E}}{dt} = \sum_{i=1}^{n} \gamma_i \left(\log\left(\frac{\gamma_i}{l_i}\right)\right)_t \leq \frac{\sum_{i=1}^{n} \gamma_i}{2} \cdot \frac{1}{\omega - t} \tag{5}$$

where ω is the time at which one, hence all, of the l_i's becomes zero.

Proof. Let

$$u_i = \left(\log\left(\frac{\gamma_i}{l_i}\right)\right)_t = -\frac{(l_i)_t}{l_i}$$

$$= -\frac{1}{l_i}\left(\frac{\gamma_i}{l_i}\cdot A_i - \frac{\gamma_{i+1}}{l_{i+1}}\cdot B_i - \frac{\gamma_{i-1}}{l_{i-1}}\cdot C_i\right),$$

where A_i, B_i, C_i are defined as before in the proof of the lemma 2.2.

We have then:

$$\begin{aligned}(u_i)_t &= \left(\frac{2\gamma_i}{l_i^3}A_i - \frac{\gamma_{i+1}}{l_{i+1}l_i^2}B_i - \frac{\gamma_{i-1}}{l_{i-1}l_i^2}C_i\right)(l_i)_t \\ &\quad - \frac{\gamma_{i+1}}{l_{i+1}l_i^2}B_i(l_{i+1})_t - \frac{\gamma_{i-1}}{l_{i-1}l_i^2}C_i(l_{i-1})_t\end{aligned}$$

and since $(l_j)_t = -u_j l_j$ for all j's it follows that:

$$(u_i)_t = 2u_i^2 + \frac{\gamma_{i+1}}{l_{i+1}l_i}B_i(u_{i+1} - u_i) + \frac{\gamma_{i-1}}{l_{i-1}l_i}C_i(u_{i-1} - u_i)$$

for all $i = 1, \dots, n$.

Thus:

$$\begin{aligned}\mathcal{E}_{tt} &= \left(\sum_{i=1}^{n}\gamma_i u_i\right)_t \\ &= \sum_{i=1}^{n} 2\gamma_i u_i^2 + \sum_{i=1}^{n}\frac{\gamma_{i+1}}{l_{i+1}}\frac{\gamma_i}{l_i}B_i(u_{i+1} - u_i) + \frac{\gamma_{i-1}}{l_{i-1}}\frac{\gamma_i}{l_i}C_i(u_{i-1} - u_i).\end{aligned}$$

And since $C_i = B_{i-1}$ by reindexing the last sum we have

$$\mathcal{E}_{tt} = \sum_{i=1}^{n} 2\gamma_i u_i^2 \geq 2\frac{\left(\sum_{i=1}^{n}\gamma_i u_i\right)^2}{\sum_{i=1}^{n}\gamma_i} = 2\frac{(\mathcal{E}_t)^2}{\sum_{i=1}^{n}\gamma_i}.$$

We used Schwartz inequality. Setting $y = \sum_{i=1}^{n}\gamma_i u_i$ yields

$$y_t \geq \frac{2}{\sum_{i=1}^{n}\gamma_i}y^2.$$

The equation (5) follows now from a comparison principle and the fact that the entropy must be infinite at time $t = \omega$. The latter is given by the definition of ω as the time when the area enclosed by the polygonal curve is zero, which coincides with the time when all the sides of the polygon become of zero length ([T]). $\square$

Proposition 2.2 *Magnify* $\mathbf{R}^2$ *by* $\mu = (2\omega - 2t)^{-1/2}$ *and make the change of variable* $\tau = -\frac{1}{2}\log(2\omega - 2t)$. *Then the evolution equations are trans-*

formed into the rescaled equations:

$$(\bar{h}_i)_\tau = (\mu h_i)_\tau = -\frac{\gamma_i}{\bar{l}_i} + \bar{h}_i \tag{6}$$

$$(\bar{l}_i)_\tau = \bar{l}_i + \frac{\gamma_i}{\bar{l}_i} A_i - \frac{\gamma_{i+1}}{\bar{l}_{i+1}} B_i - \frac{\gamma_{i-1}}{\bar{l}_{i-1}} C_i \tag{7}$$

$$\bar{L}_\tau = \bar{L} + \sum_{i=1}^{n} \left(\frac{\gamma_i}{\bar{l}_i} A_i - \frac{\gamma_{i+1}}{\bar{l}_{i+1}} B_i - \frac{\gamma_{i-1}}{\bar{l}_{i-1}} C_i \right) \tag{8}$$

$$\bar{A} = \frac{1}{2} \sum_{i=1}^{n} \gamma_i = \text{constant}. \tag{9}$$

All the equations follow by direct calculation.

We will refer to the equations (6)–(9) as the normalized equations, as the family of evolution curves encloses constant area for all time.

We have then a normalized entropy estimate which follows by direct calculation as well as from its definition and the equation (5):

Corollary 2.1

$$\bar{\mathcal{E}}_\tau(\tau) = \left(\sum_{i=1}^{n} \gamma_i \log \frac{\gamma_i}{\bar{l}_i} \right)_\tau \leq 0 \tag{10}$$

and

$$\bar{\mathcal{E}}(\tau) = \sum_{i=1}^{n} \gamma_i \log \frac{\gamma_i}{\bar{l}_i} \leq \bar{C}_{\mathcal{E}} \tag{11}$$

where $\bar{C}_{\mathcal{E}}$ depends only on the initial curve.

Following Gage and Li [GL], we now obtain a lower bound on the width of the normalized curve in terms of the entropy bound. This implies a lower bound on the inradius of the normalized curve which leads to upper bounds on its diameter and its length.

Lemma 2.3 *Let $w(\theta_0)$ be the width of the curve in the unitary direction* $(\cos(\theta_0), \sin(\theta_0))$.

Then there exists a constant, $C_{\mathcal{E},\gamma}$, depending on the initial conditions *only such that*

$$\log w(\theta_0) \geq \frac{1}{n}\left(-4c^\gamma \pi \log 2 - C_{\mathcal{E},\gamma}\right)$$

where $c^\gamma := \min_i |\theta_{i+1} - \theta_i|$ *is the minimum of the differences between two consecutive admissible directions, independent of the flow's evolution.*

Proof. This can be deduced from the bound on the entropy using Hamilton's idea: The width of a curve in a unit direction is given by taking the mixed volume with an interval in that direction.

$$2w(\theta_0) = \sum_{i=1}^{n} |\sin(\theta_i - \theta_0)| \cdot \bar{l}_i.$$

We consider first only the directions θ_0 for which $\sin(\theta_i - \theta_0) \neq 0$, for all $i = 1, \ldots, n$.

Therefore by taking the logarithm and using the inequality between the arithmetic and geometric means of positive numbers we have:

$$\begin{aligned}
\log 2w(\theta_0) &= \log\left(\sum_{i=1}^{n} |\sin(\theta_i - \theta_0)| \cdot \bar{l}_i\right) \\
&\geq \frac{1}{n}\sum_{i=1}^{n} \log\left(|\sin(\theta_i - \theta_0)| \cdot \bar{l}_i\right) \\
&= \frac{1}{n}\sum_{i=1}^{n} \left(\log|\sin(\theta_i - \theta_0)| + \log \bar{l}_i\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n} \log|\sin(\theta_i - \theta_0)| - \sum_{i=1}^{n} \log\left(\frac{1}{\bar{l}_i}\right)\right)
\end{aligned}$$

The complex-valued function $f(z) = \log|1 - z| = \log|1 - r\exp i\theta|$ is harmonic on $\mathbf{B}(0, r)$ for any $r \leq 1$. By using a mean value formula for $r = 1$, we see that:

$$\int_0^{2\pi} \log|\sin(\theta)| d\theta = -2\pi \log 2.$$

We use $\{\theta_i\}_{i=1}^{n}$ as a partition of the interval $[0, \pi/2] \cup [\pi/2, \pi] \cup [\pi, 3\pi/2] \cup [3\pi/2, 2\pi]$ for a Riemann sum bounded below by the value of the integral and obtain:

$$\begin{aligned}
-2\pi\log 2 \le\ & \frac{1}{2}\sum_{i=1,0<\theta_i-\theta_0\le\pi/2}^{n} \log(|\sin(\theta_i-\theta_0)|)\cdot|\theta_i-\theta_{i-1}| \\
&+\frac{1}{2}\sum_{i=1,\pi/2<\theta_i-\theta_0<\pi}^{n} \log(|\sin(\theta_i-\theta_0)|)\cdot|\theta_{i+1}-\theta_i| \\
&+\frac{1}{2}\sum_{i=1,\pi<\theta_i-\theta_0\le 3\pi/2}^{n} \log(|\sin(\theta_i-\theta_0)|)\cdot|\theta_i-\theta_{i-1}| \\
&+\frac{1}{2}\sum_{i=1,3\pi/2<\theta_i-\theta_0<2\pi}^{n} \log(|\sin(\theta_i-\theta_0)|)\cdot|\theta_{i+1}-\theta_i| \\
\le\ & \frac{1}{2}\sum_{i=1,\theta_i-\theta_0\ne 0,\pi,2\pi.}^{n} \log(|\sin(\theta_i-\theta_0)|)\cdot\min_{i=1,\ldots,n}|\theta_{i+1}-\theta_i| \\
=\ & \frac{1}{2}\sum_{i=1,\theta_i-\theta_0\ne 0,\pi,2\pi.}^{n} \log(|\sin(\theta_i-\theta_0)|)\cdot c^{\gamma}.
\end{aligned}$$

The lower bound on the negative of the second sum follows from the upper bound on the entropy which concludes the result for the selected unitary directions.

We have proved then that the width stays bounded from below for all, but at most 2n+1 unitary directions $\{\theta_1,\ldots,\theta_n,\theta_1+\pi,\ldots,\theta_n+\pi,\theta_1+2\pi\} =: \mathcal{N}$. However, for any $\theta_0\in\mathcal{N}$ there exists a natural number N_{θ_0} such that for all $m\ge N_{\theta_0}$ the sequence with the general term $\theta_m := \theta_0+\frac{1}{m}$ lies in the complement of $\mathcal{N}$ in the interval $[0,2\pi]$ and θ_m converges to θ_0 as m goes to infinity. Then since the width is a continuous function on $[0,2\pi]$ it follows that for any direction θ_0 in $\mathcal{N}$ we have

$$w(\theta_0)=\lim_{m\to\infty} w(\theta_m)\ge\frac{1}{n}\left(-4c^{\gamma}\pi\log 2 - C_{\mathcal{E},\gamma}\right),$$

which concludes the proof for all the unitary directions. □

The following estimates are purely geometrical and they do not depend on the evolution of the boundary of K:

Lemma 2.4 *The inradius of a convex curve (i.e. the radius of the largest* *circle inscribed inside the curve) satisfies*:

$$r_{in} \geq \frac{1}{3} \min_{\theta} w(\theta).$$

(A proof can be found in [GL] or in some classical books on convex geometry, [E] for example.)

And, since

$$\bar{A} \geq \sum_{i=1}^{n} \frac{\bar{l}_i r_{in}}{2} = \frac{r_{in}}{2} \cdot \bar{L} \geq \frac{r_{in}}{2} \cdot 2D \geq \frac{\min_\theta w(\theta)}{3} \cdot D$$

it follows

Corollary 2.2 *In addition, the diameter of the curve is bounded*

$$D \leq \frac{3\bar{A}}{\min_\theta w(\theta)}$$

and, since $\bar{L} \leq \pi D$, *so is the length of the normalized curve.*

3. Conclusions

Let

$$H(\tau, a, b) = \sum_{i=1}^{n} \gamma_i \log \bar{h}_i \tag{12}$$

where $\bar{h}_i = \bar{h}_i(\tau, a, b) = \bar{h}(\tau, \theta_i, a, b)$ is the support function relative to the point (a, b) of the i-th side of normal direction $(\cos\theta_i, \sin\theta_i)$ of a polygonal curve $p(\tau)$ which is evolving under the normalized equations.

Lemma 3.1 *The evolution of* H *is described by the following equation*:

$$H(\tau, a, b)_\tau = \sum_{i=1}^{n} \left(\gamma_i - \frac{{\gamma_i}^2}{\bar{l}_i \bar{h}_i} \right) \leq 0 \tag{13}$$

Proof. This follows by a direct calculation using the equations (6), (9) and the Schwartz inequality:

$$\sum_{i=1}^{n} \frac{{\gamma_i}^2}{\bar{l}_i \bar{h}_i} \cdot \sum_{i=1}^{n} \bar{h}_i \bar{l}_i \geq \left(\sum_{i=1}^{n} \gamma_i \right)^2.$$

□

It is necessary at this point to choose a center (a, b) such that the support functions $\bar{h}(\tau, \theta_i)$ stay positive for all time and therefore the value of $H(\tau, a, b)$ remains bounded for all time.

Let $\{h(\theta_i, 0, 0) \mid i = 1, \dots, n\}$ be the support functions of the flat sides a convex polygon with respect to the origin $(0, 0)$. Then the support function $h(\theta_i, a, b)$ is given by:

$$\begin{aligned} h(\theta_i, a, b) &= -\langle X - (a, b), N_i \rangle\rangle \\ &= h(\theta_i, 0, 0) - a\cos\theta_i - b\sin\theta_i, \end{aligned} \tag{14}$$

where $X = X(\theta_i)$ is the position vector of the convex curve in the direction $N_i = N(\theta_i) = (\cos\theta_i, \sin\theta_i)$ normal to the i-th side.

By a direct calculation one can check that

$$\begin{aligned} l_i(0,0) &= h_{i+1}(0,0)\csc\Delta\theta_{i+1} + h_i(0,0)\csc\Delta\theta_i \\ &\quad - h_i(0,0)(\cot\Delta\theta_{i+1} + \cot\Delta\theta_i) \\ &= h_{i+1}(a,b)\csc\Delta\theta_{i+1} + h_i(a,b)\csc\Delta\theta_i \\ &\quad - h_i(a,b)(\cot\Delta\theta_{i+1} + \cot\Delta\theta_i) = l_i(a,b) \end{aligned}$$

and if $\{h(\tau, \theta, 0, 0) \mid i = 1, \dots, n\}$ is a solution of the equation (2) in some time interval, so is $\{h(\tau, \theta, a, b) \mid i = 1, \dots, n\}$ in the same time interval.

Using (14) we see that the normalized support functions satisfy:

$$\begin{aligned} \bar{h}(t, \theta_i, a, b) &= \frac{h(t, \theta_i, a, b)}{(2\omega - 2t)^{-1/2}} \\ &= \frac{h(t, \theta_i, 0, 0)}{(2\omega - 2t)^{-1/2}} - \frac{a\cos\theta_i}{(2\omega - 2t)^{-1/2}} - \frac{b\sin\theta_i}{(2\omega - 2t)^{-1/2}} \end{aligned}$$

and by the definition of τ:

$$\bar{h}(t, \theta_i, a, b) = \bar{h}(t, \theta_i, 0, 0) - a\exp\tau\cdot\cos\theta_i - b\exp\tau\cdot\sin\theta_i \tag{15}$$

This means that a translation of the initial curve gives a solution whose support functions of the sides are equal to the original support functions shifted by distances which increase exponentially with τ.

Proposition 3.1 *If $\mathbf{r}_{in}$ denotes the minimum value of the inradius of the family of evolving normalized curves, then for any initial curve p_0 there is a choice of origin such that $H(\tau)$ is uniformly bounded below by $\log(\mathbf{r}_{in})\sum_{i=1}^{n}\gamma_i$ for all positive τ's.*

Also, the evolving normalized curves remain within a fixed ball.

Proof. It suffices to show that we can choose a point (a, b) within the initial curve $p(0)$ and implicitely an initial set of support functions $\{\bar{h}(0, \theta_i, a, b) \mid i = 1, \ldots, n\}$ such that $H(\tau)$ is bounded by the desired constant and then apply the continuous dependence of the solutions on initial conditions to show that $H(\tau)$ remains bounded by $\log(\mathbf{r}_{in}) \sum_{i=1}^{n} \gamma_i$.

Notice from the earlier estimates (Lemma 2.3 and Lemma 2.4) that the inradius of the normalized curve $p(\tau)$ is bounded below for all time by a positive constant $\mathbf{r}_{in}$.

Let τ_j be a sequence of times diverging to ∞ such that for each τ_j there is choice of $(\tilde{a}_j, \tilde{b}_j)$ which makes

$$\bar{h}_i(\tau_j, 0, 0) - \tilde{a}_j \cos\theta_i - \tilde{b}_j \sin\theta_i \geq \mathbf{r}_{in},$$

by simply taking $(\tilde{a}_j, \tilde{b}_j)$ the center of the inradius circle at the time τ_j.

Set now: $a_j = \tilde{a}_j \exp(-\tau_j)$ and $b_j = \tilde{b}_j \exp(-\tau_j)$, so $\bar{h}_i(\tau_j, a_j, b_j) \geq \mathbf{r}_{in}$.

Since $\bar{h}_i$ is strictly positive at a time τ_j it must be strictly positive for all earlier times, because if it would be negative in some direction θ_0 the equation (6) implies that it will stay negative in that direction. The fact that $\bar{h}_i(0, a_j, b_j)$ are all positive shows also that the point (a_j, b_j) lies inside the curve $p(0)$.

Since $H(\tau, a_j, b_j)$ is decreasing we also have:

$$H(\tau, a_j, b_j) \geq \log(\mathbf{r}_{in}) \sum_{i=1}^{n} \gamma_i, \quad \text{for all} \quad \tau \leq \tau_j.$$

The sequence (a_j, b_j) lies in a compact set, so it contains a subsequence, denoted the same for simplicity, which converges to (a_∞, b_∞).

Fixing τ we have that

$$\bar{h}_i(\tau, a_j, b_j) \longrightarrow \bar{h}_i(\tau, a_\infty, b_\infty) \quad \text{as} \quad (a_j, b_j) \to (a_\infty, b_\infty)$$

and

$$-\log \bar{h}_i(\tau, a_j, b_j) \longrightarrow -\log \bar{h}_i(\tau, a_\infty, b_\infty) \quad \text{as} \quad (a_j, b_j) \to (a_\infty, b_\infty)$$

Also, $\{-\log \bar{h}_i(\tau, a_j, b_j) \mid i = 1, \ldots, n\}$ is uniformly bounded below by the negative of the logarithm of the diameter.

It follows that:

$$\sum_{i=1}^{n} (-\gamma_i \log \bar{h}_i(\tau, a_\infty, b_\infty)) \leq \liminf_j \sum_{i=1}^{n} (-\gamma_i \log \bar{h}_i(\tau, a_j, b_j))$$

or

$$\begin{aligned}\sum_{i=1}^{n} \gamma_i \log \bar{h}_i(\tau, a_\infty, b_\infty) &\geq \limsup_{j} \sum_{i=1}^{n} (\gamma_i \log \bar{h}_i(\tau, a_j, b_j)) \\ &\geq \log(\mathbf{r}_{in}) \sum_{i=1}^{n} \gamma_i.\end{aligned}$$

□

We assume from now on that the choice of the origin is the one from the Proposition 3.1.

Corollary 3.1 *For this choice of the origin the $\bar{h}_i$'s are uniformly bounded independently of τ.*

Proof. The support function with respect to the point (a, b) as above is non-negative and bounded above by the diameter's bound. □

Proposition 3.2 *The normalized crystalline curvature is bounded for all time.*

Proof. First notice that if $\max_i \frac{1}{\bar{l}_i} < m$ at time τ_0, there is a δ depending only on m and the set of admissible directions such that $\max_i \frac{1}{\bar{l}_i} < 2m$ for all τ in $[\tau_0, \tau_0 + \delta]$. (This follows from the comparison of $\max_i \frac{1}{\bar{l}_i}$ with the solution of the equation $y_t = Cy^3$ for the appropriate constant C.)

Then the following series converges:

$$\sum_{j=0}^{\infty} \int_{j\delta}^{(j+1)\delta} \sum_{i=1}^{n} \left(\gamma_i - \frac{{\gamma_i}^2}{\bar{h}_i \bar{l}_i} \right) dt = \lim_{\tau \to \infty} H(\tau) - H(0) > -\infty.$$

Thus by the convergence criterion for series

$$\lim_{j \to \infty} \int_{j\delta}^{(j+1)\delta} \sum_{i=1}^{n} \left(\gamma_i - \frac{\gamma_i^2}{\bar{h}_i \bar{l}_i} \right) dt = 0.$$

And by the mean value theorem, there exists $\xi \in [j\delta, (j+1)\delta]$ such that

$$\frac{1}{\delta} \int_{j\delta}^{(j+1)\delta} \sum_{i=1}^{n} \left(\gamma_i - \frac{\gamma_i^2}{\bar{h}_i \bar{l}_i} \right) dt = \sum_{i=1}^{n} \left(\gamma_i - \frac{\gamma_i^2}{\bar{h}_i(\xi) \bar{l}_i(\xi)} \right)$$

So for any ϵ and j large enough there exists $\xi_j \in [j\delta, (j+1)\delta]$ such that

$$\left(\sum_{i=1}^{n} \gamma_i\right) + \epsilon \geq \frac{(\min_i \gamma_i)^2}{\max_i \bar{h}_i} \sum_{i=1}^{n} \frac{1}{\bar{l}_i(\xi_j)} \geq \frac{(\min_i \gamma_i)^2}{\max_i \bar{h}_i} \left(\frac{1}{\bar{l}_i(\xi_j)}\right)_{\max}$$

It follows that $\left(\frac{1}{\bar{l}_i(\xi_j)}\right)_{\max} < m$ where m depends on the initial curve but not on τ_j. Using the observation from the beginning of the proof we have $\left(\frac{1}{\bar{l}_i(\xi_j)}\right)_{\max} < 2m$ for all τ in $[\xi_j, \xi_j + 2\delta] \supseteq [(j+1)\delta, (j+2)\delta]$ which completes the proof. □

Let

$$J(\tau, a, b) = -\sum_{i=1}^{n} \bar{h}_i \bar{l}_i + 2\sum_{i=1}^{n} \gamma_i \log \bar{h}_i. \tag{16}$$

Lemma 3.2 *$J(\tau, a, b)$ evolves under the normalized flow by the equation:*

$$\begin{aligned} J_\tau &= -2\sum_{i=1}^{n} \frac{1}{\bar{h}_i \frac{1}{\bar{l}_i}} \left(\bar{h}_i - \frac{\gamma_i}{\bar{l}_i}\right)^2 \\ &\leq \frac{-2}{(\bar{h}_i)_{\max} \left(\frac{1}{\bar{l}_i}\right)_{\max}} \sum_{i=1}^{n} \left(\bar{h}_i - \frac{\gamma_i}{\bar{l}_i}\right)^2 \leq -C\sum_{i=1}^{n} \left(\bar{h}_i - \frac{\gamma_i}{\bar{l}_i}\right)^2, \end{aligned}$$

where C depends only on the initial conditions.

Moreover,

$$J(\tau) \geq -2\bar{A}(0) + 2\log \bar{h}_{\min} \sum_{i=1}^{n} \gamma_i.$$

These two inequalities imply that

$$\lim_{\tau \to \infty} J_\tau = 0.$$

Proof. Using the expression of the length of the i-th segment in terms of its neighboring support functions, as we did in the proof of the Lemma 2.1, we obtain the following evolution equation for $\bar{l}_i$:

$$\begin{aligned} (\bar{l}_i)_\tau(\tau) &= (\bar{h}_{i+1})_\tau(\tau) \csc \Delta\theta_{i+1} + (\bar{h}_{i-1})_\tau(\tau) \csc \Delta\theta_i \\ &\quad - (\bar{h}_i)_\tau(\tau)(\cot \Delta\theta_{i+1} + \cot \Delta\theta_i). \end{aligned}$$

This consequently implies, by reindexing two of the sums:

$$\begin{aligned} J_\tau &= -2\sum_{i=1}^{n} \bar{l}_i(\bar{h}_i)_\tau + 2\sum_{i=1}^{n} \gamma_i \frac{(\bar{h}_i)_\tau}{\bar{h}_i} = -2\sum_{i=1}^{n} \frac{\bar{l}_i}{\bar{h}_i}\left(\bar{h}_i - \frac{\gamma_i}{\bar{l}_i}\right)^2 \\ &\leq -2\frac{(\bar{l}_i)_{\min}}{(\bar{h}_i)_{\max}} \sum_{i=1}^{n}\left(\bar{h}_i - \frac{\gamma_i}{\bar{l}_i}\right)^2 \leq -C\cdot\sum_{i=1}^{n}\left(\bar{h}_i - \frac{\gamma_i}{\bar{l}_i}\right)^2 \end{aligned}$$

The last inequality follows from the bounds on $\bar{h}_i$ and $\frac{1}{\bar{l}_i}$ of the Corollary 3.1 and the Proposition 3.2. □

Theorem 3.1 *There exists a self-similar solution to the equation* (2).

Proof. Let τ_j be a sequence of times diverging to infinity. The functions $\frac{1}{\bar{l}_i(\tau)}$ and $\bar{h}_i(\tau)$ are continuous and bounded, so $\frac{1}{\bar{l}_i(\tau_j)}$ and $\bar{h}_i(\tau_j)$ must each contain a converging subsequence. By the previous lemma, the converging subsequence must converge to a solution of the equation $\bar{h}_i - \frac{\gamma_i}{\bar{l}_i} = 0$, and this is the equation which defines a self-similar solution. □

In the process of proving the Theorem 3.1, we have seen that the shape of a self-similar solution defined by the set $\{\tilde{h}_i\}_{1\leq i\leq n}$ and the set of normal admissible directions $\{\theta_i\}_{1\leq i\leq n}$ (which therefore determines the set of sides $\{\tilde{l}_i\}_{1\leq i\leq n}$ by the relation used in the proof of Lemma 2.1) is tied to the definition of the function γ on the set of admissible directions by n equalities:

$$\tilde{h}_i - \frac{\gamma_i}{\tilde{l}_i} = 0 \qquad 1 \leq i \leq n.$$

(In the above expression the self-similar solution is normalized so that the its enclosed area equals $\frac{1}{2}\sum_{i=1}^{n}\gamma_i$.)

Since there exists, at least, a self-similar solution to the crystalline flow defined by (2), we have the implication regarding the form of the function γ as in the Corollary 1.2.

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