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On the number of crossed homomorphisms

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Abstract. In this paper, we study congruences about the number of crossed homomorphisms from a finite abelian p-group to a finite p-group.

Key words: congruence, crossed homomorphism, finite p-group, group homomorphism.

1. Introduction

The purpose of this paper is to study the following conjectures concerning with congruences about the number of group homomorphisms and crossed homomorphisms between finite groups.

Let A and G be finite groups, and denote the set of group homomorphisms from A to G as $\mathrm{Hom}(A, G)$. Let C and H be finite groups such that C acts on H, and denote by ${}^{c}h$ this action of $c \in C$ on $h \in H$. We denote $Z^1(C, H)$ for the set of crossed homomorphisms from C to H; i.e.

$$Z^1(C,H) := \{\eta : C \longrightarrow H \mid \eta(cc') = \eta(c) \cdot {}^{c}\eta(c') \ \text{ for } \ c, c' \in C\}.$$

Conjecture H. Let A and G be finite groups, then

$$|\mathrm{Hom}(A, G)| \equiv 0 \mod \gcd(|A/A'|, |G|),$$

where A' is the commutator subgroup of A.

Conjecture I. Let C be a finite abelian p-group and H a finite p-group such that C acts on H. Then

$$|Z^1(C, H)| \equiv 0 \mod \gcd(|C|, |H|).$$

First, the number of group homomorphisms is studied in Yoshida [3] and, as a generalization of Frobenius Theorem ([2]), the following theorem is proved.

Main Theorem (Yoshida [3]) *Let A be a finite abelian group and G a*

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finite group, then

$$|\operatorname{Hom}(A,G)| \equiv 0 \mod \gcd(|A|,|G|).$$

As a generalization of the above theorem, Conjectures H and I are introduced in Asai-Yoshida [1] and they have the following relation.

Theorem 2.1 (Asai-Yoshida [1]) *If Conjecture* I *is true, then so is Conjecture* H.

Conjecture H and I have not been proved yet in general, but they seem to be natural and hold in some special cases.

Here, we list some results concerning with Conjecture H and I which are proved in Asai-Yoshida [1] and this paper.

Proposition 1.1 (i) *If C is a cyclic p-group, then Conjecture* I *is true.*

(ii) *If C is an elementary abelian p-group, then Conjecture* I *is true.*

(iii) *If C is a direct product of a cyclic p-group and an elementary abelian p-group, then Conjecture* I *is true.*

(iv) *If H is an abelian p-group, then Conjecture* I *is true.*

(v) *Suppose that the action of C on H is defined by a homomorphism from C to H, that is, there exists some $f \in \operatorname{Hom}(C,H)$ such that ${}^{c}h := f(c)hf(c)^{-1}$. Then Conjecture* I *is true.*

Theorem 1.2 (i) *If A/A' is a cyclic group, then Conjecture* H *is true.*

(ii)

$$|\operatorname{Hom}(A,G)| \equiv 0 \mod \gcd(((A/A') : \Phi(A/A')),|G|),$$

where A' is the commutator subgroup of A and $\Phi(A/A')$ is the Frattini subgroup of A/A'. Especially, if every Sylow subgroup of A/A' is an elementary abelian group, then Conjecture H *is true.*

(iii) *If every Sylow subgroup of A/A' is a direct product of a cyclic group and an elementary abelian group, then Conjecture* H *is true.*

The statements (i), (ii) of Proposition 1.1 and (i), (ii) of Theorem 1.2 are in Asai-Yoshida [1]. We prove (iii) of Proposition 1.1 and (iii) of Theorem 1.2 in Section 2 and (iv), (v) of Proposition 1.1 in Section 3.

2. On Conjecture I

First we extend Conjecture I as follows.

Notation Let C be a finite abelian p-group and H a finite p-group such that C acts on H. Let D be a subgroup of C. For $\mu \in Z^1(D,H)$, we denote $\boldsymbol{\mu}(D) := \{\mu(d)d \mid d \in D\} \leq HC$. Here $HC \trianglerighteq H$ is the semidirect product of H by C.

Conjecture II. Under the above notation, for any $\mu \in Z^1(D,H)$,

$$|Z^1(C,H;D,\mu)| \equiv 0 \mod \gcd\left(|C/D|, |C_H(\boldsymbol{\mu}(D))|\right),$$

where $Z^1(C,H;D,\mu) := \{\lambda \in Z^1(C,H) \mid \lambda_{|D} = \mu\}$ and $C_H(\boldsymbol{\mu}(D)) = C_{HC}(\boldsymbol{\mu}(D)) \cap H$.

Lemma 2.1 *Conjecture* II *is true if and only if Conjecture* I *is true.*

Proof. It is obvious that Conjecture II implies Conjecture I, so we show that Conjecture I implies Conjecture II. We may assume $|Z^1(C,H;D,\mu)| \neq 0$. Take any $\lambda \in Z^1(C,H;D,\mu)$, then C/D acts on $C_H(\boldsymbol{\mu}(D))$ by ${}^{cD}h := \lambda(c) \cdot {}^{c}h \cdot \lambda(c)^{-1}$ for $c \in C$ and $h \in C_H(\boldsymbol{\mu}(D))$. We consider $Z^1(C/D, C_H(\boldsymbol{\mu}(D)))$ with respect to this action, and show that there is a one to one correspondence between $Z^1(C,H;D,\mu)$ and $Z^1(C/D, C_H(\boldsymbol{\mu}(D)))$.

Here note that $\lambda(c)c \in C_{HC}(\boldsymbol{\mu}(D)) \cap Hc$ for any $c \in C$ and so

$$\begin{aligned} C_{HC}(\boldsymbol{\mu}(D)) \cap Hc &= C_{HC}(\boldsymbol{\mu}(D))\lambda(c)c \cap H\lambda(c)c \\ &= (C_{HC}(\boldsymbol{\mu}(D)) \cap H)\lambda(c)c \\ &= C_H(\boldsymbol{\mu}(D))\lambda(c)c. \end{aligned}$$

Hence we have that for any $\eta \in Z^1(C,H;D,\mu)$ and $c \in C$,

$$\begin{aligned} \eta(c)c &\in C_{HC}(\boldsymbol{\mu}(D)) \cap Hc \\ &= C_H(\boldsymbol{\mu}(D))\lambda(c)c. \end{aligned}$$

So there is some $\tilde{\eta} : C \to C_H(\boldsymbol{\mu}(D))$ such that $\eta(c) = \tilde{\eta}(c)\lambda(c)$. For c_1, c_2, $c \in C$ and $d \in D$,

$$\begin{aligned} \tilde{\eta}(c_1c_2)\lambda(c_1c_2) &= \eta(c_1c_2) \\ &= \eta(c_1){}^{c_1}\eta(c_2) \\ &= \tilde{\eta}(c_1)\lambda(c_1){}^{c_1}(\tilde{\eta}(c_2)\lambda(c_2)) \\ &= \tilde{\eta}(c_1)\lambda(c_1){}^{c_1}\tilde{\eta}(c_2){}^{c_1}\lambda(c_2) \end{aligned}$$

$$
\begin{aligned}
&= \tilde{\eta}(c_1)\lambda(c_1)^{c_1}\tilde{\eta}(c_2)\lambda(c_1)^{-1}\lambda(c_1)^{c_1}\lambda(c_2) \\
&= \tilde{\eta}(c_1)\lambda(c_1)^{c_1}\tilde{\eta}(c_2)\lambda(c_1)^{-1}\lambda(c_1c_2), \\
\tilde{\eta}(cd)\lambda(cd) &= \eta(cd) \\
&= \eta(c)^c\eta(d) \\
&= \eta(c)^c\mu(d) \\
&= \eta(c)^c\lambda(d) \\
&= \tilde{\eta}(c)\lambda(c)^c\lambda(d) \\
&= \tilde{\eta}(c)\lambda(cd).
\end{aligned}
$$

So $\tilde{\eta} \in Z^1(C/D, C_H(\boldsymbol{\mu}(D)))$. Conversely, for any $\tilde{\eta} \in Z^1(C/D, C_H(\boldsymbol{\mu}(D)))$, we define $\eta : C \to H$ by $\eta(c) := \tilde{\eta}(cD)\lambda(c)$ for $c \in C$. Then for c_1, $c_2 \in C$ and $d \in D$,

$$
\begin{aligned}
\eta(c_1c_2) &= \tilde{\eta}(c_1c_2D)\lambda(c_1c_2) \\
&= \tilde{\eta}(c_1D)\lambda(c_1)^{c_1}\tilde{\eta}(c_2D)\lambda(c_1)^{-1}\lambda(c_1)^{c_1}\lambda(c_2) \\
&= \tilde{\eta}(c_1D)\lambda(c_1)^{c_1}(\tilde{\eta}(c_2D)\lambda(c_2)) \\
&= \eta(c_1)^{c_1}\eta(c_2), \\
\eta(d) &= \tilde{\eta}(dD)\lambda(d) \\
&= \tilde{\eta}(D)\lambda(d) \\
&= \lambda(d) \\
&= \mu(d).
\end{aligned}
$$

So $\eta \in Z^1(C, H; D, \mu)$.

Thus we have that Conjecture II is true if and only if

$$
|Z^1(C/D, C_H(\boldsymbol{\mu}(D)))| \equiv 0 \mod \gcd\left(|C/D|, |C_H(\boldsymbol{\mu}(D))|\right).
$$

Hence Conjecture I implies Conjecture II. $\square$

Proposition 2.2 *If C (resp. C/D) is a cyclic p-group or an elementary abelian p-group, then Conjecture* I *(resp. Conjecture* II*) is true.*

Proof. By Proposition 1.1 (i), (ii) and Lemma 2.1, this statement holds. $\square$

Proposition 2.3 *If C (resp. C/D) is a direct product of a cyclic p-group and an elementary abelian p-group, then Conjecture* I *(resp. Conjecture* II*) is true.*

Proof. By Lemma 2.1, we need only to show that Conjecture I holds in this case. Let $C = C_1 \times C_2$ where C_1 is cyclic and C_2 is elementary abelian. Then

$$\begin{aligned} |Z^1(C,H)| &= \sum_{\mu \in Z^1(C_2,H)} |Z^1(C,H;C_2,\mu)| \\ &= \sum_{\mu \in \mathcal{X}_1} |Z^1(C,H;C_2,\mu)| + \sum_{\mu \in \mathcal{X}_2} |Z^1(C,H;C_2,\mu)|, \end{aligned}$$

where

$$\begin{aligned} \mathcal{X}_1 &:= \{\mu \in Z^1(C_2,H) \mid |C_H(\boldsymbol{\mu}(C_2))| \leq |C_1|\}, \\ \mathcal{X}_2 &:= \{\mu \in Z^1(C_2,H) \mid |C_1| < |C_H(\boldsymbol{\mu}(C_2))|\}. \end{aligned}$$

Step 1.

$$\sum_{\mu \in \mathcal{X}_1} |Z^1(C,H;C_2,\mu)| \equiv 0 \mod |H|.$$

Proof of Step 1. We define an action of H on $\mathcal{X}_1$ by conjugation, i.e.

$$\begin{array}{ccc} H \times \mathcal{X}_1 & \longrightarrow & \mathcal{X}_1 \\ (h,\mu) & \longmapsto & ({}^h\mu : c \mapsto h \cdot \mu(c) \cdot {}^c h^{-1}). \end{array}$$

Thus

$$\begin{aligned} &\sum_{\mu \in \mathcal{X}_1} |Z^1(C,H;C_2,\mu)| \\ &\qquad = \sum_{\mu \in \mathcal{X}_1/\sim_H} (H : C_H(\boldsymbol{\mu}(C_2))) \cdot |Z^1(C,H;C_2,\mu)|, \end{aligned}$$

where in the last summation μ runs over a set of complete representatives of the above action. Here C/C_2 is cyclic and $|C_H(\boldsymbol{\mu}(C_2))|$ divides $|C_1| = |C/C_2|$, so we have that

$$\begin{aligned} &(H : C_H(\boldsymbol{\mu}(C_2))) \cdot |Z^1(C,H;C_2,\mu)| \\ &\quad \equiv 0 \mod (H : C_H(\boldsymbol{\mu}(C_2))) \cdot \gcd(|C/C_2|, |C_H(\boldsymbol{\mu}(C_2))|) \\ &\quad \equiv 0 \mod |H|. \end{aligned}$$

Thus we have Step 1.

Step 2.

$$\sum_{\mu \in \mathcal{X}_2} |Z^1(C, H; C_2, \mu)| \equiv 0 \mod |C|.$$

Proof of Step 2. We may assume that H is a nontrivial p-group. Let $Z := \Omega_1(Z(HC) \cap H)$, where HC is the semidirect product of H by C. Here note that $Z \neq 1$, because H is a normal subgroup of HC. Now the group $\mathrm{Hom}(C_2, Z)$ acts on $\mathcal{X}_2$ by multiplication, i.e.

$$\begin{array}{ccc} \mathrm{Hom}(C_2, Z) \times \mathcal{X}_2 & \longrightarrow & \mathcal{X}_2 \\ (f, \mu) & \longmapsto & (f\mu : c \mapsto f(c) \cdot \mu(c)). \end{array}$$

Since this action is semi-regular and

$$|Z^1(C, H; C_2, \mu)| = |Z^1(C, H; C_2, f\mu)|$$

for any $f \in \mathrm{Hom}(C_2, Z)$ and $\mu \in \mathcal{X}_2$, we have

$$\begin{aligned} &\sum_{\mu \in \mathcal{X}_2} |Z^1(C, H; C_2, \mu)| \\ &\qquad = \sum_{\mu \in \mathcal{X}_2/\sim_{\mathrm{Hom}(C_2, Z)}} |\mathrm{Hom}(C_2, Z)| \cdot |Z^1(C, H; C_2, \mu)|, \end{aligned}$$

where in the last summation μ runs over a set of complete representatives of the above action. Since C_2 is elementary abelian and $Z \neq 1$,

$$|\mathrm{Hom}(C_2, Z)| \equiv 0 \mod |C_2|.$$

Here C/C_2 is cyclic and $|C_1| = |C/C_2|$ divides $|C_H(\boldsymbol{\mu}(C_2))|$, so we have that

$$\begin{aligned} &|\mathrm{Hom}(C_2, Z)| \cdot |Z^1(C, H; C_2, \mu)| \\ &\quad \equiv 0 \mod |C_2| \cdot \gcd\left(|C/C_2|, |C_H(\boldsymbol{\mu}(C_2))|\right) \\ &\quad \equiv 0 \mod |C|. \end{aligned}$$

Thus we have Step 2.

By Steps 1 and 2, we have

$$|Z^1(C, H)| \equiv 0 \mod \gcd(|C|, |H|).$$

$\square$

Theorem 2.4 *If every Sylow subgroup of A/A' is a direct product of a cyclic group and an elementary abelian group, then Conjecture* H *is true.*

Proof. By Proposition 2.3 and the almost same argument of the proof of Theorem 3.4, 3.5 [1], we have the theorem. $\square$

3. Some special cases

Theorem 3.1 *If H is an abelian p-group, then Conjectures* I *and* II *are true.*

Proof. By Lemma 2.1, it is enough to show that Conjecture I holds in this case. Let $C = C_1 \times \cdots \times C_n$, $C_i = \langle c_i \rangle$, be a cyclic group decomposition of C, and denote $\widehat{C}_i := \langle c_j \mid j \neq i \rangle$. Since H is abelian, $Z^1(C_i, C_H(\widehat{C}_i))$ has the following group structure:

$$\begin{array}{ccc} Z^1(C_i, C_H(\widehat{C}_i)) \times Z^1(C_i, C_H(\widehat{C}_i)) & \longrightarrow & Z^1(C_i, C_H(\widehat{C}_i)) \\ (\lambda_1, \lambda_2) & \longmapsto & (\lambda_1\lambda_2 : c_i \mapsto \lambda_1(c_i) \cdot \lambda_2(c_i)), \end{array}$$

for any i. So we let the group $\prod_{i=1}^n Z^1(C_i, C_H(\widehat{C}_i))$ act on $Z^1(C, H)$ by the rule

$$\begin{array}{rcl} (\mu_1, \ldots, \mu_n) \cdot \lambda : C & \longrightarrow & H \\ c_i & \longmapsto & \mu_i(c_i)\lambda(c_i), \end{array}$$

where $(\mu_1, \ldots, \mu_n) \in \prod_{i=1}^n Z^1(C_i, C_H(\widehat{C}_i))$ and $\lambda \in Z^1(C, H)$. This action is semi-regular and, by Proposition 2.2, we have that

$$|Z^1(C_i, C_H(\widehat{C}_i))| \equiv 0 \mod \gcd(|C_i|, |C_H(\widehat{C}_i))|).$$

So if $|C_i| \leq |C_H(\widehat{C}_i)|$ for any $1 \leq i \leq n$, then

$$|Z^1(C, H)| \equiv 0 \mod \prod_{i=1}^n |C_i| = |C|.$$

If not, there exists some i such that $|C_H(\widehat{C}_i)| < |C_i|$, and thereby,

$$\begin{aligned} |Z^1(C, H)| &= \sum_{\mu \in Z^1(\widehat{C}_i, H)} |Z^1(C, H; \widehat{C}_i, \mu)| \\ &= \sum_{\mu \in Z^1(\widehat{C}_i, H)/\sim_H} (H : C_H(\boldsymbol{\mu}(\widehat{C}_i)) \cdot |Z^1(C, H; \widehat{C}_i, \mu)|, \end{aligned}$$

where in the last summation μ runs over a set of complete representatives of orbits under the following conjugate action of H on $Z^1(\widehat{C}_i, H)$:

$$\begin{array}{ccc} H \times Z^1(\widehat{C}_i, H) & \longrightarrow & Z^1(\widehat{C}_i, H) \\ (h, \mu) & \longmapsto & ({}^h\mu : c \mapsto h \cdot \mu(c) \cdot {}^c h^{-1}). \end{array}$$

Since $C/\widehat{C}_i$ is cyclic, by Proposition 2.2,

$$|Z^1(C, H; \widehat{C}_i, \mu)| \equiv 0 \mod \gcd(|C/\widehat{C}_i|, |C_H(\boldsymbol{\mu}(\widehat{C}_i))|).$$

Here H is abelian, so $C_H(\boldsymbol{\mu}(\widehat{C}_i)) = C_H(\widehat{C}_i)$ for any $\mu \in Z^1(\widehat{C}_i, H)$. Hence we have that

$$\begin{aligned} &(H : C_H(\boldsymbol{\mu}(\widehat{C}_i)) \cdot |Z^1(C, H; \widehat{C}_i, \mu)| \\ &\equiv 0 \quad \mod (H : C_H(\widehat{C}_i)) \cdot \gcd(|C_i|, |C_H(\widehat{C}_i)|) \\ &\equiv 0 \quad \mod |H|. \end{aligned}$$

So in either case, we have

$$|Z^1(C, H)| \equiv 0 \mod \gcd(|C|, |H|).$$

□

Theorem 3.2 *Suppose that the action of C on H is defined by a homomorphism from C to H, that is, there exists some $f \in \mathrm{Hom}(C, H)$ such that ${}^c h := f(c) h f(c)^{-1}$. Then Conjectures* I *and* II *are true.*

Proof. In this case, the semidirect product HC is isomorphic to the direct product of H and C. This isomorphism is given by

$$\begin{array}{ccc} HC & \cong & H \times C \\ (h, c) & \mapsto & (hf(c), c^{-1}). \end{array}$$

So $|Z^1(C, H)| = |\mathrm{Hom}(C, H)|$. By Theorem 2.1 [3], Conjecture I and II are true. □

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