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On viscosity solutions of the Hamilton-Jacobi equation

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Abstract. Comparison and uniqueness results are obtained for viscosity solutions of Hamilton-Jacobi equations. The main objective is the characterization of the value function associated with a variational problem of the Bolza type. This is accomplished, in particular, in the presence of certain conditions reminiscent of the classical Tonelli conditions.

Key words: Hamilton-Jacobi equation, viscosity solution, Bolza problem, value function, uniqueness, comparison principle.

1. Introduction and preliminaries

The concept of a viscosity solution of a partial differential equation has been intensively studied since it was introduced by M.G. Crandall and P.-L. Lions in [14, 15], and the efforts have proved immensely successful. The voluminous body of results attests the significance, for a variety of problems, of this notion of a generalized solution.

We will here be concerned with the Hamilton-Jacobi equation which has been one of the main targets of the theory of viscosity solutions since its infancy (see e.g. [4, 5, 6, 9, 12, 15, 16, 17, 22, 24] and the review article [13]). Having its historical roots in the calculus of variations and closely allied fields, the Cauchy problem for the Hamilton-Jacobi equation admits in regular cases a variational solution which is termed the value function or, in the context of classical mechanics, the action function. In favorable circumstances the value function is by the current state of the uniqueness theory necessarily the sole viscosity solution. While the collection of uniqueness theorems is indisputably substantial, it is not yet complete. To the best of the author's knowledge, the vast majority of the available theorems, by now numerous, either fail to cover or do not give satisfactory information about certain natural problems in the calculus of variations (cf. [27]). There are however exceptions: in the article [4] uniqueness among locally Lipschitz continuous solutions that are bounded below is demonstrated for a rich

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class of finite horizon optimal control problems with unbounded control space. Despite the recent progress in [4], it still seems that the theory of viscosity solutions for problems in the category alluded to is not mature. (Incidentally, Theorem 4 below contains an analogue of the uniqueness result obtained in [4]. In Theorem 4 local Lipschitz continuity is however not a premise.)

The purpose of this paper is, accordingly, to examine the partial differential equation

$$\text{(HJ)} \qquad u_t(t,x) + H(x, u_x(t,x)) = 0, \qquad (t,x) \in (0,T] \times \mathbb{R}^n,$$

from the perspective of the calculus of variations. The Cauchy problem, in the sequel referred to as (CP), consists in finding solutions u that satisfy the initial condition

$$\text{(IC)} \qquad \lim_{t \downarrow 0} u(t,x) = \varphi(x) \qquad \text{for all} \quad x \in \mathbb{R}^n,$$

where φ is a prescribed function which may assume the value ∞. The main topic here is the delicate uniqueness problem for (CP). In contrast to the framework in most papers devoted to viscosity solutions of Hamilton-Jacobi equations in unbounded domains, the solutions will neither be assumed to be uniformly continuous nor bounded, nor will the Hamiltonian function $H(x,p)$ be restricted to enjoy uniform continuity in any of its arguments. The initial incitement for this work comes to a great extent from the articles [22, 17] in which comparison theorems were derived under certain constraints on the considered subsolutions. In the first main theorem of the present paper this idea is systematically investigated, see Theorem 3. To be more specific, Theorem 3 states a comparison result for subsolutions u and w such that $H_p(x, w_x(t,x))$ is subject to a certain growth limitation. Theorem 3 shows to advantage also when the Lagrangian does not automatically yield a well-behaved variational problem. As a matter of fact, the supplementary growth restriction turns out to be superfluous for the uniqueness problem when (CP) corresponds to a decent problem in the calculus of variations. In this important case a characterization of the value function is furnished in Theorem 4. Theorem 4 is however not the ultimate result since it employs the assumption that the considered candidate solution be bounded from below by a function of linear growth and it still remains an open problem, as far as the author knows, whether that hypothesis is redundant or not. Nevertheless, the case $H(x,p) \equiv H(p)$ with $\lim_{|p|\to\infty} H(p)/|p| = \infty$ is fully understood [28]; see also Theorem 4 (iii). Analogous results for bounded below solutions of stationary problems appear in [1].

As noted above, the value function associated with φ and the Lagrangian function L plays a distinguished role.

Definition 1 (Value function) The *value function* V is the function on $(0,T] \times \mathbb{R}^n$ that assigns to $(t,x) \in (0,T] \times \mathbb{R}^n$ the infimum of

$$\varphi(X(0)) + \int_0^t L(X(\tau), \dot{X}(\tau))\, d\tau$$

as X ranges over all Lipschitz continuous curves $X\colon [0,t] \to \mathbb{R}^n$ with $X(t) = x$.

Let us recall the definition of a viscosity solution of (HJ) which involves the following generalized differentials.

Definition 2 (Regular sub- and supergradients) Let u be a real-valued function on $(0,T]\times\mathbb{R}^n$. The *subdifferential* of u at a point $(t,x) \in (0,T]\times\mathbb{R}^n$, symbolized by $\partial^- u(t,x)$, is the set consisting of all $(\omega,p) \in \mathbb{R} \times \mathbb{R}^n$ such that

$$u(t',x') \geq u(t,x) + \omega(t'-t) + \langle p, x'-x\rangle + o(t'-t, x'-x) \quad \text{as} \quad (0,T] \times \mathbb{R}^n \ni (t',x') \to (t,x).$$

Similarly, the *superdifferential* of u at (t,x), denoted by $\partial^+ u(t,x)$, is the set of all $(\omega,p) \in \mathbb{R}\times\mathbb{R}^n$ such that the reverse inequality holds; put differently, $\partial^+ u(t,x) = -\partial^-(-u)(t,x)$.

We will often tacitly appeal to the following tractable characterization of sub- and supergradients: $(\omega,p) \in \partial^\pm u(t,x)$ if and only if there exists a C^1 function ψ such that $\pm(u-\psi)$ attains a local maximum relative to $(0,T] \times \mathbb{R}^n$ at (t,x) and $(\omega,p) = d\psi(t,x)$. In fact, the local extremum may be assumed strict.

Definition 3 (Viscosity solution) A *viscosity subsolution* of (HJ) is an upper semicontinuous function $u\colon (0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\omega + H(x,p) \leq 0 \quad \text{whenever} \quad (\omega,p) \in \partial^+ u(t,x),$$

whereas a *viscosity supersolution* of (HJ) is a lower semicontinuous function

u on $(0,T] \times \mathbb{R}^n$ such that

$$\omega + H(x,p) \geq 0 \quad \text{whenever} \quad (\omega,p) \in \partial^- u(t,x).$$

By a *viscosity solution* it is understood a function that is simultaneously a viscosity subsolution and a viscosity supersolution.

For the sake of brevity we will frequently use "subsolution" synonymously with "viscosity subsolution," etc.

Our methodology and presentation rely heavily on nonsmooth and variational analysis. As a general reference in that branch of mathematics we recommend the recent comprehensive book [26] by R.T. Rockafellar and R. J-B Wets. Of course we also draw on many of the techniques that have evolved since the seminal contributions on viscosity solutions and in particular on those developed for convex problems in the fundamental monograph [24], which treats various aspects of Hamilton-Jacobi equations thoroughly. (For an account of the developments prior to the concept of a viscosity solution refer to [8], and for connections to classical mechanics consult [2].)

The organization of the rest of the paper is as follows. In Section 2 we add further prerequisites and list technical conditions on L, H. Section 3 exhibits basic properties of the value function; it is in particular recalled that the value function solves (CP) in regular cases.

Section 4 is devoted to our main concern, namely to the comparison principle and the uniqueness problem, and contains our main results: Theorems 3 and 4. The proof of Theorem 3 borrows ideas from [17]: comparison on a compact domain in combination with restriction at infinity. The proof of Theorem 4 utilizes techniques developed in [24].

Finally, Section 5 presents briefly an application of the results of Section 4 to an initial-value problem that one encounters in optimal control theory.

We have aimed at a self-contained presentation. Accordingly, the results in Sections 2 and 3 are not of a genuinely novel nature but rather tailor-made versions of basic facts designed to support our study of uniqueness in Section 4. They should nevertheless be of independent interest.

2. Basic assumptions and further preparations

Throughout we confine our attention to autonomous convex problems. Specifically, we assume that the Hamiltonian function H is derivable from a Lagrangian function L through the Legendre-Fenchel transformation,

$$H(x,p) = \sup\{\langle p, v\rangle - L(x,v); v \in \mathbb{R}^n\} \text{ for all } (x,p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

The Lagrangian $L(x,v)$ is in its turn assumed convex and lower semicontinuous in v. Thus, reciprocally,

$$L(x,v) = \sup\{\langle p, v\rangle - H(x,p); p \in \mathbb{R}^n\} \text{ for all } (x,v) \in \mathbb{R}^n \times \mathbb{R}^n,$$

for the Legendre-Fenchel transformation acts as an involution on the set of lower semicontinuous convex functions; [25] is the standard reference for convex analysis.

We will furthermore usually impose the following mild regularity condition.

(H0) H is finite and continuous on $\mathbb{R}^n \times \mathbb{R}^n$.

For future reference we also formulate two conditions on L.

(L0) L is finite, bounded from below, and continuous on $\mathbb{R}^n \times \mathbb{R}^n$.

(L1) L is Hölder continuous on each compact subset of $\mathbb{R}^n \times \mathbb{R}^n$; $L(x,v) \geq M(|v|)$ for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$ where $M\colon [0,\infty) \to \mathbb{R}$ is convex and nondecreasing with $M(s)/s \to \infty$ as $s \to \infty$; and L obeys the following strong convexity condition in the velocity: for each compact subset K of $\mathbb{R}^n \times \mathbb{R}^n$ there exists a $\mu > 0$ such that

$$L(x,v') - L(x,v) \geq \langle p, v' - v\rangle + \mu|v' - v|^2$$

for every $(x,v') \in K$, $(x,v) \in K$, and every subgradient $p \in \partial_v L(x,v)$.

Here $\partial_v L(x,v)$ signifies the subdifferential of convex analysis of $v \mapsto L(x,v)$.

Condition (L1), akin to the classical Tonelli hypotheses, implies conditions (L0), (H0) and also that $H(x,p)$ is a smooth function of p. Several sets of hypotheses that assure well-behaved variational problems have been identified in the literature. These developments involve refinements and extensions of classical results to nonsmooth or even extended-real-valued Lagrangians. Our choice, (L1), was used in [29] and is but one among several elaborate alternatives, see e.g. [10, 11, 27]. Let us just observe for the time being that when (L0) is in force and $\inf \varphi$ is finite, then the value function V is a well-defined real-valued bounded below function. When (L1) is fulfilled and φ is lower semicontinuous and such that $\varphi(x) \geq -C(1 + |x|)$, for some constant C and all $x \in \mathbb{R}^n$, then the infimum defining $V(t,x)$ is attained for every $(t,x) \in (0,T] \times \mathbb{R}^n$ and there exists a constant C' such that $V(t,x) \geq -C'(1+|x|)$ for all $(t,x) \in (0,T] \times \mathbb{R}^n$.

Let us now turn to a certain notion of variational convergence pertinent to our methodology.

Definition 4 (Epi-convergence) In generic terms, if f and f_j are lower semicontinuous extended-real-valued functions defined on $\mathbb{R}^N$, f_j is declared *epi-convergent* to f if the following two conditions are met:

(E1) $f(x) \leq \liminf_{j\to\infty} f_j(x_j)$ whenever $x_j \to x$.

(E2) To each $x \in \mathbb{R}^N$ there corresponds a convergent sequence $y_j \to x$ with $f(x) = \lim_{j\to\infty} f_j(y_j)$.

Epi-convergence is alternatively called Γ-convergence in the literature [18]. Its basic relevance in optimization theory is due to its magical variational properties [3, 18]. A key role in the present paper will however be played by a stronger mode of convergence, namely the conjunction of pointwise convergence and epi-convergence, which amounts to requiring (E1) and $\lim_{j\to\infty} f_j(x) = f(x)$ for any $x \in \mathbb{R}^N$.

Lemma 1 *Let $f, f_j: \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous extended-real-valued functions. Then conditions* (a) *and* (b) *below are equivalent, and either of them entails condition* (c):

(a) *$f_j \to f$ in the sense of both pointwise convergence and epi-convergence as $j \to \infty$.*

(b) *For any nonempty compact subset $K \subset \mathbb{R}^N$,*

$$\min_K f_j \to \min_K f \qquad \text{as } j \to \infty.$$

(c) *If $K \subset \mathbb{R}^N$ is compact, x_j minimizes f_j over K, and $\bar{x}$ is a cluster point of (x_j), then $\bar{x}$ minimizes f over K.*

Proof. (a) $\Rightarrow$ (b): Assume (a) and let K be a nonempty compact subset of $\mathbb{R}^N$. Let x_j minimize f_j over K and extract a subsequence (x_{j_k}) such that

$$\lim_{k\to\infty} f_{j_k}(x_{j_k}) = \liminf_{j\to\infty} f_j(x_j)$$

and $x_{j_k} \to \bar{x}$. Then by (E1),

$$f(\bar{x}) \leq \liminf_{k\to\infty} f_{j_k}(x_{j_k}) = \liminf_{j\to\infty} f_j(x_j)$$

and so

$$\min_K f \le \liminf_{j\to\infty} \min_K f_j. \tag{1}$$

Choose a point $\tilde{x}$ minimizing f over K. Then

$$\min_K f = f(\tilde{x}) = \lim_{j\to\infty} f_j(\tilde{x}) \ge \limsup_{j\to\infty} \min_K f_j, \tag{2}$$

and the conjunction of (1) and (2) yields (b).

(b) $\Rightarrow$ (a): Assume (b). One learns that $f_j \to f$ pointwise by taking $K = \{x\}$, an arbitrary singleton subset of $\mathbb{R}^N$. Let $x_j \to x$ and put $K_m = \{x_j; j \ge m\} \cup \{x\}$. Then K_m is compact, $\min_{K_m} f_j \le f_j(x_j)$ if $j \ge m$, and so

$$\liminf_{j\to\infty} f_j(x_j) \ge \liminf_{j\to\infty} \min_{K_m} f_j = \min_{K_m} f.$$

Sending $m \to \infty$, one obtains (E1).

Assuming (a), (b), if x_j minimizes f_j over K, and $\bar{x}$ is a cluster point of (x_j), then, arguing similarly as above, $f(\bar{x}) \le \min_K f$ follows. □

In connection with this notion of convergence the following simple result is noteworthy. We will in what follows denote by B_ϱ the open ball centered at the origin and of radius ϱ.

Proposition 1 *Let the number*

$$\inf\{H(x,p); (x,p) \in B_\varrho \times \mathbb{R}^n\} = -\sup\{L(x,0); x \in B_\varrho\}$$

be finite for any $\varrho > 0$. Let u be a viscosity subsolution of (HJ) *which is continuous on $(0,T] \times \mathbb{R}^n$.*

(i) *Then the epi-limit and the pointwise limit of $u(t,\cdot)$ as $t \downarrow 0$ both exist and are in fact equal. The common limit is a lower semicontinuous extended-real-valued function u_0 which does not attain the value $-\infty$.*

(ii) *Let u be extended to $[0,T] \times \mathbb{R}^n$ through $u(0,\cdot) = u_0$. Then u becomes lower semicontinuous on $[0,T] \times \mathbb{R}^n$. Moreover, if u_0 is continuous at a point x, then u is continuous at $(0,x)$.*

Proof. We choose $\varrho > 0$ and let the premise on H furnish a finite number

$$c := \inf\{H(x,p); (x,p) \in B_\varrho \times \mathbb{R}^n\}.$$

Then u is a viscosity subsolution of

$$u_t(t,x) + c = 0 \qquad \text{for} \quad (t,x) \in (0,T) \times B_\varrho.$$

Thus $t \mapsto u(t,x) + ct$ is nonincreasing for any $x \in B_\varrho$ by virtue of the calculus presented in [24, App. 2]. We have hereby reduced our problem to the monotone case in which it is well-known that epi-limits and pointwise limits always exist and agree. Of course the monotonicity also implies that the limit function u_0 is lower semicontinuous and assumes values in $\mathbb{R} \cup \{\infty\}$ exclusively.

The asserted lower semicontinuity of u follows immediately from the fact that (E1) is fulfilled. In order to prove the continuity statement, let u_0 be continuous at x and let $(t_j, x_j) \to (0, x)$ where ϱ is so large that x belongs to B_ϱ. In order to establish $u(t_j, x_j) \to u_0(x)$ we need only show $u_0(x) \geq \limsup_{j\to\infty} u(t_j, x_j)$ in view of condition (E1). But this inequality follows immediately from

$$u(t_j, x_j) + ct_j \leq u_0(x_j).$$

□

Remark 1. The natural role of epi-convergence in the formulation of the initial condition (or terminal condition), indicated by the proposition, has been observed in [5, 6].

Notation 1 (i) The diagonal $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n; x = y\}$ will be denoted by Δ. (ii) The *indicator function* of a subset $A \subseteq \mathbb{R}^N$ will be signified by $\mathfrak{I}_A$. Thus $\mathfrak{I}_A(x) = 0$ if $x \in A$ while $\mathfrak{I}_A(x) = \infty$ if $x \in \mathbb{R}^N \setminus A$.

Lemma 2 *Let $\theta_j \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be lower semicontinuous functions converging pointwise as well as epi-converging to the indicator function of Δ. Let $g \colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous, K be a nonempty compact subset of $[0,T] \times \mathbb{R}^n$, and let (t_j, x_j, y_j) minimize $g(t,x,y) + \theta_j(x,y)$ over $(t,x) \in K$, $(t,y) \in K$. If $(\bar{t}, \bar{x}, \bar{y})$ is a cluster point of (t_j, x_j, y_j) as $j \to \infty$, then $\bar{x} = \bar{y}$ and $(\bar{t}, \bar{x})$ minimizes $K \ni (t,x) \mapsto g(t,x,x)$. Furthermore,*

$$\lim_{j\to\infty} \min\{g(t,x,y) + \theta_j(x,y); (t,x) \in K, (t,y) \in K\} = \min\{g(t,x,x); (t,x) \in K\} \tag{3}$$

and $\theta_j(x_j, y_j) \to 0$ as $j \to \infty$.

Such functions θ_j will serve as penalty functions in certain auxiliary maximization problems in the proof of Theorem 3.

Proof. Let

$$f_j(t,x,y) = g(t,x,y) + \theta_j(x,y) \quad \text{if} \quad (t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n,$$

and let f_j be equal to ∞ at all other points in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. It is easy to verify that f_j converges pointwise and epi-converges to the function f that agrees with g in $\{(t,x,y); t \in [0,T], x = y\}$ and takes the value ∞ elsewhere. According to Lemma 1, $(\bar{t},\bar{x},\bar{y})$ solves the constrained problem of minimizing $f(t,x,y)$ with $(t,x) \in K$, $(t,y) \in K$. But this means that $\bar{x} = \bar{y}$ and that $(\bar{t},\bar{x})$ minimizes $g(t,x,x)$ over $(t,x) \in K$, as asserted.

The stated equation (3) follows easily from Lemma 1, so we concentrate on the limit of $\theta_j(x_j,y_j)$. It is readily seen, in view of (E1), that $0 \leq \liminf_{j\to\infty} \theta_j(x_j,y_j)$. Extract a convergent subsequence $(t_{j_k},x_{j_k},y_{j_k}) \to (\tilde{t},\tilde{x},\tilde{x})$ such that

$$\limsup_{j\to\infty} \theta_j(x_j,y_j) = \lim_{k\to\infty} \theta_{j_k}(x_{j_k},y_{j_k}).$$

Then by virtue of equation (3) and the lower semicontinuity of g we learn

$$\lim_{k\to\infty} \theta_{j_k}(x_{j_k},y_{j_k}) = \mu - \lim_{k\to\infty} g(t_{j_k},x_{j_k},y_{j_k}) \leq \mu - g(\tilde{t},\tilde{x},\tilde{x}) \leq 0$$

where $\mu = \min\{g(t,x,x); (t,x) \in K\}$. □

3. Some properties of the value function

We shall in this section acquaint ourselves with the object that this paper revolves around, namely the value function. To facilitate the discussion we will start by considering the fixed endpoint problem.

Definition 5 For $(t,x,y) \in (0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, $\Theta(t,x,y)$ is defined as the infimum of the values of the integral

$$\int_0^t L(X(\tau),\dot{X}(\tau))\, d\tau$$

as X runs through all Lipschitz continuous arcs connecting $(0,y)$ with (t,x), $X(0) = y$ and $X(t) = x$.

Θ may be regarded as a fundamental solution in view of the fact that the value function associated with φ and L is given by

$$V(t,x) = \inf\{\varphi(y) + \Theta(t,x,y); y \in \mathbb{R}^n\}, \quad (t,x) \in (0,T] \times \mathbb{R}^n. \qquad (4)$$

Lemma 3 (i) *Under* (L0), Θ *is upper semicontinuous on* $(0,T]\times\mathbb{R}^n\times\mathbb{R}^n$.

(ii) *Let* (L1) *be fulfilled and let* $(t,x,y) \in (0,T] \times \mathbb{R}^n \times \mathbb{R}^n$. *Then the infimum*

$$\inf_X \int_0^t L(X(\tau), \dot{X}(\tau))\, d\tau$$

extended over all absolutely continuous arcs X *joining* $(0,y)$ *with* (t,x) *is achieved and every minimizing arc is actually* C^1.

(iii) *If* (L1) *holds, then* Θ *is locally Lipschitz continuous in* $(0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ *and, moreover,* $\Theta(t,\cdot,\cdot)$ *both converges pointwise and epi-converges to the indicator function of* Δ *as* $t \downarrow 0$.

In the presence of (L1), therefore,

$$\begin{aligned}\Theta(t,x,y) &= \min_{X\in C^1} \int_0^t L(X(\tau), \dot{X}(\tau))\, d\tau \\ &= \min_{X\in W^{1,1}} \int_0^t L(X(\tau), \dot{X}(\tau))\, d\tau,\end{aligned}$$

where in each case $X(0) = y$, $X(t) = x$. In particular, (L1) excludes the so-called Lavrentiev phenomenon.

Proof. (i) Let $(t_j, x_j, y_j) \to (t,x,y)$ in $(0,T]\times\mathbb{R}^n\times\mathbb{R}^n$, let $X: [0,t] \to \mathbb{R}^n$ be a Lipschitz continuous path joining $(0,y)$ with (t,x), and define

$$X_j(\tau) = X(t\tau/t_j) + \eta_j + (\tau/t_j)(\xi_j - \eta_j) \qquad \text{if } \tau \in [0,t_j],$$

where $\xi_j = x_j - x$, $\eta_j = y_j - y$. The admissibility of the curve X_j for the optimization problem defining $\Theta(t_j,x_j,y_j)$ yields

$$\begin{aligned}&\Theta(t_j,x_j,y_j) \le \int_0^{t_j} L(X_j(\tau), \dot{X}_j(\tau))\, d\tau \\ &= \frac{t_j}{t} \int_0^t L(X(s) + \eta_j + (s/t)(\xi_j - \eta_j), (t/t_j)\dot{X}(s) + (\xi_j - \eta_j)/t_j)\, ds,\end{aligned}$$

where a change of the variable of integration has been performed. It is elementary, utilizing the Lipschitz continuity of X and the continuity of L, to deduce that the second integrand is bounded over $s \in [0,t]$ (a.e.) uniformly over j. Therefore, by Lebesgue's dominated convergence theorem,

$$\limsup_{j\to\infty} \Theta(t_j, x_j, y_j) \leq \int_0^t L(X(s), \dot{X}(s))\, ds.$$

But this inequality is retained for every feasible curve X and so

$$\limsup_{j\to\infty} \Theta(t_j, x_j, y_j) \leq \Theta(t, x, y).$$

Concerning (ii) and (iii), the smoothness of minimizing arcs and the Lipschitz continuity of Θ (and much more) were proved in the recent article [29], see also [8, Chap. II] for related questions.

Thus we may turn to the asserted convergence of $\Theta(t, \cdot, \cdot)$ as $t \downarrow 0$. The convexity of $v \mapsto M(|v|)$ implies the lower bound $\Theta(t, x, y) \geq tM(|x-y|/t)$, where M is the function participating in (L1). Hence if $t_j \downarrow 0$ and $(x_j, y_j) \to (x, y)$, then

$$\Theta(t_j, x_j, y_j) \geq \begin{cases} t_j M(|x_j - y_j|/t_j) \to \infty & \text{if } x \neq y, \\ t_j \inf L \to 0 & \text{if } x = y. \end{cases}$$

Furthermore,

$$t \inf L \leq \Theta(t, x, x) \leq \int_0^t L(x, 0)\, d\tau = tL(x, 0)$$

and $\lim_{t\downarrow 0} \Theta(t, x, x) = 0$ ensues. □

In well-behaved instances V is a solution of (CP) and so existence is manifest.

Theorem 1 *Let the Lagrangian L satisfy* (L0) *and let $\varphi: \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be finite somewhere. Then the following two statements are true.*

(i) *If V is everywhere finite, then V is a viscosity subsolution of* (HJ). *If in addition φ happens to be upper semicontinuous and V is extended to $[0, T] \times \mathbb{R}^n$ by $V(0, \cdot) = \varphi$, then V becomes upper semicontinuous on $[0, T] \times \mathbb{R}^n$.*

(ii) *Let* (L1) *be fulfilled. Assume that φ is lower semicontinuous and that φ possesses a minorant of linear growth. Then V is a locally Lipschitz continuous viscosity solution of* (CP). *The initial condition* (IC) *holds in the sense that*

$$V(t, \cdot) \to \varphi \qquad \textit{as } t \downarrow 0$$

with respect to both pointwise convergence and epi-convergence. If V is extended to $[0,T] \times \mathbb{R}^n$ by $V(0,\cdot) = \varphi$, then V becomes lower semicontinuous on $[0,T] \times \mathbb{R}^n$.

Thus when L enjoys (L1) we may view $V(t,\cdot)$ as the initial function $V(0,\cdot) = \varphi$ propagated from time 0 to time t in a manner dictated by L. This process furnishes a regularization of φ.

Proof. (i) We start by observing that V is upper semicontinuous on $(0,T] \times \mathbb{R}^n$ as an immediate consequence of (4) and Lemma 3. Let $V(0,\cdot) = \varphi$ and let $(t_j, x_j) \to (0,x)$. Then

$$V(t_j, x_j) \le \varphi(x_j) + t_j L(x_j, 0)$$

and $\limsup_{j\to\infty} V(t_j, x_j) \le V(0,x)$ follows provided φ is upper semicontinuous.

Fix $(t,x) \in (0,T] \times \mathbb{R}^n$ and assume $(\omega, p) \in \partial^+ V(t,x)$. Let $v \in \mathbb{R}^n$ be arbitrary and choose an admissible arc X with $\dot{X} = v$ near t. Then

$$V(t-\varepsilon, X(t-\varepsilon)) + \int_{t-\varepsilon}^{t} L(X(\tau), \dot{X}(\tau))\, d\tau \ge V(t,x) \tag{5}$$

where in fact for sufficiently small $\varepsilon > 0$

$$X(\tau) = x - (t-\tau)v \quad \text{and} \quad \dot{X}(\tau) = v \quad \text{when} \quad \tau \in [t-\varepsilon, t].$$

But since $(\omega, p) \in \partial^+ V(t,x)$,

$$\begin{aligned} V(t-\varepsilon, X(t-\varepsilon)) &= V(t-\varepsilon, x - \varepsilon v) \\ &\le V(t,x) - \omega\varepsilon - \varepsilon\langle p, v\rangle + o(\varepsilon). \end{aligned} \tag{6}$$

On combining (5) and (6), dividing by ε, and then sending $\varepsilon \downarrow 0$, one finds

$$\omega + \langle p, v\rangle - \liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} L(x - (t-\tau)v, v)\, d\tau \le 0,$$

which becomes, in view of (L0),

$$\omega + \langle p, v\rangle - L(x,v) \le 0.$$

By taking the supremum over v one arrives at the inequality $\omega + H(x,p) \le 0$.

(ii) According to Lemma 3, Θ is locally Lipschitz continuous. Likewise, V is locally Lipschitz continuous. Indeed, given $(t_0, x_0) \in (0,T] \times \mathbb{R}^n$, the lower bound $\Theta(t,x,y) \ge tM(|x-y|/t)$ and the assumption $\varphi(y) \ge$

$-C(1+|y|)$ together imply the existence of a compact neighborhood $\mathcal{N}$ of (t_0,x_0), $\mathcal{N}\subset(0,T]\times\mathbb{R}^n$, and a compact set $K\subset\mathbb{R}^n$ such that

$$V(t,x)=\min\{\varphi(y)+\Theta(t,x,y);y\in K\},\qquad (t,x)\in\mathcal{N}.$$

It follows that V is Lipschitzian in $\mathcal{N}$.

Suppose $(\omega,p)\in\partial^-V(t,x)$. Let X be an optimal arc-choose first y so as to give the minimum in (4) and then choose an arc X that is optimal for $\Theta(t,x,y)$. Then

$$V(t,x)=\varphi(X(0))+\int_0^{t-\varepsilon}L(X(\tau),\dot X(\tau))\,d\tau+\int_{t-\varepsilon}^{t}L(X(\tau),\dot X(\tau))\,d\tau$$

and so

$$V(t,x)\geq V(t-\varepsilon,X(t-\varepsilon))+\int_{t-\varepsilon}^{t}L(X(\tau),\dot X(\tau))\,d\tau. \tag{7}$$

But

$$V(t-\varepsilon,X(t-\varepsilon))\geq V(t,x)-\omega\varepsilon-\int_{t-\varepsilon}^{t}\langle p,\dot X(\tau)\rangle\,d\tau+o(\varepsilon). \tag{8}$$

(The Lipschitz continuity of X ensures that the remainder is indeed of order $o(\varepsilon)$.) The conjunction of the last two inequalities, (7) and (8), yields

$$\omega+\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\{\langle p,\dot X(\tau)\rangle-L(X(\tau),\dot X(\tau))\}\,d\tau\geq o(1) \tag{9}$$

and so

$$\omega+\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}H(X(\tau),p)\,d\tau\geq o(1).$$

By letting ε approach zero one infers $\omega+H(x,p)\geq 0$.

We omit a proof of the stated convergence of $V(t,\cdot)$ to φ as $t\downarrow 0$ for that is a straightforward matter. □

Proposition 2 *Under* (L1), *let* (t_0,x_0) *be an arbitrary point in* $(0,T]\times\mathbb{R}^n$ *and let* $X\colon[0,t_0]\to\mathbb{R}^n$ *be a minimizing arc for the optimization problem defining* $V(t_0,x_0)$. *Then* X *is* C^1 *and*

$$\dot X(t)=H_p(X(t),p)\quad\text{whenever } t\in(0,t_0]\text{ and }(\omega,p)\in\partial^-V(t,X(t)).$$

Proof. We begin by recalling that for a locally Lipschitz continuous function u to be a viscosity solution of (HJ) it is actually necessary that the *equation* $\omega + H(x,p) = 0$ holds for any $(\omega,p) \in \partial^- u(t,x)$; study [20]. By the principle of optimality, the restriction of X to $[0,t]$ is optimal for $V(t,X(t))$ for any $t \in (0,t_0]$. Therefore, inequality (9) holds as long as $(\omega,p) \in \partial^- V(t,X(t))$ which comes out as

$$\omega + \langle p, \dot{X}(t)\rangle - L(X(t),\dot{X}(t)) \geq 0$$

in the limit for X is C^1. Since V is a locally Lipschitz continuous viscosity solution, $\omega + H(X(t),p) = 0$ and so

$$\langle p, \dot{X}(t)\rangle \geq L(X(t),\dot{X}(t)) + H(X(t),p)$$

which means $\dot{X}(t) = H_p(X(t),p)$ by convex analysis. □

A fundamental property of the value function is its maximality among subsolutions of (CP).

Theorem 2 *Assume* (L0) *and let u be a viscosity subsolution of* (HJ) *which is continuous on* $(0,T] \times \mathbb{R}^n$.
(i) *If $0 < s < t \leq T$ and $X\colon [s,t] \to \mathbb{R}^n$ is Lipschitz continuous, then*

$$u(t,X(t)) - u(s,X(s)) \leq \int_s^t L(X(\tau),\dot{X}(\tau))\,d\tau.$$

(ii) *Suppose* $\lim_{t\downarrow 0} u(t,x) \leq \varphi(x)$ *for every* $x \in \mathbb{R}^n$. *Then* $u \leq V$.

Proof. (i) Assume first X is piecewise affine, i.e., $\dot{X}$ is constant say equal to v_i on subintervals (t_{i-1},t_i), $i = 1,2,\ldots,I$, where $s = t_0 < t_1 < \cdots < t_I = t$. The hypothesis that u be a subsolution of (HJ) is equivalent to the inequality

$$u_t(t,x) + \langle u_x(t,x), v\rangle \leq L(x,v), \qquad (t,x) \in (0,T] \times \mathbb{R}^n,$$

holding in the viscosity sense for any $v \in \mathbb{R}^n$. Taking $v = v_i$ and invoking the result on directional derivatives obtained in [15, Thm. I.14] one deduces

$$u(t_i,X(t_i)) - u(t_{i-1},X(t_{i-1})) \leq \int_{t_{i-1}}^{t_i} L(X(\tau),v_i)\,d\tau.$$

Through summation of these inequalities, one infers the desired conclusion.

In the general case one approximates X by piecewise affine equi-Lipschitzian functions X_j such that $X_j(s) = X(s)$, $X_j(t) = X(t)$, and $X_j \to X$ uniformly while $\dot{X}_j \to \dot{X}$ a.e. as $j \to \infty$ (consult e.g. [19, Chap. X]).

The first part of the proof applied to X_j yields

$$u(t, X(t)) - u(s, X(s)) \leq \int_s^t L(X_j(\tau), \dot{X}_j(\tau))\, d\tau.$$

One completes the proof by passing to the limit as $j \to \infty$.

(ii) Let $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$ and let $X: [0, t_0] \to \mathbb{R}^n$ be a Lipschitz continuous curve with $X(t_0) = x_0$. Consider the function u^ε defined on $[0, T-\varepsilon]$ by $u^\varepsilon(t, x) = u(t+\varepsilon, x)$; u^ε is clearly a viscosity subsolution of the Hamilton-Jacobi equation in $(0, T-\varepsilon] \times \mathbb{R}^n$. Applying the result obtained in (i) to u^ε, assuming also $\varepsilon \in (0, T-t_0)$, one finds

$$u^\varepsilon(t_0, x_0) \leq u^\varepsilon(s, X(s)) + \int_s^{t_0} L(X(\tau), \dot{X}(\tau))\, d\tau$$

when $0 < s < t_0 < T - \varepsilon$. Sending, in order, $s \downarrow 0$ and $\varepsilon \downarrow 0$ leads to

$$u(t_0, x_0) \leq \varphi(X(0)) + \int_0^{t_0} L(X(\tau), \dot{X}(\tau))\, d\tau.$$

Thus $u(t_0, x_0) \leq V(t_0, x_0)$. □

4. Uniqueness and comparison results

The intricate and subtle uniqueness problem is addressed in this section. First we attempt to handle the situation where among (H0), (L0), (L1), merely (H0) is assumed. Thus, in the possible absence of (L1), the problem defining $V(t, x)$ need not have an optimal solution or for that matter be a natural problem. Nevertheless, we will exhibit sufficient conditions for the comparison principle. This will be achieved under certain additional structure conditions on the Hamiltonian function as well as certain growth limitations on the gradients of the considered subsolutions.

Second we prove a uniqueness theorem in the presence of (L1). That condition guarantees, as we have seen, a well-behaved problem inasmuch as it entails the existence of minimizing Lipschitz (even C^1) arcs in the definition of the value function and, moreover, ensures that the value function does solve (CP). It turns out that it also ensures a characterization of the value function as the unique viscosity solution of (CP) that is bounded from below by a function of linear growth.

Definition 6 (Comparison principle) Let $u, w: [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be, respectively, a viscosity subsolution and a viscosity supersolution of (HJ). Let also u and w be upper and lower semicontinuous on $[0,T]\times\mathbb{R}^n$, respectively. We say that the *comparison principle* holds for (u,w) if

$$\sup_{[0,T]\times\mathbb{R}^n}(u-w)^+=\sup_{\mathbb{R}^n}(u(0,\cdot)-w(0,\cdot))^+.$$

Here $\alpha^+=\max\{0,\alpha\}$.

The following hypothesis on H has via precursors emerged as vital and crucial for the comparison principle. This version is invariant under a transformation of the independent variables.

(H1) For every $r>0$ there exists a sequence of lower semicontinuous functions $\theta_j\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}\cup\{\infty\}$ such that (i) θ_j both converges pointwise and epi-converges to the indicator function of Δ (or $\Delta\cap(\bar{B}_r\times\bar{B}_r)$) as $j\to\infty$; and (ii) each point $(\bar{x},\bar{x})\in\Delta$, $\bar{x}\in B_r$, has a neighborhood $\mathcal{N}\subset B_r\times B_r$ throughout which θ_j is finite and differentiable eventually as j becomes large and

$$H(x,\nabla_x\theta_j(x,y))-H(y,-\nabla_y\theta_j(x,y))\geq-\Lambda(\theta_j(x,y),x,y)\tag{10}$$

holds when $(x,y)\in\mathcal{N}$ and j is sufficiently large, with $\Lambda(\theta,x,y)\to 0$ as $(\theta,x,y)\to(0,\bar{x},\bar{x})$.

A prototype for θ_j is $\theta_j(x,y)=j|x-y|^2/2$.

Remark 2. For ease of presentation we refrain from the extra generality that would be gained from choosing a sequence of functions $\theta_j(t,x,y)$ depending also on t. The counterpart of (10) should then read

$$\partial\theta_j(t,x,y)/\partial t+H(x,\nabla_x\theta_j(t,x,y))-H(y,-\nabla_y\theta_j(t,x,y))\\ \geq-\Lambda(\theta_j(t,x,y),x,y).$$

Example 1. When $H(x,p)$ is separable in accordance with

$$H(x,p)=\Phi(x)+\Psi(p),$$

and Φ is continuous, then (H1) holds trivially because

$$H(x,j(x-y))-H(y,j(x-y))=\Phi(x)-\Phi(y).$$

Hence, $\theta_j(x,y)=j|x-y|^2/2$ together with $\Lambda(\theta,x,y)=\Phi(y)-\Phi(x)$ will serve.

The following conditions on a vector field a should be familiar from the uniqueness theory for ordinary differential equations.

Example 2. (Cf. [12]) Let $H(x,p) = \langle a(x), p\rangle$ be a linear Hamiltonian. Then (H1) holds with $\theta_j(x,y) = j|x-y|^2/2$ if for each $r > 0$ there exists a constant $C_r \geq 0$ such that

$$\langle a(x) - a(y), x - y\rangle \geq -C_r|x-y|^2$$

when $x \in B_r$, $y \in B_r$. (H1) remains true if a satisfies the condition that results from replacing "$C_r|x-y|^2$" by "$|x-y|\chi_r(|x-y|)$," where $\chi = \chi_r\colon [0,\infty) \to [0,\infty)$ is any continuous function such that $\varrho \mapsto \varrho\chi(\varrho)$ is nondecreasing, $\chi(\varrho) > 0$ if $\varrho > 0$, and

$$\int_0^1 \frac{d\varrho}{\chi(\varrho)} = \infty.$$

Proof. Let

$$f(s) = \int_s^1 \frac{d\varrho}{\chi(\varrho)}, \qquad s > 0,$$

fix a sequence $\varepsilon_j \downarrow 0$, and let

$$\theta_j(x,y) = \exp\left\{f(\varepsilon_j) - 2f\left(\left(\varepsilon_j^2 + |x-y|^2\right)^{1/2}\right)\right\}.$$

Then $\theta_j \in C^1(\mathbb{R}^n \times \mathbb{R}^n)$ and a calculation yields

$$H(x, \nabla_x\theta_j(x,y)) - H(y, -\nabla_y\theta_j(x,y)) \geq -2\theta_j(x,y), \quad x, y \in B_r,$$

so one may take $\Lambda(\theta, x, y) = 2\theta$. The elementary verification that θ_j converges as required is left to the reader. □

A problem intimately related to the uniqueness problem for (CP) is to find sufficient conditions for a solution of (CP) to equal the value function in circumstances where it is a priori unknown whether the latter solves (HJ). It is, we emphasize, possible that the Cauchy problem has a solution distinct from the value function.

Example 3. Let $a\colon \mathbb{R} \to \mathbb{R}$ be continuous, $a \geq 1$, and

$$\int_{-\infty}^0 \frac{dx}{a(x)} < \infty.$$

Let $\varphi = 0$, $H(x,p) = p^2/2 + a(x)p$ and so $L(x,v) = (v - a(x))^2/2$. Then

(CP) has the solution $w = 0$ which does not coincide with the value function V since

$$V(t,x) \geq \frac{1}{2t}(t - b(x))^2 > 0 \qquad \text{when } \ t > b(x) := \int_{-\infty}^{x} \frac{d\xi}{a(\xi)}.$$

Proof. Let (t, x) be a point such that $t > b(x)$. Let X be an admissible arc, i.e. a Lipschitz continuous function on $[0, t]$ with $X(t) = x$, and set $Y(\tau) = b(X(\tau))$. Then

$$\begin{aligned}
&\int_0^t L(X(\tau), \dot{X}(\tau))\, d\tau \\
&\quad = \frac{1}{2}\int_0^t a\,(X(\tau))^2 \left(\dot{Y}(\tau) - 1\right)^2 d\tau \\
&\quad \geq \frac{t}{2}\int_0^t \left(\dot{Y}(\tau) - 1\right)^2 d\tau/t \geq \frac{t}{2}\left(\int_0^t \left(\dot{Y}(\tau) - 1\right) d\tau/t\right)^2 \\
&\quad = \frac{1}{2t}(b(x) - Y(0) - t)^2 \geq \frac{1}{2t}(t - b(x))^2.
\end{aligned}$$

In this string of inequalities we have used, in succession, $a \geq 1$, Jensen's integral inequality, and $Y(0) \geq 0$. □

The significance of conditions analogous to (H1) has been pointed out in many papers. However, (H1) accompanied by (H0) is insufficient for the comparison principle. (Nevertheless, if L obeys (L1) we may dispense with (H1) as we shall see later.)

Example 4. In Example 3 the comparison principle evidently fails for the pair (V, w), V being a subsolution of (HJ) according to Theorem 1. In that instance,

$$H_p(x, w_x(t,x)) = a(x) \quad \text{and} \quad \int_{-\infty}^{0} \frac{dx}{a(x)} < \infty.$$

If we take $a(x) = 1 + x^2$ we find that (H1) holds. In fact, (H1) is fulfilled whenever a satisfies that condition which is formulated in Example 2.

Our next example exploits [7, 28]. In particular, it illustrates the fact that failure of the comparison principle does not rule out uniqueness.

Example 5. Let $H(p) = |p|^\alpha$ with $\alpha > 1$. If $1 < \beta < \gamma$ are appropriately chosen, the function $w(t,x) = \min\{0, |x|^\beta - t|x|^\gamma\}$ becomes a supersolution of (HJ) having $w(0,x) = 0$ and $w(t,x) \to -\infty$ as $|x| \to \infty$ if $t > 0$; refer to [7]. The comparison principle fails consequently for (u,w) if u equals the constant 0. However, solutions of (CP) are unique in this case [28].

Our first theorem below implies that in the presence of (H0) and (H1), the comparison principle is true when w is such that the function

$$(t,x) \mapsto \langle H_p(x, w_x(t,x)), x\rangle$$

admits some minorant that is a summable function of t times a quadratic function of x. To be exact we have the following condition in mind.

(G) There exist continuous functions $\gamma\colon (0,T] \to [0,\infty)$ and $a\colon [0,\infty) \to (0,\infty)$ with $\varrho \mapsto \varrho a(\varrho)$ nondecreasing such that

$$\int_0^T \gamma(t)\,dt < \infty, \quad \int_1^\infty \frac{d\varrho}{a(\varrho)} = \infty,$$

and

$$\sup\{\langle v, x\rangle; v \in \partial_p H(x,p)\} \geq -\gamma(t)|x|a(|x|)$$

if $(t,x) \in (0,T] \times \mathbb{R}^n$, $p \in \partial_x^- w(t,x)$.

In one version of our results u or w will be subject to a Lipschitz condition under which, it turns out, (H1) is redundant.

(LC) For every compact $K \subset (0,T] \times \mathbb{R}^n$ there exists a constant $C \geq 0$ such that

$$|w(t,x) - w(t,y)| \leq C|x-y| \quad \text{when} \quad (t,x) \in K, \ (t,y) \in K.$$

Before stating our first theorem in this section, we present a preparatory result which is the analogue in the setting of semicontinuous functions of a lemma used already in early papers on viscosity solutions [16, Lemma 2].

Lemma 4 *Assume* (H0). *Let u and w be a subsolution and a supersolution of* (HJ), *respectively, and let $U\colon (0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be defined by*

$$U(t,x,y) = u(t,x) - w(t,y) \quad \textit{for all} \quad (t,x,y) \in (0,T] \times \mathbb{R}^n \times \mathbb{R}^n.$$

Then U is a subsolution of

$$U_t + H(x, U_x) - H(y, -U_y) = 0, \quad (t,x,y) \in (0,T] \times \mathbb{R}^n \times \mathbb{R}^n.$$

Proof. Let $\Psi \in C^1$ and suppose $U - \Psi$ has a strict local maximum relative to $(0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ at $(\bar{t},\bar{x},\bar{y})$. Choose $\mathcal{N} \subset (0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, a compact neighborhood of $(\bar{t},\bar{x},\bar{y})$, in such a manner that

$$(U-\Psi)(t,x,y) < (U-\Psi)(\bar{t},\bar{x},\bar{y}) \text{ when } (t,x,y) \in \mathcal{N} \setminus \{(\bar{t},\bar{x},\bar{y})\}.$$

Select for each $j \in \mathbb{N}$ a point (s_j,t_j,x_j,y_j) maximizing

$$(s,t,x,y) \mapsto u(t,x) - w(s,y) - \Psi(t,x,y) - j(t-s)^2/2$$

over $(t,x,y) \in \mathcal{N}$, $(s,x,y) \in \mathcal{N}$. Then (s_j,t_j,x_j,y_j) converges to $(\bar{t},\bar{t},\bar{x},\bar{y})$ as $j \to \infty$. Thus for large j

$$\begin{aligned} (\Psi_t(t_j,x_j,y_j) + j(t_j-s_j), \Psi_x(t_j,x_j,y_j)) &\in \partial^+ u(t_j,x_j), \\ (j(t_j-s_j), -\Psi_y(t_j,x_j,y_j)) &\in \partial^- w(s_j,y_j), \end{aligned}$$

and so

$$\begin{aligned} \Psi_t(t_j,x_j,y_j) + j(t_j-s_j) + H(x_j,\Psi_x(t_j,x_j,y_j)) &\leq 0, \\ j(t_j-s_j) + H(y_j,-\Psi_y(t_j,x_j,y_j)) &\geq 0. \end{aligned}$$

Subtraction of the preceding inequalities leads to

$$\Psi_t(t_j,x_j,y_j) + H(x_j,\Psi_x(t_j,x_j,y_j)) - H(y_j,-\Psi_y(t_j,x_j,y_j)) \leq 0$$

and hence

$$\Psi_t(\bar{t},\bar{x},\bar{y}) + H(\bar{x},\Psi_x(\bar{t},\bar{x},\bar{y})) - H(\bar{y},-\Psi_y(\bar{t},\bar{x},\bar{y})) \leq 0,$$

and we may conclude. □

Theorem 3 *Assume* (H0). *Let u and w be upper semicontinuous and lower semicontinuous on $[0,T] \times \mathbb{R}^n$, respectively, and assume also u and w are a viscosity subsolution and a viscosity supersolution of* (HJ), *respectively. Assume moreover that w satisfies* (G) *and that either* (H1) *holds or one of u and w satisfies the Lipschitz condition* (LC). *Then the comparison principle holds true for the pair (u,w),*

$$\sup_{[0,T]\times\mathbb{R}^n} (u-w)^+ = \sup_{\mathbb{R}^n} (u(0,\cdot) - w(0,\cdot))^+.$$

Proof. We will adopt a local approach by means of a family of nested compact sets S_β exhausting $[0,T] \times \mathbb{R}^n$. So we start by introducing the auxiliary function

$$f(t,x) = \int_0^t \gamma(\tau)\,d\tau + \int_1^{(1+|x|^2)^{1/2}} \frac{d\varrho}{a(\varrho)}, \qquad (t,x) \in [0,T] \times \mathbb{R}^n,$$

and its sublevel sets

$$S_\beta = \{(t,x) \in [0,T] \times \mathbb{R}^n; f(t,x) \le \beta\}, \qquad \beta > 0,$$

which are compact because $\int_1^\infty 1/a = \infty$. We next let $g\colon (0,\infty) \to (0,\infty)$ be nondecreasing and such that

$$\exp g(\beta) > \max\{u(t,x) - w(t,x); (t,x) \in S_\beta\}, \qquad \beta > 0, \tag{11}$$

and put

$$\psi_\beta(t,x) = \exp\{g(\beta)(1 + f(t,x) - \beta)\}.$$

The smooth functions ψ_β are chosen so as to solve the differential inequality

$$\psi_t(t,x) - \gamma(t)a(|x|)|\psi_x(t,x)| \ge 0, \qquad (t,x) \in (0,T] \times \mathbb{R}^n, \tag{12}$$

and to satisfy

$$\lim_{\beta\to\infty} \psi_\beta(t,x) = 0 \qquad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^n. \tag{13}$$

It will suffice to prove that

$$u(t,x) - w(t,x) - \psi_\beta(t,x) \le \sup_{\mathbb{R}^n}(u(0,\cdot) - w(0,\cdot))^+ \qquad \text{for any } \beta > 0,\ (t,x) \in S_\beta. \tag{14}$$

Indeed, if (14) were true and $(t,x) \in [0,T] \times \mathbb{R}^n$, then one would arrive at the desired inequality

$$u(t,x) - w(t,x) \le \sup_{\mathbb{R}^n}(u(0,\cdot) - w(0,\cdot))^+$$

by sending $\beta \to \infty$ and utilizing (13). In order to establish statement (14) we argue by contradiction and assume that it is false so that for a certain β, fixed hereafter, there exists a $c > 0$ such that

$$A := \max_{(t,x)\in S}(u(t,x) - w(t,x) - \psi(t,x) - ct) > \sup_{\mathbb{R}^n}(u(0,\cdot) - w(0,\cdot))^+ =: B \tag{15}$$

and B is finite. The subscripts β have been suppressed in order to simplify the notation, i.e., S and ψ are used in place of S_β and ψ_β, respectively. As has become standard, let us approximate the left-hand maximization problem in (15) and in doing so we consider

$$\Phi_j(t,x,y) = u(t,x) - w(t,y) - \psi(t,y) - ct - \theta_j(x,y),$$

where the θ_j's serve as penalty functions. Condition (H1), when assumed, furnishes the choice of θ_j (let first $r > 0$ be so large that $(t,x) \in S$ implies $x \in B_r$ and then let (H1) supply θ_j), while $\theta_j(x,y) = j|x-y|^2/2$ if (H1) is not a premise. Select a point (t_j, x_j, y_j) maximizing Φ_j over the compact set $\{(t,x,y); (t,x) \in S, (t,y) \in S\}$. If necessary by extracting a convergent subsequence, we may assume that (t_j, x_j, y_j) tends to a limit necessarily of the form $(\bar{t}, \bar{x}, \bar{x})$ (see Lemma 2). We claim moreover that $\bar{t} \in (0,T]$ and $f(\bar{t}, \bar{x}) < \beta$. In the remaining part of the proof we may therefore assume, without loss of generality, that $t_j \in (0,T]$, $f(t_j, x_j) < \beta$, $f(t_j, y_j) < \beta$, and that θ_j is differentiable at (x_j, y_j).

To avoid obscuring the idea of the proof we will accept this as true for the moment and proceed by taking advantage of Lemma 4 to find that

$$\begin{aligned} c + \psi_t(t_j, y_j) &+ H(x_j, \theta_{jx}(x_j, y_j)) \\ &- H(y_j, -\theta_{jy}(x_j, y_j) - \psi_y(t_j, y_j)) \le 0. \end{aligned} \tag{16}$$

(A clarifying comment about the notation might be in order: $\theta_{jx}(x_j, y_j)$ stands for the partial gradient $\nabla_x \theta_j(x_j, y_j)$, etc.) Let us break up (16) as

$$\begin{aligned} \psi_t(t_j, y_j) + H(y_j, -\theta_{jy}(x_j, y_j)) &- H(y_j, -\theta_{jy}(x_j, y_j) - \psi_y(t_j, y_j)) \\ + c + H(x_j, \theta_{jx}(x_j, y_j)) &- H(y_j, -\theta_{jy}(x_j, y_j)) \le 0. \end{aligned} \tag{17}$$

To treat the terms on the first line of the preceding inequality, notice that

$$y \mapsto w(t_j, y) + \theta_j(x_j, y) + \psi(t_j, y)$$

attains a local minimum at y_j so that

$$p_j := -\theta_{jy}(x_j, y_j) - \psi_y(t_j, y_j) \in \partial_y^- w(t_j, y_j). \tag{18}$$

Therefore, by (G) and the differential inequality (12),

$$\begin{aligned} &\psi_t(t_j, y_j) + H(y_j, -\theta_{jy}(x_j, y_j)) - H(y_j, -\theta_{jy}(x_j, y_j) - \psi_y(t_j, y_j)) \\ &\quad = \psi_t(t_j, y_j) + H(y_j, p_j + \psi_y(t_j, y_j)) - H(y_j, p_j) \\ &\quad \ge \psi_t(t_j, y_j) - \gamma(t_j) a(|y_j|) |\psi_y(t_j, y_j)| \ge 0. \end{aligned} \tag{19}$$

(We have taken into account that the gradient $\psi_y(t,y)$ is of the form $\mu(t,y)y$ with $\mu(t,y) \geq 0$, which was required to invoke (G).) The inequalities (17) and (19) jointly imply

$$c + H(x_j, \theta_{jx}(x_j, y_j)) - H(y_j, -\theta_{jy}(x_j, y_j)) \leq 0. \tag{20}$$

In the event (H1) is assumed, therefore,

$$0 \geq c - \Lambda(\theta_j(x_j, y_j), x_j, y_j) \to c, \qquad j \to \infty,$$

since $(\theta_j(x_j, y_j), x_j, y_j) \to (0, \bar{x}, \bar{x})$ as $j \to \infty$ according to Lemma 2. (Here of course Λ denotes the function furnished by (H1).) Hence $c \leq 0$, in violation with our choice of c.

If instead w satisfies the partial Lipschitz condition (LC) (one argues similarly if u satisfies (LC)), and (H1) is not a hypothesis, then $\theta_j(x,y) = j|x-y|^2/2$ and p_j has a convergent subsequence since the p_j's form a bounded sequence as a consequence of (18) and (LC). Then $j(x_j - y_j)$ too has a convergent subsequence as $j \to \infty$. By passing to the limit in (20), i.e. in

$$c + H(x_j, j(x_j - y_j)) - H(y_j, j(x_j - y_j)) \leq 0,$$

we again reach the contradiction $c \leq 0$.

It remains to establish the claim about the limit of (t_j, x_j, y_j) as $j \to \infty$ whose proof was deferred. Toward this end, assume the contrary to the claim, i.e., either $\bar{t} = 0$ or $f(\bar{t}, \bar{x}) = \beta$. Lemma 2 tells us that

$$A = u(\bar{t}, \bar{x}) - w(\bar{t}, \bar{x}) - \psi(\bar{t}, \bar{x}) - c\bar{t}. \tag{21}$$

If $\bar{t} = 0$, (21) implies

$$A = u(0, \bar{x}) - w(0, \bar{x}) - \psi(0, \bar{x}) \quad \text{and so} \quad A < B$$

which is in conflict with (15). On the other hand, $f(\bar{t}, \bar{x}) = \beta$ implies $\psi(\bar{t}, \bar{x}) = \exp g(\beta)$ and hence, by (11),

$$A = u(\bar{t}, \bar{x}) - w(\bar{t}, \bar{x}) - \exp g(\beta) - c\bar{t} < 0$$

and we are once again confronted with an inconsistency. □

Our ultimate goal is to identify the value function.

Corollary 1 *In the presence of* (L0), (H0), *suppose* $w \in C([0,T] \times \mathbb{R}^n)$ *solves* (CP) *and satisfies* (G). *If in addition either* (H1) *is fulfilled or one* *of w and V satisfies the Lipschitz condition* (LC), *then $w = V$.*

Proof. One has $w \le V$ in view of the maximality property of the value function stated in Theorem 2. The reverse inequality follows from the comparison principle: apply Theorem 3 to the pair (V, w), the value function V being a subsolution according to Theorem 1. □

Corollary 2 *Assume* (H0), (H1), *and*

$$H(x, p + \lambda x) - H(x, p) \ge -\lambda |x| a(|x|) \quad \text{for all } x, p \in \mathbb{R}^n, \ \lambda \ge 0,$$

where a has the properties stated in (G). *Let φ be continuous. Then the initial-value problem* (CP) *has at most one viscosity solution $u \in C([0,T] \times \mathbb{R}^n)$.*

We proceed to attend to the case where L enjoys (L1). First we give a comparison result for bounded regions in which we do not insist, as an exception from the general hypotheses of the paper, that $H(x,p)$ be a convex function of p.

Proposition 3 *Assume* (H0). *Let B be a bounded open subset of $\mathbb{R}^n$, and let $u, w \colon [0,T] \times \bar{B} \to \mathbb{R}$ be upper semicontinuous and lower semicontinuous, respectively. Let also u and w be a viscosity subsolution and a viscosity supersolution of $u_t + H(x, u_x) = 0$ in $(0,T] \times B$, respectively. Assume moreover either that* (H1) *holds or that one of u and w satisfies a Lipschitz condition on each compact subset of $(0,T] \times B$. If furthermore $u \le w$ on the "lower boundary"*

$$\{(t,x); (t,x) \in \{0\} \times \bar{B} \ \text{ or } \ (t,x) \in (0,T] \times \partial B\},$$

then $u \le w$ everywhere in $[0,T] \times \bar{B}$.

Proof. Assume on the contrary that $u > w$ somewhere, which implies the existence of a $c > 0$ such that

$$\max\{u(t,x) - w(t,x) - ct; (t,x) \in [0,T] \times \bar{B}\} > 0. \tag{22}$$

Next introduce

$$\Phi_j(t,x,y) = u(t,x) - w(t,y) - ct - \theta_j(x,y),$$

and consider a point (t_j, x_j, y_j) that maximizes $\Phi_j(t,x,y)$ over $(t,x) \in [0,T] \times \bar{B}$, $(t,y) \in [0,T] \times \bar{B}$. (The functions θ_j are either supplied by condition (H1) or are otherwise taken as $\theta_j(x,y) = j|x-y|^2/2$.) One may assume $(t_j, x_j, y_j) \to (\bar{t}, \bar{x}, \bar{x})$ as $j \to \infty$. It is straightforward to verify that $(\bar{t}, \bar{x}) \in (0,T] \times B$ so that $(t_j, x_j, y_j) \in (0,T] \times B \times B$ eventually as j becomes large. Indeed, similarly as above one sees that

$$\begin{aligned} &u(\bar{t},\bar{x}) - w(\bar{t},\bar{x}) - c\bar{t} \\ &\qquad = \max\{u(t,x) - w(t,x) - ct; (t,x) \in [0,T] \times \bar{B}\}, \end{aligned}$$

which is inconsistent with the conjunction of (22) and the hypothesis that $u \leq w$ on the lower boundary unless $(\bar{t}, \bar{x}) \in (0,T] \times B$. Thus, via Lemma 4,

$$c + H(x_j, \nabla_x \theta_j(x_j, y_j)) - H(y_j, -\nabla_y \theta_j(x_j, y_j)) \leq 0. \tag{23}$$

If u or w satisfies a Lipschitz condition, and $\theta_j(x,y) = j|x-y|^2/2$, then $\nabla_x \theta_j(x_j, y_j) = -\nabla_y \theta_j(x_j, y_j) = j(x_j - y_j)$ remains bounded as $j \to \infty$, and a passage to the limit along a convergent subsequence in (23) leads to $c \leq 0$. On the other hand, if (H1) holds then $c \leq 0$ follows similarly as in the proof of Theorem 3. But $c \leq 0$ contradicts the choice of c above. □

Remark 3. It is known that the Cauchy problem for the linear partial differential equation determined by $H(x,p) = \langle a(x), p \rangle$ may have multiple compactly supported viscosity solutions when (H1) fails; see [15, 12].

In order to avoid a cumbersome notation, let us adopt the abbreviation

$$J(X; s, t) = \int_s^t L(X(\tau), \dot{X}(\tau))\, d\tau.$$

Theorem 4 *In the presence of* (L1), *let u be a viscosity solution of* (CP), *with φ a lower semicontinuous extended-real-valued function. Then the following three assertions are true.*

(i) *Let $\varrho > 0$. Then*

$$\begin{aligned} u(t,x) &= \inf\{u(s,y) + \Theta(t-s,x,y); \\ &\qquad (s,y) \in (\{0\} \times \bar{B}_\varrho) \cup ((0,t) \times \partial B_\varrho)\} \end{aligned}$$

for all $(t,x) \in (0,T] \times B_\varrho$. Moreover, u is locally Lipschitz continuous.

(ii) *Suppose furthermore u admits a minorant of linear growth, i.e., there exists a constant C such that $u(t,x) \geq -C(1+|x|)$ for all $(t,x) \in (0,T] \times \mathbb{R}^n$. Then $u = V$.*

(iii) $u = V$ *if* Θ *satisfies the following condition: for every* $x \in \mathbb{R}^n$ *and every* $\varepsilon \in (0, T)$

$$\lim_{|y|\to\infty} \inf\{\Theta(t_1, x, y) - \Theta(t_2, x, y); t_1, t_2 \in (0, T],\ t_2 - t_1 \geq \varepsilon\} = \infty.$$

Proof. Let us first also assume u is finite and continuous on $[0, T] \times \mathbb{R}^n$. Define for $\varrho > 0$,

$$\begin{aligned} w^\varrho(t,x) &= \inf\{u(s, X(s)) + J(X; s, t); s \in [0, t), X\colon [s, t] \to \mathbb{R}^n \\ &\quad \text{is Lipschitzian, } (s, X(s)) \in (\{0\} \times \bar{B}_\varrho) \cup ((0, t) \times \partial B_\varrho), X(t) = x\} \\ &= \inf\{u(s, y) + \Theta(t - s, x, y); (s, y) \\ &\qquad \in (\{0\} \times \bar{B}_\varrho) \cup ((0, t) \times \partial B_\varrho)\} \end{aligned}$$

for all $(t, x) \in (0, T] \times B_\varrho$. (Note that, unlike the approach in [24, Chap. 11], [9], no state constraint is incorporated into the definition of w^ϱ.) By arguing similarly as in the proof of Theorem 1, we deduce that $w^\varrho_t + H(x, w^\varrho_x) = 0$ is fulfilled in $(0, T] \times B_\varrho$, and also that w^ϱ is locally Lipschitz continuous. In addition, w^ϱ extends to a continuous function on $[0, T] \times \bar{B}_\varrho$ which agrees with u on $(\{0\} \times \bar{B}_\varrho) \cup ((0, T] \times \partial B_\varrho)$. In order to establish this claim, let us first observe that the inequality

$$u(t, X(t)) - u(s, X(s)) \leq J(X; s, t),$$

which is imported from Theorem 2 (i), implies $w^\varrho \geq u$ in $(0, T] \times B_\varrho$ (cf. the "compatibility condition" in [8, 24]). Let $(t_0, x_0) \in (0, T] \times \partial B_\varrho$. Directly from the definition of w^ϱ we see that

$$w^\varrho(t, x) \leq u(s, x_0) + \Theta(t - s, x, x_0), \qquad 0 < s < t.$$

Hence

$$\limsup_{(t,x)\to(t_0,x_0)} w^\varrho(t, x) \leq u(s, x_0) + \Theta(t_0 - s, x_0, x_0), \quad 0 < s < t_0,$$

and sending $s \uparrow t_0$ leads to

$$\limsup_{(t,x)\to(t_0,x_0)} w^\varrho(t, x) \leq u(t_0, x_0).$$

If instead $(t, x) \to (0, x_0)$, $x_0 \in \bar{B}_\varrho$, then

$$w^\varrho(t, x) \leq u(0, x) + \Theta(t, x, x) \leq u(0, x) + tL(x, 0)$$

and so

$$\limsup_{(t,x)\to(0,x_0)} w^\varrho(t,x) \le u(0,x_0).$$

We conclude that

$$\lim_{(t,x)\to(t_0,x_0)} w^\varrho(t,x) = u(t_0,x_0) \qquad \text{if } (t_0,x_0) \in (\{0\}\times \bar B_\varrho) \cup ((0,T]\times \partial B_\varrho),$$

as claimed. Now the equation $w^\varrho = u$ in $(0,T]\times B_\varrho$ follows by calling upon Proposition 3. In particular, u is locally Lipschitz continuous which completes the proof of (i).

As regards (ii), it suffices to prove $u \ge V$. Let $(t,x) \in (0,T]\times \mathbb{R}^n$. We have demonstrated that $u(t,x) = w^\varrho(t,x)$ as long as $\varrho > |x|$, i.e.,

$$\begin{aligned} u(t,x) \;=\; & \inf\{u(0,y) + \Theta(t,x,y); y \in \bar B_\varrho\} \\ & \wedge \inf\{u(s,y) + \Theta(t-s,x,y); (s,y) \in (0,t)\times \partial B_\varrho\} \end{aligned}$$

which we rewrite in short as $u(t,x) = f^\varrho(t,x) \wedge g^\varrho(t,x)$. By utilizing the assumption that $u(s,y) \ge -C(1+|y|)$, as well as the monotonicity of $\lambda \mapsto \lambda(M(\tau/\lambda) - M(0))$ and that of M, we easily derive the lower bound

$$\begin{aligned} g^\varrho(t,x) \;\ge\; & \inf\{u(s,y) + (t-s)M(|x-y|/(t-s)); s \in (0,t), |y| = \varrho\} \\ \ge\; & -C'(1+\varrho) + tM((\varrho - |x|)/t). \end{aligned}$$

It is now clear that $g^\varrho(t,x) \to \infty$ as $\varrho \to \infty$ and hence $u(t,x) = f^\varrho(t,x)$ for large ϱ. But $f^\varrho(t,x) \ge V(t,x)$ and so $u(t,x) \ge V(t,x)$ which was to be shown.

(iii) Let $(t,x) \in (0,T)\times \mathbb{R}^n$. Theorem 2 (i) implies

$$u(T,x) - u(s,y) \le \Theta(T-s,x,y)$$

and so

$$\begin{aligned} g^\varrho(t,x) \;\ge\; & u(T,x) \\ & + \inf\{\Theta(t-s,x,y) - \Theta(T-s,x,y); s \in (0,t), |y| = \varrho\}. \end{aligned}$$

Hence, by the additional assumption in (iii), $\lim_{\varrho\to\infty} g^\varrho(t,x) = \infty$ and the proof is completed similarly as in (ii).

We have now demonstrated the theorem under the assumption that u belongs to $C([0,T]\times\mathbb{R}^n)$. In the general case, we form translations in time:

for $\varepsilon \in (0,T)$ we put

$$u^\varepsilon(t,x) = u(t+\varepsilon, x) \qquad \text{for all } \ (t,x) \in [0, T-\varepsilon] \times \mathbb{R}^n.$$

Then u^ε is continuous on $[0, T-\varepsilon] \times \mathbb{R}^n$, $u^\varepsilon(0,x) = u(\varepsilon, x)$, and

$$u_t^\varepsilon(t,x) + H(x, u_x^\varepsilon(t,x)) = 0 \qquad \text{for } \ (t,x) \in (0, T-\varepsilon] \times \mathbb{R}^n$$

in the viscosity sense. Let us focus on (ii). By the above considerations, u^ε is equal to the value function determined by the initial function $u(\varepsilon, \cdot)$, i.e.

$$u^\varepsilon(t,x) = \inf\{u(\varepsilon, y) + \Theta(t,x,y); y \in \mathbb{R}^n\}, \qquad (t,x) \in (0, T-\varepsilon] \times \mathbb{R}^n.$$

For every $(t,x) \in (0,T) \times \mathbb{R}^n$ there exists a compact subset $K \subset \mathbb{R}^n$ such that

$$u^\varepsilon(t,x) = \min\{u(\varepsilon, y) + \Theta(t,x,y); y \in K\}, \quad \varepsilon \in (0, T-t],$$

for $u(\varepsilon, y) \geq -C(1+|y|)$, $\Theta(t,x,y)/|y| \to \infty$ as $|y| \to \infty$, and $u^\varepsilon(t,x)$ is a bounded function of ε. By virtue of Proposition 1, $u(\varepsilon, \cdot) \to \varphi$ in the sense of both pointwise convergence and epi-convergence as $\varepsilon \downarrow 0$ and one finds, via Lemma 1,

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x) = \min\{\varphi(y) + \Theta(t,x,y); y \in K\} \geq V(t,x),$$

where also the continuity of Θ has been taken into account. But $u(t,x) = \lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x)$ and one infers $u = V$ in $(0,T) \times \mathbb{R}^n$. Thus $u = V$ in $(0,T] \times \mathbb{R}^n$ through continuity. □

Remark 4. The condition employed in (iii) (in conjunction with (L1)) ensures uniqueness. This condition might seem opaque but it is known to be true when $H(x,p) \equiv H(p)$, consult [28]. It is not difficult to see that it also holds when $L(x,v)$ has a certain homogeneity property in v, but we refrain from presenting such versions here. Note however the following easy perturbation result: if the condition is satisfied for a certain Lagrangian L, then it remains true for $\tilde{L} = L + B$ if B is a bounded function.

5. A concluding remark

Our results in the preceding section encompass of course not only problems from the calculus of variations. By way of illustration, let us consider the following Bolza problem arising in optimal control theory. Given $(t,x) \in (0,T] \times \mathbb{R}^n$ we are to minimize

$$\varphi(X(0)) + \int_0^t \mathcal{L}(X(\tau), Z(\tau))\, d\tau$$

over state-control pairs of functions $X: [0,t] \to \mathbb{R}^n$, $Z: [0,t] \to \mathbb{R}^m$, X absolutely continuous, Z measurable, obeying the ordinary differential equation and the terminal condition

$$\dot{X}(\tau) = f(X(\tau), Z(\tau)) \quad \text{a.e. } \tau \in [0,t], \qquad X(t) = x,$$

respectively, as well as the constraint $Z(\tau) \in \mathcal{Z}$ for a.e. $\tau \in [0,t]$. Let us denote the value of this optimization problem by $\mathcal{V}(t,x)$. The Hamiltonian for the problem at hand is, as opposed to previous sections,

$$\mathcal{H}(x,p) = \sup\{\langle p, f(x,z)\rangle - \mathcal{L}(x,z); z \in \mathcal{Z}\}.$$

We assume that $\mathcal{Z}$, the control space, is a compact subset of $\mathbb{R}^m$. (This simplifies the analysis considerably and in fact means a departure from the type of problems that this paper has mainly endeavored to shed light on.) The associated Hamilton-Jacobi equation is in sufficiently decent cases satisfied by $\mathcal{V}$,

$$\begin{aligned} \mathcal{V}_t(t,x) + \mathcal{H}(x, \mathcal{V}_x(t,x)) &= 0, \qquad (t,x) \in (0,T] \times \mathbb{R}^n, \\ \mathcal{V}(0,x) &= \varphi(x), \qquad x \in \mathbb{R}^n; \end{aligned}$$

see e.g. [24, 23]. We claim now that uniqueness of viscosity solutions belonging to $C([0,T] \times \mathbb{R}^n)$ of this initial-value problem is assured under the following additional hypotheses:

(C0) $\varphi: \mathbb{R}^n \to \mathbb{R}$ and $f, \mathcal{L}: \mathbb{R}^n \times \mathcal{Z} \to \mathbb{R}$ are all continuous.

(C1) For every $r > 0$ there exists a constant $C \geq 0$ such that

$$\langle f(x,z) - f(y,z), x - y\rangle \geq -C|x-y|^2$$

for all $x \in B_r$, $y \in B_r$, and $z \in \mathcal{Z}$.

(C2) There exists a function a whose properties are described in (G) such that $\langle f(x,z), x\rangle \geq -|x|a(|x|)$ for all $(x,z) \in \mathbb{R}^n \times \mathcal{Z}$.

We divide the proof into three steps. Firstly, $\mathcal{H}(x,p)$ is convex in p, being the supremum of a family of affine functions. Moreover, $\mathcal{Z}$ being compact, it is clear that $\mathcal{H}$ is continuous in $\mathbb{R}^n \times \mathbb{R}^n$ in view of (C0).

Secondly, we verify that (H1) too holds. Let $r > 0$, $x \in B_r$, $y \in B_r$, $j \in \mathbb{N}$, and choose $\bar{z} \in \mathcal{Z}$ so as to give the maximum in the problem defining $\mathcal{H}(y, j(x-y))$. Then, in denoting by μ a modulus of continuity for the restriction of $\mathcal{L}$ to the compact set $\bar{B}_r \times \mathcal{Z}$,

$$\begin{aligned}\mathcal{H}(x, j(x-y)) &- \mathcal{H}(y, j(x-y)) \\ &\geq \langle f(x,\bar{z}) - f(y,\bar{z}), j(x-y)\rangle + \mathcal{L}(y,\bar{z}) - \mathcal{L}(x,\bar{z}) \\ &\geq -Cj|x-y|^2 - \mu(|x-y|),\end{aligned}$$

so (H1) holds with $\Lambda(\theta, x, y) = 2C\theta + \mu(|x-y|)$.

Thirdly, we find similarly, via (C2),

$$\mathcal{H}(x, p + \lambda x) - \mathcal{H}(x,p) \geq -\lambda|x|a(|x|) \quad \text{for all} \quad x, p \in \mathbb{R}^n, \ \lambda \geq 0.$$

Finally, we conclude by calling upon Corollary 2.

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References

[1] Alvarez O., *Bounded from below viscosity solutions of Hamilton-Jacobi equations.* Differential and Integral Equations **10** (1997), 419–436.

[2] Arnold V.I., *Mathematical methods of classical mechanics.* Graduate Texts in Mathematics **60**, Springer-Verlag, New York, 1989.

[3] Attouch H., *Variational convergence for functions and operators.* Applicable Mathematics Series. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1984.

[4] Bardi M. and Da Lio F., *On the Bellman equation for some unbounded control problems.* NoDEA Nonlinear Differential Equations Appl. **4** (1997), 491–510.

[5] Barron N. and Jensen R., *Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians.* Comm. Partial Differential Equations **15** (1990), 1713–1742.

[6] Barron N. and Jensen R., *Optimal control and semicontinuous viscosity solutions.* Proc. Amer. Math. Soc. **113** (1991), 397–402.

[7] Barles G., *Uniqueness for first-order Hamilton-Jacobi equations and Hopf formula.* J. Differential Equations **69** (1987), 346–367.

[8] Benton S.H., *The Hamilton-Jacobi equation. A global approach.* Mathematics in Science and Engineering, Vol. **131**, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1977.

[9] Capuzzo-Dolcetta I. and Lions P.-L., *Hamilton-Jacobi equations with state constraints.* Trans. Amer. Math. Soc. **318** (1990), 643–683.

[10] Clarke F.H. and Loewen P.D., *An intermediate existence theory in the calculus of variations.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **16** (1989), 487–526 (1990).

[11] Clarke F.H. and Vinter R.B., *Regularity properties of solutions to the basic problem in the calculus of variations.* Trans. Amer. Math. Soc. **289** (1985), 73–98.

[12] Crandall M.G., Ishii H. and Lions P.L., *Uniqueness of viscosity solutions of Hamilton-Jacobi equations revisited.* J. Math. Soc. Japan **39** (1987), 581–596.

[13] Crandall M.G., Ishii H. and Lions P.L., *User's guide to viscosity solutions of second order partial differential equations.* Bull. Amer. Math. Soc. (N.S.) **27** (1992), 1–67.

[14] Crandall M.G. and Lions P.L., *Condition d'unicité pour les solutions généralisés des équations de Hamilton-Jacobi du premier ordre.* C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), 183–186.

[15] Crandall M.G. and Lions P.L., *Viscosity solutions of Hamilton–Jacobi equations.* Trans. Amer. Math. Soc. **277** (1983), 1–42.

[16] Crandall M.G. and Lions P.L., *On existence and uniqueness of solutions of Hamilton-Jacobi equations.* Nonlinear Anal. **10** (1986), 353–370.

[17] Crandall M.G. and Lions P.L., *Remarks on the existence and uniqueness of unbounded viscosity solutions of Hamilton-Jacobi equations.* Illinois J. Math. **31** (1987), 665–688.

[18] Dal Maso G., *An introduction to* Γ*-convergence.* Progress in Nonlinear Differential Equations and their Applications **8**, Birkhäuser Boston, Inc., Boston, MA, 1993.

[19] Ekeland I. and Temam R., *Convex analysis and variational problems.* Studies in Mathematics and its Applications, Vol. **1**, North-Holland Publishing Co., Amsterdam–Oxford; American Elsevier Publishing Co., Inc., New York, 1976.

[20] Frankowska H., *Hamilton-Jacobi equations: viscosity solutions and generalized gradients.* J. Math. Anal. Appl. **141** (1989), 21–26.

[21] Ioffe A.D. and Tihomirov V.M., *Theory of extremal problems.* Studies in Mathematics and its Applications, Vol. **6**, North-Holland Publishing Co., Amsterdam-New York, 1979.

[22] Ishii H., *Uniqueness of unbounded viscosity solution of Hamilton-Jacobi equations.* Indiana Univ. Math. J. **33** (1984), 721–748.

[23] Ishii H., *Viscosity solutions and their applications.* Sugaku Expositions **10** (1997), 123–141.

[24] Lions P.-L., *Generalized solutions of Hamilton-Jacobi equations.* Research Notes in Mathematics 69, Pitman (Advanced Publishing Program), Boston-London, 1982.

[25] Rockafellar R.T., *Convex analysis.* Princeton Mathematical Series, No. **28** Princeton University Press, Princeton, N.J. 1970.

[26] Rockafellar R.T. and Wets R. J-B, *Variational analysis*. Grundlehren der Mathematischen Wissenschaften **317**, Springer-Verlag, Berlin, 1998.
[27] Rockafellar R.T. and Wolenski P.R., *Envelope representations of value functions in Hamilton-Jacobi theory*. Preprint, November 1997.
[28] Strömberg T., *Hopf's formula gives the unique viscosity solution*. Preprint, University of Lund, May 1998.
[29] Sychev M.A. and Mizel V.J., *A condition on the value function both necessary and sufficient for full regularity of minimizers of one-dimensional variational problems*. Trans. Amer. Math. Soc. **350** (1998), 119–133.

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