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On some classes of regularization methods for minimization problem of quadratic functional on a half-space

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Abstract. This paper deals with some classes of a regularization methods of quadratic functional minimization problem on a half-space of real Hilbert space. We prove the convergence of the regularized solution. Under additional conditions, we obtain an estimate for convergence rate of the presented methods.

Key words: quadratic functional, stability, regularization.

1. Introduction

We consider the following extremal problem:

$$J(u)=\|Au-f\|^{2}\rightarrow\inf,\quad u\in U=\{u\in H:\langle c,u\rangle\leq\beta\}\tag{1}$$

Here H and F are real Hilbert spaces; $A: H\rightarrow F$ is continuous linear operator; $f\in F$, $c\in H$, $c\neq 0$, are some fixed elements from the corresponding spaces; β is given real number.

In practice, instead of the exact operator A and the elements f, c, we deal with their approximations $A_{\mu}\in\mathcal{L}(H,F)$, $f_{\delta}\in F$, and $c_{\sigma}\in H$, such that

$$\|A-A_{\mu}\|\leq\mu,\quad\|f-f_{\delta}\|\leq\delta,\quad\|c-c_{\sigma}\|\leq\sigma,$$

where μ, δ and σ are small positive real numbers.

Generally speaking, the problem (1) is unstable with respect to the perturbations of the initial data A, f, c and the regularization method are required to solve it [3], [4], [5], [6], [7]. Remark that the regularization methods for the minimization problems

$$J(u)=\|Au-f\|^{2}\rightarrow\inf,\quad u\in H\tag{2}$$

and

$$J(u)=\|Au-f\|^{2}\rightarrow\inf,\quad u\in U=\{u\in H:\|u\|\leq R\}$$

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have been studied in [1], [4], [7] and in [3], [7].

In further, we suppose that the sets of the solutions of the problems (1) and (2) are not empty.

Let us introduce the following notation: $R(A)$ is the range of the operator A, P is the orthogonal projecting operator from H on $\overline{R(A)^*}$, u_* is the normal solution (i.e. the solution with the minimal norm) of the problem (1), and u_∞ is the normal solution of the problem (2).

According to the optimality conditions [7], we have that u_∞ satisfy the operator equality,

$$A^*Au_\infty = A^*f. \tag{3}$$

while for the element u^* there exists $\lambda^* \geq 0$ such that

$$A^*Au_* - A^*f + \lambda^*c = 0 \tag{4}$$

$$\lambda^*(\langle c, u_*\rangle - \beta) = 0 \tag{5}$$

Following the Tikhonov idea of regularization of unstable problem, we can take the (unique) solution of the problem

$$Q_\alpha(u) = \|A_\mu u - f_\delta\| + \alpha\|u\|^2 \to \inf, \quad u \in H \tag{6}$$

for small positive real number $\alpha = \alpha(\delta, \mu)$, δ, μ, as an approximation of a solution of the problem (1), Remark that the solution of the problem (6) can be represented by

$$u_\alpha = \bar{g}_\alpha(A_\mu^* A_\mu)A_\mu^* f_\delta,$$

where $\bar{g}_\alpha(t) = (\alpha + t)^{-1}$. We say that the method (6) is generated by the system of functions $\{\bar{g}_\alpha\}$. The generalizations of the previous method for the problem (1) were observed in [6]. These generalized methods were generated with the system of continuous functions $g_\alpha : [0, a] \to R$, $\alpha > 0$, such that

$$(\forall t \in [0, a])(\forall \alpha > 0)1 - tg_\alpha(t) \geq 0 \tag{7}$$

$$\sup\{t^p(1 - tg_\alpha(t)) : t \in [0, a]\} \leq \gamma_p\alpha^p, \qquad (0 \leq p \leq p_0,\ p_0 > 0,\ \gamma_p \equiv \text{const}) \tag{8}$$

Real number p_0 is called the qualification of the system $\{g_\alpha\}$. The functions $\bar{g}_\alpha(t) = (\alpha + t)^{-1}$ satisfy the conditions (7)–(8) with $p_0 = 1$. However, the method (6) generated with the functions $\bar{g}_\alpha$ is not suitable for describing some algorithms of choosing of the parameter α and for studying of some iterative methods of regularization. In order to study these problems we should consider the system of functions $\{g_\alpha\}$, which satisfy the conditions (7)–(8) with the qualification $p_0 > 1$. Notice that the regularizatiom methods based on the functions $\{g_\alpha$ for the minimization problem without constraints were properly observed in [5] and [4]. In this paper we show that the similar class of the functions can be also used for the regularization of the problem (1).

2. Algorithms and auxiliary results

It turns out that for studying the extremal problems with constraints, beside (7), (8), we need an additional condition

$$(\exists\beta > 0)(\forall t \in [0, a])(\forall\alpha > 0) \quad (t + \beta\alpha)^{-1} \leq g_\alpha(t) \leq (\beta\alpha)^{-1} \tag{9}$$

The examples of the functions that satisfy (9) can be found in [6].

Since

$$\beta\alpha\|u - v\|^2 \leq \langle g_\alpha^{-1}(A_\mu^* A_\mu)(u - v), u - v\rangle, \quad u, v \in H,$$

it follows that the extremal problem

$$T_\alpha(u) = \|g_\alpha^{-\frac{1}{2}}(A_\mu^* A_\mu)u - g_\alpha^{\frac{1}{2}}(A_\mu^* A_\mu)A_\mu^* f_\delta\|^2 \to \inf, \quad u \in H$$

has the unique solution w_α. Then $T'_\alpha(w_\alpha) = 0$ i.e $w_\alpha = g_\alpha(A_\mu^* A_\mu)A_\mu f_\delta$. We shall prove that the following estimate is true

$$\|A_\mu(w_\alpha - u_\infty)\|^2 \leq k(\alpha + \mu)\|w_\alpha - u_\infty\|, \quad k > 0. \tag{10}$$

Using the conditions $J'(u_\infty) = 0$ and $T'_\alpha(w_\alpha) = 0$, we have that

$$g_\alpha^{-1}(A_\mu^* A_\mu)w_\alpha - A_\mu^* f_\delta - A^* A u_\infty + A^* f = 0.$$

Multiplying the previous equality by $w_\alpha - u_\infty$ and using the properties of the function g_α, we obtain the inequality

$$\begin{aligned} \|A_\mu(w_\alpha - u_\infty)\|^2 \leq\ & \langle (A_\mu^* A_\mu - g_\alpha^{-1}(A_\mu^* A_\mu))u_\infty, w_\alpha - u_\infty\rangle \\ & + \langle (A^* A - A_\mu^* A_\mu)u_\infty, w_\alpha - u_\infty\rangle \\ & + \langle (A_\mu^* f_\delta - A^* f), w_\alpha - u_\infty\rangle \end{aligned}$$

that implies the estimation (10).

In [6] Theorem 2.4, p. 100, it is proved that if parameter $\alpha = \alpha(\delta, \mu)$ satisfies

$$\alpha(\delta, \mu) \to 0, \ \frac{\mu + \delta^2}{\alpha(\delta, \mu)} \to 0, \quad (\delta, \mu \to 0)$$

then

$$w_{\alpha(\delta,\mu)} \to u_\infty, \quad (\delta, \mu \to 0) \tag{11}$$

Lemma 1 *Suppose that the parameter $\alpha = \alpha(\mu)$ is such that*

$$\alpha(\mu) \to 0, \ \frac{\mu}{\alpha(\mu)} \to 0, \quad (\mu \to 0)$$

Then, for all $x \in H$, we have

i) $(I - A_\mu^* A_\mu g_\alpha(A_\mu^* A_\mu))Px \to 0$, $(\mu \to 0)$,

ii) $\beta\alpha g_\alpha(A_\mu^* A_\mu)x \to (I - P)x$, $(\mu \to 0)$

Proof. i) The proof of this part of Lemma can be found in [6], Lemma 2.2., p. 99.

ii) The family of operators $\{\beta\alpha g_\alpha(A_\mu^* A_\mu)\}$ is uniformly bounded, because

$$\|\beta\alpha g_\alpha(A_\mu^* A_\mu)\| \le \sup\{\beta\alpha g_\alpha(t) : t \in [0, a]\} \le 1.$$

The elements $u = A^*Aw$, $w \in H$, generate a dense subspace in $\overline{R(A^*A)}$ and

$$\begin{aligned}
&\|\beta\alpha g_\alpha(A_\mu^* A_\mu)A^*Aw\| \\
&\quad \le \|\beta\alpha g_\alpha(A_\mu^* A_\mu)(A^*A - A_\mu^* A_\mu)w\| + \|\beta\alpha g_\alpha(A_\mu^* A_\mu)A_\mu^* A_\mu w\| \\
&\quad \le k(\mu + \alpha) \to 0
\end{aligned}$$

when $\mu, \delta, \sigma \to 0$.

By virtue of Banach-Steinhaus theorem, we have

$$\beta\alpha g_\alpha(A_\mu^* A_\mu)Px \to 0, \quad (\mu, \delta, \sigma \to 0).$$

Since, $0 \le 1 - \beta\alpha g_\alpha(t) \le \frac{t}{\beta\alpha}$, it follows that

$$\begin{aligned}
\|(I - \beta\alpha g_\alpha(A_\mu^* A_\mu))(I - P)x\| &\le \frac{\|A_\mu^* A_\mu(I - P)x\|}{\beta\alpha} \\
&= \frac{\|(A_\mu^* A_\mu - A^*A)(I - P)x\|}{\beta\alpha} \le k\frac{\mu}{\alpha} \to 0 \quad (\mu \to 0).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\beta\alpha g_\alpha(A_\mu^* A_\mu)x &= \beta\alpha g_\alpha(A_\mu^* A_\mu)Px \\ &\quad + \beta\alpha g_\alpha(A_\mu^* A_\mu)(I-P)x \to (I-P)x, \quad (\mu \to 0)\end{aligned}$$

□

Lemma 2 *If the parameter $\alpha = \alpha(\mu, \sigma)$ is chosen such that*

$$\alpha(\mu,\sigma) \to 0, \ \frac{\mu+\sigma}{\alpha(\mu,\sigma)} \to 0 \quad (\mu, \sigma \to 0)$$

then

i) $\beta\alpha g_\alpha(A_\mu^* A_\mu)c_\sigma \to (I-P)c \ (\mu, \sigma \to 0)$

ii) *If* $c = A^*Ah$, $h \in \overline{R(A^*A)}$, *then* $g_\alpha(A_\mu^* A_\mu)c_\sigma \to h \ (\mu, \sigma \to 0)$.

Proof. We have that

$$g_\alpha(A_\mu^* A_\mu)c_\sigma = g_\alpha(A_\mu^* A_\mu)(c_\sigma - c) + g_\alpha(A_\mu^* A_\mu)c.$$

This equality, the estimate

$$\|g_\alpha(A_\mu^* A_\mu)(c_\sigma - c)\| \le k\frac{\mu}{\alpha},$$

and Lemma 1 imply i).

We also may write

$$\begin{aligned}&g_\alpha(A_\mu^* A_\mu)c_\sigma - h \\ &\quad = g_\alpha(A_\mu^* A_\mu)(c_\sigma - c) + g_\alpha(A_\mu^* A_\mu)c - h \\ &\quad = g_\alpha(A_\mu^* A_\mu)(c_\sigma - c) + g_\alpha(A_\mu^* A_\mu)(A^*A - A_\mu^* A_\mu)h \\ &\qquad - (I - A_\mu^* A_\mu g_\alpha(A_\mu^* A_\mu))h.\end{aligned}$$

Therefore, we obtain the inequality

$$\begin{aligned}\|g_\alpha(A_\mu^* A_\mu)c_\sigma - h\| &\le \|g_\alpha(A_\mu^* A_\mu)\| \cdot \|c_\sigma - c\| \\ &\quad + \|g_\alpha(A_\mu^* A_\mu)\| \cdot \|A^*A - A_\mu^* A_\mu\| \cdot \|h\| \\ &\quad + \|(I - A_\mu^* A_\mu g_\alpha(A_\mu^* A_\mu))h\| \\ &\le k\Big(\frac{\mu+\sigma}{\alpha} + \|I - A_\mu^* A_\mu g_\alpha(A_\mu^* A_\mu)h\|\Big).\end{aligned}$$

This inequality, Lemma 1, i) and the properties of the functions g_α, imply ii). □

Taking into account the optimality conditions (3)–(5) and Lemma 2, it is easy to prove the following relationship between normal solutions u_* and u_∞.

Lemma 3 i) *If* $c \in R(A^*A)$, *i.e.* $c = A^*Ah$ *for some* $h \in \overline{R(A^*A)}$, *then* $u_* = u_\infty - \lambda_* h$, *where*

$$\lambda^* = \begin{cases} 0, & \langle u_\infty, c\rangle \le \beta \\ \dfrac{\langle u_\infty, c\rangle - \beta}{\|(I-P)c\|^2}, & \langle u_\infty, c\rangle > \beta \end{cases}$$

ii) *If* $c \notin R(A^*A)$, *then* $u_* = u_\infty - \gamma_*(I-P)c$, *where*

$$\gamma_* = \begin{cases} 0, & \langle u_\infty, c\rangle \le \beta \ \ or \ \ (I-P)c = 0 \\ \dfrac{\langle u_{,\infty}, c\rangle - \beta}{\|(I-P)c\|^2}, & \langle u_\infty, c\rangle > \beta \ \ and \ \ (I-P)c \ne 0 \end{cases}$$

As an approximation of the solution of the problem (1), one can take the element

$$u_\alpha = g_\alpha(A_\mu^* A_\mu)(A_\mu^* A_\mu f_\delta - \lambda_\alpha c_\sigma)$$

where

$$\lambda_\alpha = \begin{cases} 0, & \langle w_\alpha, c_\sigma\rangle \le \beta \\ \dfrac{\langle w_\alpha, c_\sigma\rangle - \beta}{\langle g_\alpha(A_\mu^* A_\mu)c_\sigma, c_\sigma\rangle}, & \langle w_\alpha, c_\sigma\rangle > \beta \end{cases}$$

and $w_\alpha = g_\alpha(A_\mu^* A_\mu)A_\mu^* f_\delta$, is the solution of the extremal problem $T_\alpha(u) \to \inf$, $u \in H$. The element u_α satisfies the equalities

$$T'_\alpha(u_\alpha) + \lambda_\alpha c_\sigma = 0$$

$$\lambda_\alpha(\langle c_\sigma, u_\alpha\rangle - \beta) = 0.$$

Therefore, by virtue of Kuhn-Tucker theorem, we deduce that u_α is the solution of the following extremal problem

$$T_\alpha(u) \to \inf, \ \ u \in U_\sigma = \{u \in H : \langle c_\sigma, u\rangle \le \beta\}.$$

3. Convergence and rate of convergence

Theorem 1 *Let $c_\sigma, c \in H$, $f, f_\delta \in F$, $A, A_\mu \in L(H,F)$, are such that*

$$\|A - A_\mu\| \le \mu, \ \|A_\mu\|^2 \le a, \ \|f - f_\delta\| \le \delta, \ \|c - c_\sigma\| \le \sigma.$$

Assume that the system of the functions $\{g_\alpha\}$ satisfies the conditions (7)–(9).

If the parameter $\alpha = \alpha(\mu, \delta, \sigma)$ is chosen such that:

$$\alpha(\mu,\delta,\sigma) \to 0, \ \frac{\mu + \delta^2 + \sigma}{\alpha(\mu,\delta,\sigma)} \to 0 \ \ (\mu, \delta, \sigma \to 0)$$

then

$$u_{\alpha(\mu,\delta,\sigma)} \to u_* \ \ (\mu, \delta, \sigma \to 0) \tag{12}$$

If, in addition, the elements u_∞ and Pc can be represented in the form

$$u_\infty = (A^*A)^p v, \ Pc = (A^*A)^{q+1} w, \ v, w \in H, \ 0 < p, \ q \le p_0 \tag{13}$$

then for

$$\alpha = d(\mu + \delta + \sigma)^{\min\{\frac{1}{p+1}, \frac{1}{q+1}, \frac{1}{2}\}}, \quad d \equiv \mathrm{const} > 0 \tag{14}$$

the following inequality is valid

$$\|u_* - u_\alpha\| \le d_p(\mu + \delta + \sigma)^{\min\{\frac{p}{p+1}, \frac{q}{q+1}, \frac{1}{2}\}}. \tag{15}$$

Proof. Suppose that $(I - P)c = 0$ and $c \notin R(A^*A)$. Then $\langle u_\infty, c\rangle \le \beta$, i.e. $u_* = u_\infty$. Denote by v_α the solution of the extremal problem $T_\alpha(u) \to \inf, u \in U$. Let us estimate the value $\|v_\alpha - u_\alpha\|$. The element v_α is determined by

$$v_\alpha = g_\alpha(A_\mu^* A_\mu)(A_\mu^* f_\delta - s_\alpha c)$$

where

$$s_\alpha = \begin{cases} 0, & \langle w_\alpha, c\rangle \le \beta \\ \dfrac{\langle w_\alpha, c\rangle - \beta}{\langle g_\alpha(A_\mu^* A_\mu)c, c\rangle}, & \langle w_\alpha, c\rangle > \beta \end{cases}$$

Hence

$$g_\alpha^{-1}(A_\mu^* A_\mu)v_\alpha - A_\mu^* f_\delta + s_\alpha c = 0 \tag{16}$$

$$s_\alpha(\langle v_\alpha, c\rangle - \beta) = 0 \tag{17}$$

$$g_\alpha^{-1}(A_\mu^* A_\mu)u_\alpha - A_\mu^* f_\delta + \lambda_\alpha c_\sigma = 0 \tag{18}$$

$$\lambda_\alpha(\langle u_\alpha, c_\sigma\langle -\beta) = 0 \tag{19}$$

Using the equalities (16)–(19) and taking into account the properties of the functions g_α we obtain the following inequality

$$\|v_\alpha - u_\alpha\| \le k\frac{\sigma}{\alpha}$$

Multiplying the equalities

$$v_\alpha - u_* = w_\alpha - u_\infty - s_\alpha g_\alpha(A_\mu^* A_\mu)c$$

by $g_\alpha(A_\mu^* A_\mu)^{-1}(v_\alpha - u_*)$ and using again the properties of the functions G_α, we have

$$\beta\alpha\|v_\alpha - u_*\|^2 \le \|A_\mu(w_\alpha - u_\infty)\|^2 + \beta\alpha\|w_\alpha - u_\infty\|^2.$$

Combining these inequalities we obtain

$$\begin{aligned}\|u_\alpha - u_*\| &\le \|u_\alpha - v_\alpha\| + \|v_\alpha - u_*\| \\ &\le \frac{1}{\sqrt{\alpha\beta}}\|A_\mu(w_\alpha - u_\infty)\| + \|w_\alpha - u_\infty\| + k\frac{\sigma}{\alpha}.\end{aligned}$$

It follows from (10) and (11) that $u_{\alpha(\mu,\delta,\sigma}$ tend to u_* when $\mu, \delta, \sigma \to 0$.

Let us consider the case $c \in R(A^*A)$. Then the convergence (12) follows from the equality

$$\begin{aligned}u_\alpha - u_\infty &= w_\alpha - u_\infty + \frac{\langle u_\infty, c\rangle - \beta}{\|Ah\|^2}h \\ &- \frac{\langle w_\alpha, c_\sigma) - \beta}{\langle g_\alpha(A_\mu^* A_\mu)c_\sigma, c_\sigma\rangle}g_\alpha(A_\mu^* A_\mu)c_\sigma.\end{aligned}$$

Lemma 2, ii) and (11).

Finally, let be $(I - P)c \ne 0$. Then the convergence (12) may be derived as the consequence of the equality

$$\begin{aligned}u_\alpha - u_\infty &= w_\alpha - u_\infty + \frac{\langle u_\infty, c\rangle - \beta}{\|(I-P)c\|^2}(I - P)c \\ &- \frac{\langle w_\alpha, c_\sigma) - \beta}{\langle \alpha\beta g_\alpha(A_\mu^* A_\mu)c_\sigma, c_\sigma\rangle}\alpha\beta g_\alpha(A_\mu^* A_\mu)c_\sigma.\end{aligned}$$

Lemma 2, i) and the relation (11).

Thus, it remains to prove the inequality (15), under the additional assumptions (13) and (14). Firstly, we note that in [6] (Theorem 2.4, pp. 100), under the condition $u_\infty = (A^*A)^p v$, was proved the following estimate for $\|w_\alpha - u_\infty\|$:

$$\|w_\alpha - u_\infty\| \leq d_p\Big(\alpha^p + \frac{\mu+\delta}{\alpha}\Big) \tag{20}$$

If $\langle u_\infty, c\rangle < \beta$, then (11) implies that $\langle w_\alpha, c_\sigma\rangle < \beta$ for small enough μ, δ and σ. Thus, we have that $u_\alpha = w_\alpha$. Hence, the inequality (20) is held in this case, also.

Let us consider the case $\langle u_\infty, c\rangle \geq \beta$. In the same way as in [6] (Theorem 2.4., pp 100), for $c \in R(A^*A)$ we get

$$\|u_\alpha - u_*\| \;\leq\; d_p\Big(\|w_\alpha - u_\infty\| + (1+\ln|\,\mu\,|)\mu^{\min\{1,2q\}} + \alpha^q + \frac{\mu+\sigma}{\alpha}\Big) \quad \text{for} \;\; c \in R(A^*A) \tag{21}$$

and

$$\|u_\alpha - u_*\| \leq d_p\Big(\|w_\alpha - u_\infty\| + \alpha + \frac{\mu+\sigma}{\alpha}\Big) \quad \text{for} \;\; (I-P)c \neq 0. \tag{22}$$

Then, the condition (14) and the inequalities (20)–(22) imply the estimate (15). This completes the proof of Theorem 1. □

Theorem 2 *If $R(A)$ is closed subspace of space H, and if*

$$\alpha = d(\mu+\delta+\sigma)^{\frac{1}{2}}, \quad d \equiv \text{const} > 0,$$

then

$$\|u_\alpha - u_*\| \leq k(\mu+\delta+\sigma)^{\frac{1}{2}}.$$

4. Iterative methods of regularization

In Theorem 1 ill-posed problem (1) was regularuzed so that it was embedded in a family of well-posed problems. The solution of the source problem was obtained as a limit of the solutions of the regularized problems, when the parameter of regularization α tends to 0. In iterative methods of regularization, the parameter of regularization is the number of iterations.

Namely, in this iterative process at every step we solve a well-posed problem. Under some conditions the obtained sequence of the solutions of these well-posed problems tend to normal solution of the given problem (1).

Here, we study a class of iterative methods, generated by a continuous function $g : [0, a] \to R$ which satisfies some special conditions.

Let us consider the following iterative process:

$$\begin{aligned} &v_0 = 0, \ w_n = w_{n-1} - g(A_\mu^* A_\mu)(A_\mu^* A_\mu w_{n-1} - A_\mu^* f_\delta) \\ &h_0 = 0, \ h_n = h_{n-1} - g(A_\mu^* A_\mu)(A_\mu^* A_\mu h_{n-1} - c_\sigma) \\ &u_n = w_n - \lambda_n h_n \end{aligned}$$

where

$$\lambda_n = \begin{cases} 0, & \langle w_n, c_\sigma \rangle \le \beta \\ \dfrac{\langle w_n, c_\sigma \rangle - \beta}{\langle h_n, c_\sigma \rangle}, & \langle w_n, c_\sigma \rangle > \beta \end{cases}$$

It is easy to see that $\langle u_n, c_\sigma \rangle \le \beta$. By induction it can be proved that

$$\begin{aligned} v_n &= \sum_{j=0}^{n-1} (I - A_\mu^* A_\mu g(A_\mu^* A_\mu))^j g_(A_\mu^* A_\mu) A_\mu^* f_\delta, \\ h_n &= \sum_{j=0}^{n-1} (I - A_\mu^* A_\mu g(A_\mu^* A_\mu))^j g_(A_\mu^* A_\mu) c_\sigma, \end{aligned}$$

Introducing the functions

$$g_n(t) = \sum_{j=0}^{n-1} (1 - tg(t))^j = t^{-1}[1 - (1 - tg(t))^n], \ \ 0 < t \le a \tag{23}$$

the equalities given above, can be rewritten in the form

$$w_n = g_n(A_\mu^* A_\mu) A_\mu^* f_\delta, \ h_n = g_n(A_\mu^* A_\mu) c_\sigma, \ v_n = w_n - \lambda g_n(A_\mu^* A_\mu) c_\sigma.$$

The following Lemma shows that the system of functions $\{g_n\}$ satisfies the conditions (7)–(9).

Lemma 4 *Assume that $g : [0, a] \to R$ is a continuous function such that*

$$(t + \gamma/t)^{-1} \le g(t) \le 1/t, \ t \in [0, a], \quad \gamma = \max\{g(t) : t \in [0, a]\}.$$

Then the functions g_n the following conditions:

$$1 - tg_n(t) \geq 0,\ \max\{g_n(t) : t \in [0,a]\} \leq \gamma n,\ g_n(t) \geq n\gamma(n\gamma t + 1)^{-1}$$

$$\sup\{t^p(1 - tg_n(t)) : t \in [0,a]\} \leq \gamma_p n^{-p}, \qquad (0 \leq p \leq n, p_0 > 0, \gamma_p \equiv \text{const})$$

Proof. Since $(t + \gamma/t)^{-1} \leq g(t)$, it is obvious that $1 - tg(t) \leq 1/(1 + \gamma t)$, which, taken with Bernoulli inequality and (23) implies $g_n(t) \geq n\gamma(n\gamma t + 1)^{-1}$. The remain statements were proved in [6], Lemma 4.1, pp. 37. This completes the proof of Lemma 4. □

On every step of the iterative procedure the corresponding minimization problem can be solved only approximately. Therefore the practical iterative process can be presented by the following scheme:

$$\begin{aligned}
\bar{w}_n &= 0,\ \bar{w}_n = \bar{w}_{n-1} - g(A_\mu^* A_\mu)(A_\mu^* A_\mu \bar{w}_{n-1} - A_\mu^* f_\delta) + p_n \\
\bar{h}_0 &= 0,\ \bar{h}_n = \bar{h}_{n-1} - g(A_\mu^* A_\mu)(A_\mu^* A_\mu \bar{h}_{n-1} - c_\sigma) + q_n \\
\bar{u}_n &= \bar{w}_n - \bar{\lambda}_n \bar{c}_n
\end{aligned}$$

$$\bar{\lambda}_n = \begin{cases} 0, & \langle \bar{w}_n, c_\sigma \rangle \leq \beta \\ \dfrac{\langle \bar{w}_n, c_\sigma \rangle - \beta}{\langle \bar{h}_n, c_\sigma \rangle}, & \langle \bar{w}_n, c_\sigma \rangle > \beta \end{cases}$$

By induction, one can prove the equalities:

$$\bar{w}_n = w_n + \bar{p}_n,\ \bar{h}_n = h_n + \bar{q}_n,\ u_n = w_n - \bar{\lambda}_n h_n + \bar{p}_n - \bar{\lambda}_n \bar{q}_n$$

where

$$\begin{aligned}
\bar{p}_n &= \sum_{j=0}^{n-1} (I - A_\mu^* A_\mu g((A_\mu^* A_\mu)^j p_{n-j}, \\
\bar{q}_n &= \sum_{j=0}^{n-1} (I - A_\mu^* A_\mu g_n(A_\mu^* A_\mu))^j q_{n-j}
\end{aligned}$$

Using the same technique as in the proof of Theorem 1, one can prove the following result:

Theorem 3 *Let the following conditions be satisfied:*

1) $c_\sigma, c \in H,\ f, f_\delta \in F,\ A, A_\mu \in L(H,F)$ *and*

$$\|A - A_\mu\| \le \mu,\ \|A_\mu\|^2 \le a,\ \|f - f_\delta\| \le \delta\|c - c_\sigma\| \le \sigma;$$

2) $p_n, q_n \in H,\ n = 1,2,\dots,\max\{p_n, q_n\} \le k(\mu + \delta^2 + \sigma),\ n = 1,2,\dots, k = \text{const} > 0;$

3) $n = n(\mu,\delta,\sigma) \to +\infty,\ (\mu + \delta^2 + \sigma) \to 0$ *when* $\mu,\delta,\sigma \to 0;$

4) $g : [0,a] \to R$ *is a continuous function with the properties*

$$(t + \gamma/t)^{-1} \le g(t) \le 1/t,\ t \in [0,a], \quad \gamma = \max\{g(t) : t \in [0,a]\},$$

and

$$g_n(t) = \sum_{j=0} n - 1(1 - tg(t))^j,\ 0 < t \le a, \quad n = 1,2,\dots .$$

Then

$$\overline{w}_n \to u_* \quad \textit{when} \quad \mu, \delta\sigma \to 0,$$

where u_* *is the normal solution of the problem* (1).

If, in addition,

5) *the elements* u_∞ *and* Pc *can be represented in the form* (13);

6) $n = \left[d(\mu + \delta + \sigma)^{\min\{\frac{1}{p+1},\frac{1}{q+1},\frac{1}{2}\}}\right]^{-1}.$

Then

$$\|u_* - \overline{w}_\alpha\| \le d_p(\mu + \delta + \sigma)^{\min\{\frac{p}{p+1},\frac{q}{q+1},\frac{1}{2}\}}, \quad n = 1,2,\dots .$$

5. Conclusion

At the end, note that Theorem 1 gives the potential possibilities of the method. In practice, we do not have the information of type (13) for the properties of the solution u_∞ and the element c. In this case, the choice of the parameter α is not at all easy. In [6], for the operator equations, it was considered so called aposterior choice of the parameter of regularization α. This choice does not include any information about the properties of the solution u_∞. Let us remark that the aposterior choice of the parameter α for the problem (1) is also possible.

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