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Strong almost convergence and almost λ-statistical convergence

Ekrem SAVAŞ

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Abstract. The purpose of this paper is to define almost λ-statistical convergence by using the notion of (V,λ)-summability to generalize the concept of statistical convergence.

Key words: statistical convergence, almost statistical convergence, almost λ-statistical convergence, strongly almost convergence.

1. Introduction

Let s be the set of all real or complex sequences and let l_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences $x=(x_k)$, respectively normed as usual by $||x||=\sup_k |x_k|$. Let D be the shift operator on s, that is $D((x_k))=(x_{k+1})$. It may be recalled that Banach limit L (Banach [1]) is a linear functional on l_∞ such that

(i) $L(x)\geq 0$ if $x_k\geq 0$, $k\geq 0$,

(ii) $L(Dx)=L(x)$ for all $x\in l_\infty$

(iii) $L(e)=1$ where $e=(1,1,1,\ldots)$.

A sequence $x\in l_\infty$ is said to be almost convergent (Lorentz [13]) if all Banach limits of x coincide. Let $\hat{c}$ and $\hat{c}_0$ denote the sets of all sequences which are almost convergent and almost convergent to zero. Lorentz [13] proved that,

$$\hat{c}=\left\{x:\lim_n \frac{1}{n}\sum_{k=1}^{n} x_{k+m} \text{ exists uniformly in } m\right\}$$

Several authors including Lorentz [13], Duran [4] and King [10] have studied almost convergent sequences.

A sequence $x=(x_k)$ is said to be summable $(C,1)$ if and only if

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} x_k \text{ exists}$$

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A sequence $x = (x_k)$ is said to be strongly (Cesáro) summable if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0$$

Spaces of strongly Cesáro summable sequences were discussed by Kuttner [11] and some others and this concept was generalised by Maddox [14].

Just as convergence give rise to strongly convergence, it was quite natural to expect that almost convergence must give rise to a new type of convergence, namely strong almost convergence and this concept was introduced and discussed by Maddox [14]. If $[\hat{c}]$ denotes the set of all strongly almost convergent sequences, then Maddox defined,

$$[\hat{c}] = \left\{ x = (x_k) : \text{for some } L, \ \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - L| = 0, \right. \left. \text{uniformly in } m \right\}$$

Let $\lambda = (\lambda_k)$ be a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$.

Generalized de la Valee Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$

A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L [12] if $t_n(x) \to L$ as $n \to \infty$.

If $\lambda_n = n$, then (V, λ)-summability is reduced to $(C, 1)$ summability.

We write,

$$[\hat{V}, \lambda] = \left\{ x = (x_k) : \text{for some } L, \ \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |x_{k+m} - L| = 0, \right. \left. \text{uniformly in } m \right\}$$

for the set of sequences $x = (x_k)$ which is strongly almost (V, λ)-summable to L, i.e., $x_k \to L[\hat{V}, \lambda]$.

The idea of statistical convergence was introduced by Fast [6]. Over the years and under different names, statistical convergence has been discussed in number theory [5], and trigonometric series [19] and summability theory [3].

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \le n : |x_k - L| \ge \varepsilon\}| = 0.$$

where the vertical bars indicate the number of elements in the enclosed sets. In this case we write $s - \lim x = L$ or $x_k \to L(s)$ and s denotes the set of all statistically convergent sequences.

This paper extends the definition of the statistical convergence to the concepts of almost statistical convergence and almost λ-statistical convergence and finds its relation with $[\hat{V}, \lambda]$ and $\hat{s}$.

We have

Definition 1 A sequence $x = (x_k)$ is said to be almost statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \le n : |x_{k+m} - L| \ge \varepsilon\}| = 0, \quad \text{uniformly in } m$$

In this case we write $\hat{s} - \lim x = L$ or $x_k \to L(\hat{s})$ and $\hat{s}$ denotes the set of all almost statistically convergent sequences.

Before giving some promised inclusion relations we will give a new definition.

Definition 2 A sequence $x = (x_k)$ is said to be almost λ-statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |x_{k+m} - L| \ge \varepsilon\}| = 0, \quad \text{uniformly in } m.$$

In this case we write $\hat{s}_\lambda - \lim x = L$ or $x_k \to L(\hat{s}_\lambda)$ and

$$\hat{s}_\lambda = \{x : \text{ for some } L,\ \hat{s}_\lambda - \lim x = L\}$$

If $\lambda_n = n$, then $\hat{s}_\lambda$ is same as $\hat{s}$.

2. In this section we give some inclusion relations between $\hat{s}_\lambda$ and $[\hat{V}, \lambda]$ and $[\hat{c}]$.

Theorem 1 *If a sequence is almost strongly summable to L, then it is* *almost statistically convergent to L.*

The proof of Theorem 1 uses ideas similar to those used in proving Theorem 1 of Connor [2].

We have

Theorem 2 *Let $\lambda = (\lambda_n)$ be same as in the above, then*

(i) $x_k \to L[\hat{V}, \lambda] \Rightarrow x_k \to L(\hat{s}_\lambda)$.

and the inclusion $[\hat{V}, \lambda] \subseteq (\hat{s}_\lambda)$ is proper.

(ii) *If $x \in l_\infty$ and $x_k \to L(\hat{s}_\lambda)$, then $x_k \to L[\hat{V}, \lambda]$ and hence $x_k \to L[\hat{c}]$ provided $x = (x_k)$ is not eventually constant,*

(iii) $\hat{s}_\lambda \cap l_\infty = [\hat{V}, \lambda] \cap l_\infty$,

Proof. Let $\varepsilon > 0$ and $x_k \to L[\hat{V}, \lambda]$. Since

$$\sum_{k \in I_n} |x_{k+m} - L| \geq \sum_{\substack{k \in I_n \\ |x_{k+m} - L| \geq \varepsilon}} |x_{k+m} - L| \geq \varepsilon \left| \{ k \in I_n : |x_{k+m} - L| \geq \varepsilon \} \right| .$$

Therefore $x_k \to L[\hat{V}, \lambda] \Rightarrow x_k \to L(\hat{s}_\lambda)$.

It is easy to see that the inclusion $[\hat{V}, \lambda] \subseteq (\hat{s}_\lambda)$ is proper.

(ii) Suppose that $x_k \to L(\hat{s}_\lambda)$ and $x \in l_\infty$, say $|x_{k+m} - L| \leq M$ for all k and m. Given $\varepsilon > 0$, we have

$$\begin{aligned} &\frac{1}{\lambda_n} \sum_{k \in I_n} |x_{k+m} - L| \\ &= \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |x_{k+m} - L| \geq \varepsilon}} |x_{k+m} - L| + \frac{1}{\lambda_n} \sum_{\substack{k \in I_n \\ |x_{k+m} - L| < \varepsilon}} |x_{k+m} - L| \\ &\leq \frac{M}{\lambda_n} |\{ k \in I_n : |x_{k+m} - L| \geq \varepsilon \}| + \varepsilon, \end{aligned}$$

which implies that $x_k \to L[\hat{V}, \lambda]$.

Further, we have

$$\frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - L| = \frac{1}{n} \sum_{k=1}^{n-\lambda_n} |x_{k+m} - L| + \frac{1}{n} \sum_{k \in I_n} |x_{k+m} - L$$

$$\leq \frac{1}{\lambda_n} \sum_{k=1}^{n-\lambda_n} |x_{k+m} - L| + \frac{1}{\lambda_n} \sum_{k \in I_n} |x_{k+m} - L$$
$$\leq \frac{2}{\lambda_n} \sum_{k \in I_n} |x_{k+m} - L|.$$

Hence $x_k \to L[\hat{c}]$, since $x_k \to L[\hat{V}, \lambda]$.

(iii) This immediately follows from (i) and (ii). □

3. It is easy to see that $\hat{s}_\lambda \subseteq \hat{s}$ for all λ, since λ_n/n is bounded by 1. Now we have

Theorem 3 $\hat{s} \subseteq \hat{s}_\lambda$ *if and only if*

$$\liminf_n \frac{\lambda_n}{n} > 0 \tag{1}$$

Proof. For given $\varepsilon > 0$ we get,

$$\{k \leq n : |x_{k+m} - L| \geq \varepsilon\} \supset \{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}.$$

Hence,

$$\begin{aligned} \frac{1}{n}|\{k \leq n : |x_{k+m} - L| \geq \varepsilon\}| &\geq \frac{1}{n}|\{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}|, \\ &\geq \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k \in I_n : |x_{k+m} - L| \geq \varepsilon\}|. \end{aligned}$$

Taking limit as $n \to \infty$ and using (1), we have

$$x_k \to L(\hat{s}) \Rightarrow x_k \to L(\hat{s}_\lambda).$$

Conversely, suppose that $\liminf_n \frac{\lambda_n}{n} = 0$.

As in [7] we can choose a subsequence $(n(j))_j$ such that $\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}$.

Define a sequence $x = (x_i)$ as follows:

$$x_i = \begin{cases} 1 & \text{if } i \in I_{n(j)}, \quad j = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in [\hat{c}]$ and hence Theorem 1, $x \in \hat{s}$. But on the other hand $x \notin [\hat{V}, \lambda]$ and Theorem 2 (ii) implies that $x \notin \hat{s}_\lambda$. Hence (1) is necessary □

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References

[1] Banach S., *Theorie des Operations Linearies.* Warsazwa (1932).

[2] Connor J., *The statistical and strong p-Cesaro convergence of sequences.* Analysis **8** (1988), 47–63.

[3] Connor J., *On strong matrix summability with respect to a modulus and statistical convergence.* Canad. Math. Bul. **32** (2) (1989), 194–198.

[4] Duran J.P., *Infinite matrices and almost convergence.* Math. Zeit. **128** (1972), 75–83.

[5] Erdös P. and Tenenbaum, *Sur les densities de certaines suites d'entiers.* Proc. London Math. soc. (3), **59** (1989] 417–438.

[6] Fast H., *Sur la convergence statistique.* Colloq. Math. **2** (1951), 241–244.

[7] Freedman A.R., Sember J.J. and Raphael M., *Some Cesaro-type summability spaces.* Proc. London Math. Soc. **37** (1978), 508–520.

[8] Freedman A.R. and Sember J.J., *Densities and summability.* Pacific J. Math. **95** (2) (1981), 293–305.

[9] Fridy J.A., *On statistical convergence.* Analysis **5** (1985), 301–313.

[10] King J.P., *Almost summable sequences.* Proc. Amer. Math. Soc. **16** (6), (1966), 1219–1225.

[11] Kuttner B., *Note on strong summability.* J. London Math. Soc. **21** (1946), 118-9122.

[12] Leindler L., *Uber de la Vallee Poursinsche Summierbarkeit allgemeiner Orthogonalreihen.* Acta Math. Acad. Sci. Hung. **16** (1965), 375–378.

[13] Lorentz G.G., *A contribution to the theory of divergent sequences.* Acta Math. **80** (1948), 167–190.

[14] Maddox I.J., *Spaces of strongly summable sequences.* Quart. J. Math. **18** (1967), 345–355.

[15] Maddox I.J., *A new type of convergence.* Math. Proc. Camb. Phil. Soc. **83** (1978) 61–64

[16] Maddox I.J., *Inclusions between FK-spaces and Kuttner's theorem.* Math. Proc. Camb. Phil. Soc. **101** (1987), 523–527.

[17] Rath D. and Tripaty B.C., *On Statistically convergent and statistically Cauchy sequences.* Indian J. Pure. appl. Math. **25** (4), (1994), 381–386.

[18] Salat T., *On statistically convergent sequences of real numbers.* Math. Slovaca **30**, 2 (1980), 139–150.

[19] Zygmund A., *Trigonometric series.* Second Ed. Cambridge University Press (1979).

Department of Mathematics
Yüzüncü Yil University
Van, Turkey
E-mail: esavas@ahtamara.yyu.edu.tr