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On the Schur indices of the irreducible characters of $SL(n, q)$

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Abstract. We shall give some sufficient conditions subject for that the Schur indices of irreducible characters of the special linear groups over finite fields are equal to one.

Key words: special linear groups, irreducible characters, Schur index.

Introduction

Let S denote the special linear group $SL(n, q)$ of degree $n \geqq 2$ over a finite field $\mathbb{F}_q$ with q elements of characteristic p. If χ is a complex irreducible character of a finite group and K is a field of characteristic 0, then $m_K(\chi)$ denotes the Schur index of χ with respect to K, where we consider χ as a character over some algebraically closed extension of K. Then the following results are known:

Theorem A (R. Gow [3]) *For any (complex) irreducible character χ of S, we have $m_{\mathbb{Q}}(\chi) \leqq 2$.*

Theorem B (A.V. Zelevinsky [15]) *Assume that $p = 2$. Then, for any irreducible character χ of S, $m_{\mathbb{Q}}(\chi) = 1$.*

Theorem C (Z. Ohmori [9]) *Assume that $p \neq 2$ and n is odd. Then, for any irreducible character χ of S, $m_{\mathbb{Q}}(\chi) = 1$.*

Theorem D (Gow [3]) *Assume that $p \neq 2$, n is even, and* $\mathrm{ord}_2 n > \mathrm{ord}_2(p-1)$. *Then, for any irreducible character χ of S, $m_{\mathbb{Q}}(\chi) = 1$.*

Theorem E (Gow [3]) *Assume that $p \neq 2$, n is even,* $\mathrm{ord}_2 n \leqq \mathrm{ord}_2(p-1)$, *and q is an even power of p. Let χ be any irreducible character of S. Then, if $\chi(-1_n) = \chi(1_n)$, we have $m_{\mathbb{Q}}(\chi) = 1$. If $\chi(-1_n) = -\chi(1_n)$, then, for any prime number $r \neq p$, we have $m_{\mathbb{Q}_r}(\chi) = 1$.*

Theorem F (Gow [3]) *Assume that $p \neq 2$ and $n = 4m$ for some positive*

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integer m. Then, for any irreducible character χ of S, $m_{\mathbb{R}}(\chi) = 1$.

The purpose of this paper is to give some more sufficient conditions subject for that $m_{\mathbb{Q}}(\chi) = 1$ for irreducible characters χ of S (Propositions 2, 3).

In §1 we shall review Zelevinsky's result which states that, for any irreducible character η of $G = GL(n,q)$, there is a linear character ϕ of a Sylow p-subgroup U of G such that $(\phi^G, \eta)_G = 1$ (see Theorem 1). In §§2, 3 we shall show that Theorems A, B, C, D, E, F are easy cosequences of Theorem 1; the idea of our proofs is originally due to Gow ([2]). In §3 we shall give some more detailed results concerning Theorem E. The contents of §§4, 5 are our main results.

1. A Zelevinsky's theorem

The purpose of this section is to review a result of A.V. Zelevinsky which is proved in [15, 12.5, p.141] (see Theorem 1 below).

Let m be a non-negative integer. If $m_1, \ldots, m_s$ are non-negative integers such that $m = m_1 + \cdots + m_s$, then the symbol $[m_1, \ldots, m_s]$ will be called a partition of m; if $m'_1, \ldots, m'_s$ is any permutation of $m_1, \ldots, m_s$, then we should have $[m'_1, \ldots, m'_s] = [m_1, \ldots, m_s]$; we also have

$$[m_1, \ldots, m_s, 0, 0, \ldots, 0] = [m_1, \ldots, m_s].$$

If a partition μ of m has r_1 parts equal to 1, r_2 parts equal to 2, r_3 parts equal to $3, \ldots$, then we shall often write $\mu = [1^{r_1} 2^{r_2} e^{r_3} \cdots]$. If $m = 0$, 0 will denote the partition of the number 0. P_m will denote the set of all partitions of m; P_m has a lexicographical ordering: for $\mu = [m_1, \ldots, m_s] \in P_m$ with $m_1 \geqq \cdots \geqq m_s \geqq 0$ and $\nu = [n_1, \ldots, n_s] \in P_m$ with $n_1 \geqq \cdots \geqq n_s \geqq 0$, we have $\mu > \nu$ either if $m_1 > n_1$ or if $m_1 = n_1, \ldots, m_i = n_i$ and $m_{i+1} > n_{i+1}$ for some $i \geqq 1$. If $\mu \in P_m$, then we write $|\mu| = m$. Let $\mu \in P_m$, and arrange its parts in descending order: $\mu = [m_1, \ldots, m_s]$ with $m_1 \geqq \cdots \geqq m_s > 0$; then we define $\widetilde{\mu} = [s^{m_s}(s-1)^{m_{s-1}-m_s}(s-2)^{m_{s-2}-m_{s-1}} \cdots 1^{m_1-m_2}]$; $\widetilde{\mu}$ is called the conjugate partition of μ. We put $P = \bigcup_{m \geqq 0} P_m$.

Let m, m' be non-negative integers. Let $\mu = [m_1, \ldots, m_s] \in P_m$ with $m_1 \geqq \cdots \geqq m_s \geqq 0$, and $\mu' = [m'_1, \ldots, m'_{s'}] \in P_{m'}$ with $m'_1 \geqq \cdots \geqq m'_{s'} \geqq 0$. Assume that $s \geqq s'$. Then we define $\mu \cdot \mu' = [m_1 + m'_1, \ldots, m_{s'} + m'_{s'}, m_{s'+1}, \ldots, m_s]$. For example, if $\mu = [3, 1, 2]$ and $\mu' = [2, 4, 1, 5]$, then $\mu \cdot \mu' = [3, 2, 1, 0] \cdot [5, 4, 2, 1] = [3+5, 2+4, 1+2, 0+1] = [8, 6, 3, 1]$.

We also define $\mu + \mu' = [m_1, \dots, m_s, m'_1, \dots, m'_{s'}]$. We note that $\mu \cdot \mu'$, $\mu + \mu' \in P_{m+m'}$.

Let d be a positive integer and v a non-negative integer; if $\pi = [p_1, \dots, p_s] \in P_v$, then we define $d \cdot \pi = [dp_1, \dots, dp_s] \in P_{dv}$.

Let x be a variable over $\mathbb{F}_q$. Let F be the set of all irreducible polynomials $f = f(x)$ over $\mathbb{F}_q$ other than the polynomial x; we write $d(f)$ for the degree of f. Call C the set of all functions $\nu : F \to P$ such that $\sum_{f \in F} |\nu(f)| d(f) = n$; for $\nu \in C$, set (symbolically)

$$c = c_\nu = (\cdots f^{\nu(f)} \cdots) = (f_1^{\nu_1} \cdots f_N^{\nu_N}),$$

where $f_1, \dots, f_N$ are all the $f \in F$ such that $\nu(f) \neq 0$, and, for $1 \leqq i \leqq N$, $\nu_i = \nu(f_i)$. Then, by the theory of Jordan canonical forms over $\mathbb{F}_q$, we know that the symbols c_ν, $\nu \in C$, parametrize the conjugacy classes of $G = GL(n,q)$. We identify c_ν with the corresponding conjugacy class of G. In particular, the classes $((x-1)^\lambda)$, $\lambda \in P_n$, are the unipotent classes of G; if a unipotent element u of G belongs to a class $((x-1)^\lambda)$, then we shall say that u is a unipotent element of G of type λ; if $\lambda = [d_1, \dots, d_s]$ with $d_i \neq 0$ for all i, then u is conjugate in G to the element of the form:

$$\begin{pmatrix} \underbrace{\begin{matrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & & 1 \end{matrix}}_{d_1} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\begin{matrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & & 1 \end{matrix}}_{d_s} \end{pmatrix}.$$

Let s be a positive integer. Then a set $g = \{h, hq, hq^2, \ldots, hq^{s-1}\}$ of integers will be called an s-simplex with the roots $h, hq, hq^2, \ldots, hq^{s-1}$ if they are distinct modulo $q^s - 1$; we identify g with $qg = \{hq, hq^2, hq^3, \ldots, hq^s\}$; we write $d(g) = s$. Call Σ the set of all s-simplexes for $s \geqq 1$. Let X be the set of all functions $\nu : \Sigma \to P$ such that $\sum_{g \in \Sigma} |\nu(g)| d(g) = n$; for $\nu \in X$, set (symbolically)

$$\eta_\nu = (\cdots g^{\nu(g)} \cdots) = (g_1^{\nu_1} \cdots g_N^{\nu_N}),$$

where $g_1, \ldots, g_N$ are all the $g \in \Sigma$ such that $\nu(g) \neq 0$, and, for $1 \leqq i \leqq N$, $\nu_i = \nu(g_i)$. Then the symbols η_ν, $\nu \in X$, parametrize the irreducible characters of G (see J.A. Green [4]). We identify η_ν with the corresponding irreducible character of G.

Let U be the upper-triangular maximal unipotent subgroup of G; U is a Sylow p-subgroup of G. Let $\psi : \mathbb{F}_q^+ \to \mathbb{C}^\times$ be a fixed non-trivial additive character of the additive group $\mathbb{F}_q^+$ of $\mathbb{F}_q$. For $a \in \mathbb{F}_q$, we define $\psi_a \in \mathrm{Hom}(\mathbb{F}_q^+, \mathbb{C}^\times)$ by $\psi_a(y) = \psi(ay)$, $y \in \mathbb{F}_q$. Then $\{\psi_a \mid a \in \mathbb{F}_q\} = \mathrm{Hom}(\mathbb{F}_q^+, \mathbb{C}^\times)$. Let U' be the derived group (i.e. the commutator subgroup) of U; for $u = (u_{ij}) \in U$, $u \in U'$ if and only if $u_{12} = u_{23} = \cdots = u_{n-1,n} = 0$; so $U/U' = \prod^{n-1} \mathbb{F}_q^+$. Thus, for any linear character ϕ of U, there are $a_1, \ldots, a_{n-1} \in \mathbb{F}_q$ such that $\phi(u) = \psi(a_1 u_{12} + a_2 u_{23} + \cdots + a_{n-1} u_{n-1,n})$, $u = (u_{ij}) \in U$; for such ϕ, define $u_\phi = (u_{ij}) \in U$ by $u_{i,i+1} = a_i$, $1 \leqq i \leqq n-1$, and $u_{ij} = 0$, $1 \leqq i < j-1 < n$; then we shall say that ϕ is of type λ if u_ϕ is of type λ.

Now we can state Zelevinsky's result:

Theorem 1 (Zelevinsky [15, 12.5]) *Let η be any irreducible character of $G = GL(n,q)$, and suppose that $\eta = (g_1^{\nu_1} \cdots g_N^{\nu_N})$. Let $\mu = (d(g_1) \cdot \widetilde{\nu}_1) \cdot \cdots \cdot (d(g_N) \cdot \widetilde{\nu}_N)$. Let ϕ be a linear character of U of type λ. Then we have $(\phi^G, \eta)_G = 1$ if $\lambda = \mu$, and 0 if $\lambda > \mu$.*

We should compare the statement in Theorem 1 with that of Theorem 1 of [10]. In fact, if we use the character-theory of G by Green [4], we can prove Theorem 1 by a method similar to that in the proof of Theorem 1 of [10].

2. Some remarks

In this and next sections, we shall show that Theorems A, B, C, D, E, F are easy consequences of Theorem 1.

Proof of Theorem B. Assume that $p = 2$. Then U/U' is an elementary abelian 2-group, so that any linear character of U is realizable in $\mathbb{Q}$. Thus, for any linear character ϕ of U, ϕ^S is also realizable in $\mathbb{Q}$. Let χ be any irreducible character of S. Then, by Clifford theory, there is an irreducible character η of G such that $\eta \mid S = \chi_1 + \cdots + \chi_h$, where $\chi_1, \ldots, \chi_h$ are the G-conjugates of χ (cf. G/S is a cyclic group). By Theorem 1, there is a linear character ϕ of U such that $(\phi^G, \eta)_G = 1$, so that, by Frobenius reciprocity law, we have $(\phi^S, \eta \mid S)_S = (\phi^S, \chi_1 + \cdots + \chi_h)_S = 1$, and $(\phi^S, \chi_i)_S = 1$ for some i.

The following property of the Schur index is well known, and is proved, for instance, in Feit's book [1] ([1, (11.4), p.62]).

Lemma 1 *Let H be a finite group, and let K be a field of characteristic 0. Let θ be an irreducible character of H and ξ a character of H which is realizable in K. Then $m_K(\theta)$ divides $(\xi, \theta)_H$.*

By Lemma 1, we have $m_{\mathbb{Q}}(\chi_i) = 1$. As χ is G-conjugate to χ_i, we therefore have $m_{\mathbb{Q}}(\chi) = 1$.

Remark Let H be a finite group, N a normal subgroup of H, and K a field of characteristic 0. Let θ be an irreducible character of H. Then, by Clifford theory, there are irreducible characters $\rho_1, \ldots, \rho_h$ of N and a positive integer e such that $\theta \mid N = e(\rho_1 + \cdots + \rho_n)$ (see, e.g., Feit [1, (9.10), p.53]). For $1 \leqq i \leqq h$, there is an element x_i of H such that $\rho_i = \rho^{x_i}$ ($\rho = \rho_1$, $\rho^{x_i}(y) = \rho(x_i y x_i^{-1})$, $y \in N$). Set $K_i = K(\rho_i) = K(\rho_i(y),\ y \in N)$, $1 \leqq i \leqq h$. Then we clearly have $K_1 = \cdots = K_h$. For $1 \leqq i \leqq h$, let A_i be the simple component of the group algebra $K_i[N]$ of N over K_i associated with ρ_i; there is a finite dimensional central simple division algebra D_i over K_i and a positive integer r_i such that A_i is isomorphic over K_i to the total matrix algebra $M_{r_i}(D_i)$ over D_i; we have $[D_i : K_i] = m_{K_i}(\rho_i)^2 = m_K(\rho_i)^2$; the mapping $y \to x_i y x_i^{-1}$, $y \in N$, induces an isomorphism of $K_1[N]$ onto $K_i[N]$, which clearly maps A_1 onto A_i, so that A_1 is isomorphic to A_i as rings; thus we must have $m_K(\rho) = m_{K_1}(\rho) = m_{K_i}(\rho_i) = m_K(\rho_i)$.

Proof of Theorem C. Assume that $p \neq 2$ and n is odd. We fix a primitive element ν of the prime field $\mathbb{F}_p$ of $\mathbb{F}_q$. Let $\widetilde{\nu}$ be a fixed integer such that $\widetilde{\nu} \bmod p\mathbb{Z} = \nu$ in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, let ζ_p be a fixed primitive p-th root of unity in $\mathbb{C}$, and let α be the automorphism of $\mathbb{Q}(\zeta_p)$ over $\mathbb{Q}$ given by $\zeta_p^\alpha = \zeta_p^{\widetilde{\nu}}$. Suppose that $n = 2m+1$ with $m \geqq 1$, and let

$$t = \mathrm{diag}(\nu^m, \nu^{m-1}, \ldots, \nu, 1, \nu^{-1}, \nu^{-2}, \ldots, \nu^{-m}).$$

Then $t \in S$, $t^{p-1} = 1$, and, for $u \in U$, $tut^{-1} \equiv u^{\widetilde{\nu}} \pmod{U'}$.

Let ϕ be any linear character of U. We show that ϕ^S is realizable in $\mathbb{Q}$. In fact, if $\phi = 1_U$, then ϕ^S is clearly realizable in $\mathbb{Q}$. Suppose therefore that $\phi \neq 1_U$. Then, for $u \in U$, $\phi^t(u) = \phi(tut^{-1}) = \phi(u^{\widetilde{\nu}}) = \phi(u)^{\widetilde{\nu}} = \phi(u)^\alpha$. Let $M = U\langle t\rangle$. Then, as U is a normal subgroup of M, $\phi^M = 0$ outside of U, and $\sum_{i=1}^{p-1} \phi^{\alpha^i}$ on U. It follows that ϕ^M is a $\mathbb{Q}$-valued irreducible character of M. Moreover, by Propositions 3.4, 3.5 of T. Yamada [14], we see that the simple component A of $\mathbb{Q}[M]$ associated with ϕ^M is isomorphic over $\mathbb{Q}$ to the cyclic algebra $(1, \mathbb{Q}(\zeta_p), \alpha)$ over $\mathbb{Q}$. The latter cyclic algebra clearly splits in $\mathbb{Q}$, so that we have $m_{\mathbb{Q}}(\phi^M) = 1$. Therefore ϕ^M is realizable in $\mathbb{Q}$, hence $\phi^S = (\phi^M)^S$ is realizable in $\mathbb{Q}$.

Thus we have proved that ϕ^S is realizable in $\mathbb{Q}$ for any linear character ϕ of U. Therefore, by the argument in the proof of Theorem B, we can prove that $m_{\mathbb{Q}}(\chi) = 1$ for any irreducible character χ of S.

Proof of Theorem D. Assume that $p \neq 2$ and n is even. Let

$$t = \mathrm{diag}(\nu^{n-1}\xi, \nu^{n-2}\xi, \ldots, \nu\xi, \xi), \quad \xi \in \mathbb{F}_p^\times.$$

Then t is an element of G, of order $p-1$, such that $\phi^t = \phi^\alpha$ for any linear character ϕ of U. As Gow has observed in [3, p.140], we have $t \in S$ if and only if $\mathrm{ord}_2 n > \mathrm{ord}_2(p-1)$. Suppose that this is the case, and put $M = U\langle t\rangle$. Then, as in the proof of Theorem C, we see that ϕ^M is realizable in $\mathbb{Q}$ for any linear character ϕ of U. Thus the argument goes as in the proof of Theorem C.

Proof of Theorem F. Assume that $p \neq 2$ and $n = 4m$ for some positive integer m. Let

$$t = \mathrm{diag}(1, -1, 1, -1, \ldots, 1, -1).$$

Then t is an element of S, of order 2, such that, for $u \in U$, $tut^{-1} \equiv u^{-1}$ (mod U'). Let ϕ be any non-principal linear character of U. Then we see that $\phi^t = \phi^{-1}$. Let $M = U\langle t\rangle$. Then we see easily that ϕ^M is a real-valued irreducible character of M. Moreover we see that the simple component of $\mathbb{R}[M]$ associated with ϕ^M is isomorphic over $\mathbb{R}$ to the cyclic algebra $(1,\mathbb{C},\iota)$ over $\mathbb{R}$ where $Gal(\mathbb{C}/\mathbb{R}) = \langle\iota\rangle$. Thus $m_{\mathbb{R}}(\phi^M) = 1$, and ϕ^M is realizable in $\mathbb{R}$. Thus, for any linear character ϕ of U, ϕ^S is realizable in $\mathbb{R}$. Therefore, by an argument similar to that in the proof of Theorem B, we can prove that $m_{\mathbb{R}}(\chi) = 1$ for any irreducible character χ of S.

Proof of Theorem A. Assume that $p \neq 2$, $n = 2m$ is even, and $\mathrm{ord}_2 n \leqq \mathrm{ord}_2(p-1)$. Let

$$t = \mathrm{diag}(\nu^{2m-1}, \nu^{2m-3}, \ldots, \nu, \nu^{-1}, \nu^{-3}, \ldots, \nu^{-(2m-1)})$$

Then $t \in S$, $t^{(p-1)/2} = -1_n$, and, for $u \in U$, $tut^{-1} = u^{\tilde{\nu}^2}$ (mod U'). Let $M = U\langle t\rangle$, and let ϕ be any non-principal linear character of U. Then $\phi^M = 0$ outside of U, and $2\sum_{i=1}^{(p-1)/2}\phi^{\alpha^{2i}}$ on U. Let $k = \mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$. For $j = 0, 1$, call ϕ_j the linear character of $U\langle -1_n\rangle$ defined by $\phi_j \,|\, U = \phi$ and $\phi_j(-1_n) = (-1)^j$. Then we see that $(\phi_0)^M$ and $(\phi_1)^M$ are k-valued irreducible characters of M, and $\phi^M = (\phi_0)^M + (\phi_1)^M$. Moreover we see that, for $j = 0, 1$, the simple component of $k[M]$ associated with $(\phi_j)^M$ is isomorphic over k to the cyclic algebra $((-1)^j, k(\zeta_p), \alpha^2)$ over k.

Lemma 2 (G.J. Janusz [5, Proposition 3]) *$m_k((\phi_0)^M) = 1$, so that $(\phi_0)^M$ is realizable in k. Assume that $p \equiv 1$ (mod 4). Then $m_{\mathbb{R}}((\phi_1)^M) = 2$, and, for any finite place v of k, $m_{k_v}((\phi_1)^M) = 1$ (k_v is the completion of k at v). If $p \equiv -1$ (mod 4), then $m_k((\phi_1)^M) = 1$, and $(\phi_1)^M$ is realizable in k.*

Let χ be any irreducible character of S. Let $\chi_1, \ldots, \chi_h$ be the G-conjugates of χ, and let η be an irreducible character of G such that $\eta \,|\, S = \chi_1 + \cdots + \chi_h$. Let ϕ be a linear character of U such that $(\phi^G, \eta)_G = 1$ (Theorem 1). Then $(\phi^S, \chi_i)_S = 1$ for some i. If $\phi = 1_U$, then ϕ^S is realizable in $\mathbb{Q}$, so that $m_{\mathbb{Q}}(\chi_i) = 1$, and $m_{\mathbb{Q}}(\chi) = 1$. Assume therefore that $\phi \neq 1_U$. Then $((\phi_0)^S + (\phi_1)^S, \chi_i)_S = 1$, so $((\phi_0)^S, \chi_i)_S = 1$ or $((\phi_1)^S, \chi_i)_S = 1$; in view of Schur's lemma, as -1_n is a central element of S, we have $((\phi_j)^S, \chi_i)_S = 1$ if and only if $\chi_i(-1_n) = (-1)^j\chi_i(1_n)$. Suppose that $j = 0$. Thus as $(\phi_0)^S$ is realizable in k (Lemma 2), by Lemma 1, we have $m_k(\chi_i) = 1$. Let $k(\chi_i) = k(\chi_i(y),\ y \in S)$. Then, as $[k : \mathbb{Q}] = 2$, $[k(\chi_i) : \mathbb{Q}(\chi_i)] \leqq 2$. Thus $m_{\mathbb{Q}}(\chi_i) = m_{\mathbb{Q}(\chi_i)}(\chi_i) \leqq 2$. Suppose that $j = 1$. Then $m_{\mathbb{R}}(\chi_i) \leqq 2$ (of course). Let r be any prime number, and let v be a place of k lying above r. Then, by Lemma 2, $(\phi_1)^S$ is realizable in k_v, so that $m_{k_v}(\chi_i) = 1$. But, as

$[k_v : \mathbb{Q}_r] \leqq 2$, we have $k_{\mathbb{Q}_r}(\chi_i) \leqq 2$. As $m_{\mathbb{Q}}(\chi_i)$ is equal to the least common multiple of the $m_{\mathbb{Q}_s}(\chi_i)$, where s runs over all places of $\mathbb{Q}$, we must have $m_{\mathbb{Q}}(\chi_i) \leqq 2$. Thus, in any case, we have $m_{\mathbb{Q}}(\chi) \leqq 2$. This completes the proof of Theorem A.

We make here some comments about Theorem A. Let the notation be as in the proof of Theorem A. We assume that $\eta = (g_1^{\nu_1} \cdots g_N^{\nu_N})$, and that ϕ is of type $(d(g_1) \cdot \widetilde{\nu}_1) \cdots (d(g_N) \cdot \widetilde{\nu}_N)$ (see Theorem 1). As to the local index $m_{\mathbb{R}}(\chi)$ of χ, if χ is not real, then $m_{\mathbb{R}}(\chi) = 1$. This is a consequence of the fact that the Brauer group of $\mathbb{C}$ is trivial. Suppose that χ is real. If χ is linear, then $m_{\mathbb{Q}}(\chi) = 1$. Therefore we may assume that χ is not linear, which forces that $\phi \neq 1_U$. [In fact, suppose that $\phi = 1_U$. Then ϕ is of type $[1^n]$, so that $N = 1$, $d(g_1) = 1$, and $\nu_1 = [n]$, i.e., $\eta = (g_1^{[n]})$. Then, by Remark (3) of Green [4, p.444], η must be linear.] By Schur's lemma, we must have $\chi(-1_n) = \chi(1_n)$ or $-\chi(1_n)$. Assume that $p \equiv 1 \pmod 4$. Then, if $\chi(-1_n) = \chi(1_n)$, we must have $((\phi_0)^S, \chi_i)_S = 1$, so that, as $(\phi_0)^S$ is realizable in $\mathbb{R}$, we have $m_{\mathbb{R}}(\chi_i) = 1$, and $m_{\mathbb{R}}(\chi) = 1$. Suppose that $\chi(-1_n) = -\chi(1_n)$. Then we must have $((\phi_1)^S, \chi_i)_S = 1$ and $m_{\mathbb{R}}((\phi_1)^M) = 2$. As $\mathbb{R}(\chi_i) = \mathbb{R}((\phi_1)^M) = \mathbb{R}$, by Proposition 3.8 of [14, p.29], we have $m_{\mathbb{R}}(\chi_i) = 2$, and $m_{\mathbb{R}}(\chi) = 2$. When $p \equiv -1 \pmod 4$, if q is an even power of p, then the same results hold as we shall see in the next section. These results have been already observed by Gow ([3, Theorem 3.4(d)]).

However, as far as the author knows, the character table of S is not obtained if $n \geqq 4$, so that to know whether χ is real or not is generally a difficult problem (but, cf. §3 below).

We next consider other local indices of χ. Suppose that $p \equiv 1 \pmod 4$. Then $k = \mathbb{Q}(\sqrt{p})$. If $\mathbb{Q}(\chi) \supset k$, then we have $m_{\mathbb{Q}_r}(\chi) = 1$ for any prime number r (cf. G.I. Lehrer [8]). Generally, let r be any prime number, and let v be a place of k that lies above r. Then we see that, for $r \neq p$, $k_v = \mathbb{Q}_r$ if and only if the Legendre symbol $\left(\frac{p}{r}\right) = 1$, and if this is the case, then we have $m_{\mathbb{Q}_r}(\chi) = 1$. Suppose that $p \equiv -1 \pmod 4$. Then $k = \mathbb{Q}(\sqrt{-p})$. Then, if $\mathbb{Q}(\chi) \supset k$, we have $m_{\mathbb{Q}}(\chi) = 1$. Generally, for a prime number $r \neq p$, and for a place v of k that lies above r, we have $k_v = \mathbb{Q}_r$ if and only if $\left(\frac{-p}{r}\right) = 1$, and if this is the case, we have $m_{\mathbb{Q}_r}(\chi) = 1$.

3. Proof of Theorem E and some comments

We assume that $p \neq 2$, $n = 2m$ is even, $\mathrm{ord}_2 n \leqq \mathrm{ord}_2(p-1)$, and q is an even power of p. Then there is an element ε of $\mathbb{F}_q$ such that $\varepsilon^2 = \nu$. Let

$$t = \mathrm{diag}(\varepsilon^{2m-1}, \varepsilon^{2m-3}, \ldots, \varepsilon, \varepsilon^{-1}, \varepsilon^{-3}, \ldots, \varepsilon^{-(2m-1)}).$$

Then $t \in S$, $t^{p-1} = -1_n$, and, for $u \in U$, $tut^{-1} \equiv u^{\widetilde{\nu}} \pmod{U'}$. Let $M = U\langle t\rangle$. Let ϕ be any non-principal linear character of U; for $j = 0, 1$, let ϕ_j be as in the proof of Theorem A. Then we see that $(\phi_0)^M$ and $(\phi_1)^M$ are $\mathbb{Q}$-valued irreducible characters of M, and $\phi^M = (\phi_0)^M + (\phi_1)^M$. Moreover we see that, for $J = 0, 1$, the simple component of $\mathbb{Q}[M]$ assiciated with $(\phi_j)^M$ is isomorphic over $\mathbb{Q}$ to the cyclic algebra $((-1)^j, \mathbb{Q}(\zeta_p), \alpha)$ over $\mathbb{Q}$.

Lemma 3 (Janusz [5, Proposition 2]) *We have $m_{\mathbb{Q}}((\phi_0)^M) = 1$. And, we have $m_{\mathbb{R}}((\phi_1)^M) = m_{\mathbb{Q}_p}((\phi_1)^M) = 2$ and $m_{\mathbb{Q}_r}((\phi_1)^M) = 1$ for any prime number $r \neq p$.*

Thus by using the argument similar to that in the proof of Theorem A, we can prove Theorem E.

As far as we are concerned with the proof of Theorem E, the above argument is sufficient. But in our case we can get more detailed results.

Let the assumption be as above. Let χ be any irreducible character of S, let $\chi_1, \ldots, \chi_h$ be the G-conjugates of χ, let η be an irreducible character of G such that $\eta \,|\, S = \chi_1 + \cdots + \chi_h$, and let ϕ be a linear character of U such that $(\phi^G, \eta)_G = 1$; we may assume that $\eta = (g_1^{\nu_1} \cdots g_N^{\nu_N})$ and ϕ is of type $(d(g_1) \cdot \widetilde{\nu}_1) \cdot \cdots \cdot (d(g_N) \cdot \widetilde{\nu}_N)$ (see Theorem 1). Call Z the centre of S, let z be a generator of Z, call c the order of Z, i.e. $c = (n, q-1)$, and let ζ_c be a fixed primitive c-th root of unity in $\mathbb{C}$. Let t be as above, and let $L = UZ\langle t\rangle$. For $j = 1, \ldots, c$, let η_j be the linear character of $Z = \langle z\rangle$ given by $\eta_j(z) = \zeta_c^j$, and let $\mu_j = \mathrm{Ind}_{UZ}^L(\phi\eta_j)$. We may assume that χ is not linear, so that $\phi \neq 1_U$. For $j = 1, \ldots, c$, let $k_j = \mathbb{Q}(\zeta_c^j)$. We see that $\mu_1, \ldots, \mu_c$ are different irreducible characters of L and $\phi^L = \mu_1 + \cdots + \mu_c$. Moreover, we see that, for $j = 1, \ldots, c$, the simple component B_j of $k_j[L]$ associated with μ_j is isomorphic over k_j to the cyclic algebra $((-1)^j, k_j(\zeta_p), \alpha_j)$ over k_j, where k_j is the extension of α to $k_j(\zeta_p)$ over k_j (note that μ_j is k_j-valued).

We have $(\phi^S, \chi_i)_S = 1$ for some i, and we assume that $\chi = \chi_i$. Then $((\mu_1)^S + \cdots + (\mu_c)^S, \chi)_S = 1$, and $((\mu_j)^S, \chi)_S = 1$ for some j; by Schur's lemma, we must have $\chi(z) = \zeta_c^j \chi(1_n)$. (We note that one can calculate the value $\eta(z)$ explicitely (cf. [12]), so that $\chi(z)$ can be known.) Assume that j is even. Then $B_j \sim (1, k_j(\zeta_p), \alpha_j) \sim k_j$, so that $(\mu_j)^S$ is realizable in k_j, hence $m_{\mathbb{Q}}(\chi) = m_{k_j}(\chi) = 1$. This case corresponds to the case that $\chi(-1_n) = \chi(1_n)$. Suppose that j is odd (i.e. $\chi(-1_n) = -\chi(1_n)$). If χ is not real, we have $m_{\mathbb{R}}(\chi) = 1$. In particular, if $c > 2$, and if $j \neq c/2$, then $m_{\mathbb{R}}(\chi) = 1$. Suppose that χ is real. Then we must have $j = c/2$. Then $m_{\mathbb{R}}(\mu_j) = 2$, so that we must have $m_{\mathbb{R}}(\chi) = 2$.

We next investigate the local index $m_{\mathbb{Q}_p}(\chi)$ of χ. As j is odd, $B_j \simeq (-1, k_j(\zeta_p), \alpha_j) \sim k_j \otimes_{\mathbb{Q}} (-1, \mathbb{Q}(\zeta_p), \alpha)$. Let v be a place of k_j that lies above p, and let $f_j = [(k_j)_v : \mathbb{Q}_p]$. Then the Hasse invariant of B_j at v is $\equiv f_j \times \frac{1}{2}$ (mod 1).

Lemma 4 ([11, Lemma 11]) *Let $q = p^{2^a s}$ with $(2, s) = 1$. Then f_j is odd if and only if any odd prime divisor of $c/(c, j)$ divides $p^s - 1$.*

Thus we get

Proposition 1 *Assume that $p \neq 2$, n is even, $\mathrm{ord}_2 n \leqq \mathrm{ord}_2(p-1)$, and q is an even power of p; let $q = p^{2^a s}$ with $(2, s) = 1$. Let χ be an irreducible character of S such that $\chi(-1_n) = -\chi(1_n)$. Let z be a generator of the centre Z of S, let c be the order of Z, and let ζ_c be a primitive c-th root of unity. Suppose that $\chi(z) = \zeta_c^j \chi(1_n)$, and that $c > 2$. Then*:

(i) *If $j \not\equiv \frac{c}{2}$ (mod c), we have $m_{\mathbb{R}}(\chi) = 1$.*

(ii) *If some odd prime divisor of $c/(c, j)$ does not divides $p^s - 1$, then $m_{\mathbb{Q}_p}(\chi) = 1$.*

Theorem G *Assume that $p = 2$, or n is odd, or $p \neq 2$ and $\mathrm{ord}_2 n > \mathrm{ord}_2(p-1)$, or q is an even power of p. Then, for any irreducible character χ of $PSL(n, q)$, we have $m_{\mathbb{Q}}(\chi) = 1$.*

Remark Let p be an odd prime number such that $p \equiv -1$ (mod 4) and $(6, p-1) = 2$. Then $PSL(6, p)$ has irreducible characters χ such that $m_{\mathbb{R}}(\chi) = 2$ (see D. Prasad [13, pp. 309–310]).

4. Some sufficient condition (I)

Assume that $p \neq 2$. Let η be an irreducible character of G, and suppose that $\eta = (g_1^{\nu_1} \cdots g_N^{\nu_N})$. Let $\mu = (d(g_1) \cdot \widetilde{\nu}_1) \cdot \cdots \cdot (d(g_N) \cdot \widetilde{\nu}_N) = [m_1, \ldots, m_s]$ with $m_i \neq 0$ for all i. Let $\eta \,|\, S = \chi_1 + \cdots + \chi_h$, where $\chi_1, \ldots, \chi_h$ are some irreducible characters of S. Let ϕ be a linear character of U of type

μ. Suppose that $\mu \neq [1^n]$, i.e. $\phi \neq 1_U$. Let, for $u = (u_{ij}) \in U$, $\phi(u) = \psi(a_1u_{12} + a_2u_{23} + \cdots + a_{n-1}u_{n-1,n})(a_1, \ldots, a_{n-1} \in \mathbb{F}_q)$ (see §1). We assume that $a_1 \neq 0, \ldots, a_{m_1-1} \neq 0$, $a_{m_1} = 0$, $a_{m_1+1} \neq 0, \ldots, a_{m_1+m_2-1} \neq 0$, $a_{m_1+m_2} = 0, \ldots, a_{m_1+\cdots+m_{s-1}} = 0$, $a_{m_1+\cdots+m_{s-1}+1 \neq 0, \ldots, a_{m_1+\cdots+m_{s-1}+m_s-1}} \neq 0$.

Lemma 5 *Let $\{1, \ldots, s\} = X_1 \coprod \cdots \coprod X_{\tilde{s}}$ (disjoint union); by taking a permutation of $m_1, \ldots, m_s$ if necessary, we assume that $X_1 = \{1, \ldots, s_1\}$, $X_2 = \{s_1 + 1, \ldots, s_1 + s_2\}, \ldots, X_s = \{s_1 + \cdots + s_{\tilde{s}-1} + 1, \ldots, s_1 + \cdots + s_{\tilde{s}-1} + s_{\tilde{s}}\}$. For $1 \leqq j \leqq \tilde{s}$, put $\tilde{m}_j = \sum_{i \in X_j} m_i$. Assume that*

$$GCD\{\tilde{m}_1, \ldots, \tilde{m}_s, p-1\} \Bigm| \sum_{j=1}^{\tilde{s}} \frac{\tilde{m}_j(\tilde{m}_j - 1)}{2}.$$

Then we can find a diagonal element t of S of order $p-1$ such that $\phi^t = \phi^\alpha$. (For integers $c_1, \ldots, c_k, GCD\{c_1, \ldots, c_k\}$ is the greatest common divisor of $c_1, \ldots, c_k$.)

Proof. Let $\xi_1, \ldots, \xi_s \in \mathbb{F}_p^\times$, and put

$$\begin{aligned} t = \operatorname{diag}(&\nu^{\tilde{m}_1-1}\xi_1, \nu^{\tilde{m}_1-2}\xi_1, \ldots, \nu\xi_1, \xi_1; \\ &\nu^{\tilde{m}_2-1}\xi_2, \nu^{\tilde{m}_2-2}\xi_2, \ldots, \nu\xi_2, \xi_2; \\ &\ldots\ldots\ldots\ldots\ldots \qquad ; \\ &\nu^{\tilde{m}_{\tilde{s}}-1}\xi_{\tilde{s}}, \nu^{\tilde{m}_{\tilde{s}}-2}\xi_{\tilde{s}}, \ldots, \nu\xi_{\tilde{s}}, \xi_{\tilde{s}}). \end{aligned}$$

Then t is an element of G of order $p-1$ and $\phi^t = \phi^\alpha$. For $1 \leqq j \leqq \tilde{s}$, let $\xi_j = \nu^{i_j}$. Then

$$\det(t) = \prod_{j=1}^{\tilde{s}} \nu^{\tilde{m}_j(\tilde{m}_j-1)/2 + i_j\tilde{m}_j}.$$

Thus t belongs to S if and only if

$$\sum_{j=1}^{\tilde{s}} \frac{\tilde{m}_j(\tilde{m}_j - 1)}{2} \equiv \sum_{j=1}^{\tilde{s}} (-i_j)\tilde{m}_j \pmod{p-1},$$

that is, if and only if

$$GCD\{\widetilde{m}_1,\dots,\widetilde{m}_s,p-1\} \Bigm| \sum_{j=1}^{\widetilde{s}} \frac{\widetilde{m}_j(\widetilde{m}_j-1)}{2}$$

□

Proposition 2 *Assume that $p \neq 2$. Let η be an irreducible character of $GL(n,q)$, and suppose that $\eta = (g_1^{\nu_1}\cdots g_N^{\nu_N})$. Let $(d(g_1)\cdot\widetilde{\nu}_1)\cdot\ \cdots\ \cdot(d(g_N)\cdot\widetilde{\nu}_N) = [m_1,\dots,m_s]$ with $m_i \neq 0$ for all i. Let $\chi_1,\dots,\chi_h$ be the irreducible characters of S such that $\eta\,|\,S = \chi_1+\cdots+\chi_h$. Suppose that there is a decomposition $\{1,\dots,s\} = X_1\coprod\cdots\coprod\chi_{\widetilde{s}}$ such that, if we put $\widetilde{m}_j = \sum_{i\in X_j} m_i$, $1 \leqq j \leqq \widetilde{s}$, then $GCD\{m_1,\dots,m_s,p-1\}$ divides $\sum_{j=1}^{\widetilde{s}}\frac{\widetilde{m}_j(\widetilde{m}_j-1)}{2}$. Then $m_{\mathbb{Q}}(\chi_1) = \cdots = m_{\mathbb{Q}}(\chi_h) = 1$.*

Proof. We may assume that $X_1,\dots,X_{\widetilde{s}}$ are as in Lemma 5. Let ϕ be a linear character of U as above. Let $M = U\langle t\rangle$, where t is as in Lemma 5. Then ϕ^M is realizable in $\mathbb{Q}$ (see §2), and ϕ^S is realizable in $\mathbb{Q}$. By Theorem 1, we have $(\phi^G,\eta)_G = 1$ $(G = GL(n,q))$, so that $(\phi^S,\chi_1+\cdots+\chi_h)_S = 1$, and $(\phi^S,\chi_i)_S = 1$ for some i. Thus, by Lemma 1, we have $m_{\mathbb{Q}}(\chi_i) = 1$, hence $m_{\mathbb{Q}}(\chi_1) = \cdots = m_{\mathbb{Q}}(\chi_s) = 1$. □

Remark As to the parametrization of the irreducible characters of S, see the paper of Lehrer [6].

The condition of Proposition 2 is satisfied if, for instance, (i) some $m_i = 1$, or (ii) all the m_i are odd.

5. Some sufficient condition (II)

Lemma 6 ([9]) *Let η be an irreducible character of $G = GL(n,q)$, and suppose that $\eta = (g_1^{\nu_1}\cdots g_N^{\nu_N})$. Let $\mu = (d(g_1)\cdot\nu_1)\cdot\ \cdots\ \cdot(d(g_N)\cdot\nu_N)$. Let u be a unipotent element of G of type μ. Then $\eta(u)$ is equal to the p-part of $\eta(1)$ up to ± 1.*

Proposition 3 *Assume that $p \neq 2$. Let η be an irreducible character of $G = GL(n,q)$, and let $\chi_1,\dots,\chi_h$ be the irreducible characters of S such that $\eta\,|\,S = \chi_1+\cdots+\chi_h$. Suppose that $\eta = (g_1^{\nu_1}\cdots g_N^{\nu_N})$, and let $\mu = (d(g_1)\cdot\nu_1)\cdot\ \cdots\ \cdot(d(g_N)\cdot\nu_N) = [d_1,\dots,d_s]$ with $d_i \neq 0$ for all i. Suppose that there is a decomposition $\{1,\dots,s\} = X_1\coprod\cdots\coprod X_{\widetilde{s}}$ such that, if we put $\widetilde{d}_i =$*

$\sum_{i\in X_j} d_i$, $1 \leqq j \leqq \tilde{s}$, *then* $GCD\{\tilde{d}_1, \ldots, \tilde{d}_s, p-1\}$ *divides* $\sum_{j=1}^{\tilde{s}} \frac{\tilde{d}_j(\tilde{d}_j-1)}{2}$. *Then* $m_{\mathbb{Q}}(\chi_1) = \cdots = m_{\mathbb{Q}}(\chi_h) = 1$.

Proof. By permutating $d_1, \ldots, d_s$ if necessary, we may assume that $X_1 = \{1, \ldots, s_1\}$, $X_2 = \{s_1+1, \ldots, s_1+s_2\}, \ldots, X_s = \{s_1+\cdots+s_{\tilde{s}-1}+1, \ldots, s_1+\cdots+s_{\tilde{s}-1}+s_{\tilde{s}}\}$. For $1 \leqq i \leqq s$, let $S_i = SL(d_i, q)$, and let $H = S_1 \times \cdots \times S_s$. For $1 \leqq i \leqq s$, let U_i be the upper-triangular maximal unipotent subgroup of S_i, and let $V = U_1 \times \cdots \times U_s$. Call Λ the set of all linear characters of V. Take any $i, 1 \leqq i \leqq h$, and set $\chi = \chi_i$. Let

$$\chi \,|\, V = \sum_{\lambda\in\Lambda} a_\lambda \lambda + \sum_{\rho\in R} b_\rho \rho,$$

where R is the set of all non-linear irreducible characters of V. By an argument similar to that in the proof of Lemma 5, we can find a diagonal element t of S of order $p-1$ such that $\lambda^t = \lambda^\alpha$ for all $\lambda \in \Lambda$. Thus, for any $\lambda \in \Lambda$, λ^S is realizable in $\mathbb{Q}$, so that, by Lemma 1, $m_{\mathbb{Q}}(\chi)$ divides $a_\lambda = (\lambda^S, \chi)_S$.

Let u be an element of V of type μ. Then u is a regular unipotent element of H, so that, by a theorem of Lehrer [7], we have $\rho(u) = 0$ for all $\rho \in R$. Thus we have the expression

$$\chi(u) = \sum_{\lambda\in\Lambda} a_\lambda \lambda(u).$$

Let $\lambda \in \Lambda$. Then $a_{\lambda^t} = (\chi \,|\, V, \lambda^t)_V = (\chi^t \,|\, V, \lambda^t)_V = (\chi \,|\, V, \lambda)_V = a_\lambda$. Thus

$$\begin{aligned} \chi(u)^\alpha &= \sum_{\lambda\in\Lambda} a_\lambda \lambda(u)^\alpha \\ &= \sum_{\lambda\in\Lambda} a_\lambda \lambda^t(u) \\ &= \sum_{\lambda\in\Lambda} a_{\lambda^t} \lambda^t(u) \\ &= \chi(u), \end{aligned}$$

so that $\chi(u) \in \mathbb{Q}$, hence $\chi(u) \in \mathbb{Z}$. Now in the expression

$$\chi(u)/m_{\mathbb{Q}}(\chi) = \sum_{\lambda\in\Lambda} (a_\lambda/m_{\mathbb{Q}}(\chi))\lambda(u),$$

the right hand side is an algebraic integer and the left hand side is a rational number. Hence $m_{\mathbb{Q}}(\chi)$ divides $\chi(u)$.

Put $m = m_{\mathbb{Q}}(\chi_1) = \cdots = m_{\mathbb{Q}}(\chi_h)$. Then, as we have seen above, m divides all $\chi_i(u)$, $1 \leqq i \leqq h$, so that m divides $\eta(u) = \chi_1(u) + \cdots + \chi_h(u)$. By Lemma 6, $\eta(u)$ is a power of p, so that m divides a power of p. On the other hand, by Theorem A, m divides 2. Therefore, as p is odd, we conclude that $m = 1$. $\square$

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