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Putnam's theorems for w-hyponormal operators

Ariyadasa ALUTHGE and Derming WANG

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Abstract. Three theorems on hyponormal operators due to Putnam are generalized to apply to the broader class of w-hyponormal operators. In particular, it is shown that if an operator T is w-hyponormal and the spectrum of $|T^{*}|$ is not an interval, then T has a nontrivial invariant subspace.

Key words: p-, log- and w-hyponormal operators, approximate point spectrum, invariant subspace.

1. Introduction

Let T be a bounded linear operator on a Hilbert space H with inner product $(\cdot, \cdot)$ and $p>0$. The operator T is said to be p-hyponormal if $(T^{*}T)^{p} \geq (TT^{*})^{p}$. A p-hyponormal operator is said to be hyponormal if $p=1$, semi-hyponormal if $p=1/2$. It is a consequence of the well-known Löwner-Heinz inequality that if T is p-hyponormal, then it is q-hyponormal for any $0<q\leq p$. An invertible operator T is said to be log-hyponormal if $\log|T| \geq \log|T^{*}|$. Clearly, every invertible p-hyponormal operator is log-hyponormal. Let $T=U|T|$ be the polar decomposition of the operator T. Following [1], we define $\widetilde{T}=|T|^{1/2}U|T|^{1/2}$. An operator T is said to be w-hyponormal if

$$|\widetilde{T}| \geq |T| \geq |\widetilde{T}^{*}|. \tag{1.1}$$

Inequalities (1.1) show that if T is w-hyponormal, then $\widetilde{T}$ is semi-hyponormal. The classes of log- and w-hyponormal operators were introduced and their spectral properties studied in [2]. It was shown in [2] and [3] that the class of w-hyponormal operators contains both the p- and log-hyponormal operators. Log-hyponormal operators were independently introduced by Tanahashi in the paper [8]. There he gave an example of a log-hyponormal operator which is not p-hyponormal for any $p>0$. Thus, neither the class of p-hyponormal operators nor the class of log-hyponormal operators contains the other. In [4], we pointed out that if T is the

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Tanahashi operator on H, then $T \oplus 0$ on $H \oplus H$ is a w-hyponormal operator which is neither log-hyponormal nor p-hyponormal for any $p > 0$. Thus, the class of w-hyponormal operators properly contains both the p- and log-hyponormal operators.

Putnam [7] proved, among other things, three theorems concerning the spectral properties of hyponormal operators. These theorems were recently generalized to p-hyponormal operators by others. Here we generalize further these theorems to w-hyponormal operators. In Section 2, we prove the first generalization concerning points in the approximate point spectrum of a w-hyponormal operator. The second generalization, proven in Section 3, concerns the relationship between the spectra of T and $|T|$ of a w-hyponormal operator T. Finally, drawing on the results obtained in Sections 2 and 3, we prove the third generalization that if a w-hyponormal operator T is such that the spectrum of $|T^*|$ is not an interval, then T has a nontrivial invariant subspace.

2. The Approximate Point Spectrum

A complex number $\lambda \in \mathbb{C}$ is said to be in the approximate point spectrum $\sigma_a(T)$ of the operator T if there is a sequence $\{x_n\}$ of unit vectors satisfying $(T - \lambda)x_n \to 0$. The boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$ of an operator T is a subset of $\sigma_a(T)$. For bounded linear operators S and T, it is known that the nonzero points of $\sigma(ST)$ and $\sigma(TS)$ are identical. Thus, if $T = U|T|$ is the polar decomposition of T, then the facts that $|T^*| = U|T|U^*$ and $|T| = U^*U|T|$ imply that the nonzero points of $\sigma(|T^*|)$ and $\sigma(|T|)$ are identical.

In this section we prove a result concerning the approximate point spectrum of a w-hyponormal operator. Two consequences of this result will be drawn. The first (Corollary 1) is a generalization of a theorem, due to Putnam, concerning the boundary points of the spectrum of a hyponormal operator. The second consequence (Theorem 3) will be given in Section 4. The main result of this paper, concerning the existence of nontrivial invariant subspaces for w-hyponormal operators, is based in part on this second result. Two observations are needed in order to prove the main result of this section.

Let T be a bounded linear operator and $\lambda \in \mathbb{C}$. One readily checks that the following equations hold.

$$(|T|+|\lambda|)(|T|-|\lambda|)=T^*(T-\lambda)+\lambda(T^*-\overline{\lambda}). \tag{2.1}$$

$$(|T^*|+|\lambda|)(|T^*|-|\lambda|)=T(T^*-\overline{\lambda})+\overline{\lambda}(T-\lambda). \tag{2.2}$$

Stronger than its statement [9, Theorem 2.5, p.12], Xia actually proved the following:

Lemma 1 (Xia) *Let T be semi-hyponormal and $\lambda \in \mathbb{C}$. If the sequence $\{x_n\}$ of unit vectors is such that $(T-\lambda)x_n \to 0$, then $(T^*-\overline{\lambda})x_n \to 0$.*

Theorem 1 *Let $T=U|T|$ be w-hyponormal and $\lambda \neq 0$. If the sequence $\{x_n\}$ of unit vectors is such that $(T-\lambda)x_n \to 0$, then $(|T^*|-|\lambda|)x_n \to 0$. If in addition, T is invertible, then $(T^*-\overline{\lambda})x_n \to 0$.*

Proof. Since $\|(T-\lambda)x_n\| \geq |\,|\lambda|-\|Tx_n\|\,|$, passing to a subsequence if necessary, we may assume that the sequence $\{\|Tx_n\|\}=\{\|\,|T|x_n\|\}$ is bounded away from 0. Let $y_n=|T|^{1/2}x_n$. The bounded sequence $\{\|y_n\|\}$ is bounded away from 0 and $(\widetilde{T}-\lambda)y_n \to 0$. Since $\widetilde{T}$ is semi-hyponormal, it follows from Lemma 1 that $(\widetilde{T}^*-\overline{\lambda})y_n \to 0$. Since $|\widetilde{T}|+|\lambda|$ and $|\widetilde{T}^*|+|\lambda|$ are invertible, (2.1) and (2.2), with $\widetilde{T}$ in place of T, imply that $(|\widetilde{T}|-|\lambda|)y_n \to 0$, and $(|\widetilde{T}^*|-|\lambda|)y_n \to 0$. By (1.1), we have

$$\begin{aligned} 0 &\leq ((|T|-|\widetilde{T}^*|)y_n, y_n) \\ &\leq \{((|\widetilde{T}|-|\lambda|)y_n, y_n)-((|\widetilde{T}^*|-|\lambda|)y_n, y_n)\} \to 0, \end{aligned}$$

and hence

$$(|T|-|\widetilde{T}^*|)y_n \to 0.$$

Therefore,

$$(|T|-|\lambda|)y_n=\{(|T|-|\widetilde{T}^*|)y_n+(|\widetilde{T}^*|-|\lambda|)y_n\} \to 0,$$

and

$$|T|(|T|-|\lambda|)x_n=|T|^{1/2}(|T|-|\lambda|)y_n \to 0. \tag{2.3}$$

Multiplying each side of (2.1) on the left by $\lambda^{-1}|T|$, it follows from (2.3) that $|T|(T^*-\overline{\lambda})x_n \to 0$, and that

$$T(T^*-\overline{\lambda})x_n=U|T|(T^*-\overline{\lambda})x_n \to 0. \tag{2.4}$$

Since $|T^*|+|\lambda|$ is invertible, (2.2) together with (2.4) imply $(|T^*|-$

$|\lambda|)x_n \to 0$. If T is invertible, it follows from (2.4) that $(T^* - \overline{\lambda})x_n \to 0$. The proof is complete. □

Corollary 1 *Let T be w-hyponormal. If $\lambda \neq 0$ is such that $\lambda \in \sigma_a(T)$, then $|\lambda| \in \sigma(|T|) \cap \sigma(|T^*|)$.*

Corollary 2 *Let $T = U|T|$ be p-hyponormal. If $\lambda \in \sigma_a(T)$, then $|\lambda| \in \sigma(|T|) \cap \sigma(|T^*|)$.*

Proof. Since $\|Tx\| = \|\,|T|x\,\|$ for any vector x, if $0 \in \sigma_a(T)$, then $0 \in \sigma(|T|)$. The assumption that T is p-hyponormal implies $0 \in \sigma(|T^*|)$. This proves the corollary for the case $\lambda = 0$. For the case $\lambda \neq 0$, the result follows from Corollary 1. □

With the added assumption that the polar factor U is unitary, Corollary 2 was proven for $\lambda \in \partial\sigma(T)$ in the case T is hyponormal by Putnam [7, Theorem 1], and the case T is p-hyponormal, by Chō, Huruya and Itoh [5, Theorem 2].

3. The Spectra of T and $|T|$

Let $T = U|T|$ be a p-hyponormal operator. Does it follow that if $z \in \sigma(T)$, then $|z| \in \sigma(|T|)$? Apparently, by Corollary 2, the answer is in the affirmative if $z \in \sigma_a(T)$. In general, the answer to the question is in the negative [7] even if T is hyponormal and the polar factor U is unitary. However, the converse is true for p-hyponormal operators. Indeed, the following Lemma 2 was proven for the case T is hyponormal by Putnam [7], for the case T is semi-hyponorma by Xia [9], and the general case by Chō and Itoh [6].

Lemma 2 *If T is p-hyponormal, then $\sigma(|T|) \subset \rho(\sigma(T))$, where $\rho : \mathbb{C} \to \mathbb{R}$ is defined by $\rho(z) = |z|$.*

In this section we extend this result to w-hyponormal operators with connected spectra. Recall that the numerical range $W(T)$ of an operator T is defined by

$$W(T) = \{(Tx, x) : x \in H \text{ is a unit vector}\}.$$

Let $\overline{W}(T)$ denote the closure of $W(T)$. It is known that for any operator T, $W(T)$ is a convex set and $\sigma(T) \subset \overline{W}(T)$. Moreover, if T is normal, then

$\overline{W}(T) = \operatorname{conv}\sigma(T)$, the convex hull of $\sigma(T)$. The next lemma is well-known; its proof is therefore omitted.

Lemma 3 *If $T = U|T|$ is the polar decomposition of the operator T, and $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, then $\sigma(T) = \sigma(\widetilde{T})$.*

Lemma 4 *If T is w-hyponormal, then $\overline{W}(|\widetilde{T}|) \subset \overline{W}(|\widetilde{T}^*|)$.*

Proof. Let $\widetilde{T} = V|\widetilde{T}|$ be the polar decomposition of $\widetilde{T}$. The nonzero points of $\sigma(|\widetilde{T}^*|)$ and $\sigma(|\widetilde{T}|)$ are identical. Since T is w-hyponormal, $|\widetilde{T}| \geq |\widetilde{T}^*|$. It follows that $0 \in \sigma(|\widetilde{T}^*|)$ if $0 \in \sigma(|\widetilde{T}|)$. Therefore, $\sigma(|\widetilde{T}|) \subset \sigma(|\widetilde{T}^*|)$, and hence

$$\overline{W}(|\widetilde{T}|) = \operatorname{conv}\sigma(|\widetilde{T}|) \subset \operatorname{conv}\sigma(|\widetilde{T}^*|) = \overline{W}(|\widetilde{T}^*|).$$

□

Lemma 5 *If T is w-hyponormal, then $\sigma(|T|) \subset \overline{W}(|\widetilde{T}^*|)$.*

Proof. The assumption that T is w-hyponormal implies

$$(|\widetilde{T}|x, x) \geq (|T|x, x) \geq (|\widetilde{T}^*|x, x)$$

for any unit vector x. By Lemma 4, $(|\widetilde{T}|x, x) \in W(|\widetilde{T}|) \subset \overline{W}(|\widetilde{T}^*|)$. The convexity of $W(|\widetilde{T}^*|)$ and the above inequalities imply $(|T|x, x) \in \overline{W}(|\widetilde{T}^*|)$, and hence $\sigma(|T|) \subset \operatorname{conv}\sigma(|T|) = \overline{W}(|T|) \subset \overline{W}(|\widetilde{T}^*|)$. □

Theorem 2 *If T is w-hyponormal and $\sigma(T)$ is connected, then $\sigma(|T|) \subset \rho(\sigma(T))$, where $\rho : \mathbb{C} \to \mathbb{R}$ is defined by $\rho(z) = |z|$.*

Proof. Since $\widetilde{T}$ is semi-hyponormal, it follows from Lemma 2 and Lemma 3 that

$$\sigma(|\widetilde{T}|) \subset \rho(\sigma(T)).$$

Since the nonzero points of $\sigma(|\widetilde{T}^*|)$ and $\sigma(|\widetilde{T}|)$ are identical, and since $0 \in \sigma(|\widetilde{T}^*|)$ implies that $\widetilde{T}^*$ is not invertible, and hence $0 \in \sigma(T)$ by Lemma 3, the above containment may be modified to become

$$\sigma(|\widetilde{T}^*|) \subset \rho(\sigma(T)).$$

Now, since $\sigma(T)$ is compact and connected, $\rho(\sigma(T))$ is a closed convex subset of $\mathbb{R}$. Therefore, Lemma 5 implies

$$\sigma(|T|) \subset \overline{W}(|\widetilde{T}^*|) = \operatorname{conv} \sigma(|\widetilde{T}^*|) \subset \operatorname{conv} \rho(\sigma(T)) = \rho(\sigma(T)).$$

The proof is complete. □

4. Invariant Subspaces

Putnam [7, Theorem 10] proved that if T is hyponormal and $\sigma(|T^*|)$ is not an interval, then T has a nontrivial invariant subspace. This result was generalized to hold for p-hyponormal operators by Chō, Huruya and Itoh [5, Theorem 4]. If T is p-hyponormal, then $0 \in \sigma(|T|)$ implies $0 \in \sigma(|T^*|)$. Consequently, if $\sigma(|T|)$ is not an interval, then $\sigma(|T^*|)$ is not. Thus, Putnam's result holds if one assumes instead that $\sigma(|T|)$ is not an interval. In this section we give a further generalization to w-hyponormal operators.

A complex number λ is in the compression spectrum $\sigma_c(T)$ of an operator T if the range of $T - \lambda$ is not dense in H. It is known that $\sigma(T) = \sigma_a(T) \cup \sigma_c(T)$ for any operator T. Moreover, if $\lambda \in \sigma_c(T)$, then it is readily seen that the closure of the range of $T - \lambda$ is a nontrivial invariant subspace of T.

Theorem 3 *Let T be w-hyponormal. If there is a $\lambda \in \sigma(T)$, $\lambda \neq 0$, for which $|\lambda| \notin \sigma(|T|) \cap \sigma(|T^*|)$, then T has a nontrivial invariant subspace.*

Proof. By Corollary 1, $\lambda \notin \sigma_a(T)$. Therefore, $\lambda \in \sigma_c(T)$, and hence T has a nontrivial invariant subspace. □

Theorem 4 *Let T be w-hyponormal. If either $\sigma(|T|)$ or $\sigma(|T^*|)$ is not an interval, then T has a nontrivial invariant subspace.*

Proof. We will only give the proof for the case $\sigma(|T^*|)$ is not an interval, for the proof can be easily modified to apply to the other case. If $\sigma(T)$ is not connected, then clearly the theorem is proven. Thus assume $\sigma(T)$ is connected. The assumption that $\sigma(|T^*|)$ is not an interval implies there exist s, $t \in \sigma(|T^*|)$, $0 \leq s < t$ for which the open interval (s,t) is such that

$$(s,t) \cap \sigma(|T^*|) = \emptyset. \tag{4.1}$$

Let $N = \{z : s < |z| < t\}$. Since the nonzero points of $\sigma(|T|)$ and $\sigma(|T^*|)$ are identical, Theorem 2 implies there is a $\nu \in \sigma(T)$ for which $|\nu| = t$.

Similarly, if $s > 0$, then there is a $\mu \in \sigma(T)$ for which $|\mu| = s$. On the other hand, if $s = 0$, then T^* is not invertible and hence $0 \in \sigma(T)$. In either case, both the outer and inner boundaries of the annulus N contain a point of $\sigma(T)$. Since $\sigma(T)$ is connected; we must have $N \cap \sigma(T) \neq \emptyset$. Therefore, there is a $\lambda \in N \cap \sigma(T)$. It follows that $|\lambda| \in (s,t)$, and hence $|\lambda| \notin \sigma(|T^*|)$ by (4.1). Thus, T has a nontrivial invariant subspace by Theorem 3. The proof is complete. □

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Ariyadasa Aluthge
Department of Mathematics
Marshall University
Huntington, WV 25775-2560
U.S.A.
E-mail: aluthge@marshall.edu

Derming Wang
Department of Mathematics
California State University, Long Beach
Long Beach, CA 90840-1001
U.S.A.
E-mail: dwang1@csulb.edu