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An abstract degenerate hyperbolic equation with application to mixed problems

Piero D'ANCONA and Mariagrazia Di FLAVIANO

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Abstract. We prove an existence result for the Cauchy problem associated to an abstract degenerate hyperbolic equation. Moreover we show several applications to mixed initial boundary value problems for weakly hyperbolic equations.

Key words: nonlinear weakly hyperbolic equations, abstract equations, degenerate equations, mixed initial boundary value problem.

1. Introduction

Let H be a Hilbert space with norm $|\cdot|$, and $\mathbf{B}=(B_1,\dots,B_n)$ an n-tuple of *selfadjoint* operators on H, with (dense) domains $D(B_j)$. For any multiindex $\alpha=(\alpha_1,\dots,\alpha_n)$ and integer s we shall use the notation

$$\mathbf{B}^{\alpha}=B_1^{\alpha_1}\circ\cdots\circ B_n^{\alpha_n}.$$

The subspaces H^s are defined as follows:

$$H^s=\bigcap_{1\leq j_i\leq n} D(B_{j_1}\circ\cdots\circ B_{j_s}).$$

We can obviously endow H^s with a Hilbert space structure with norm

$$|u|_s^2=\sum_{\substack{0\leq k\leq s\\ 1\leq j_i\leq n}} |B_{j_1}\circ\cdots\circ B_{j_k}u|^2.$$

We shall solve the following Cauchy problem on H:

$$u''+\sum_{|\alpha|=2m} a_{\alpha}(t)\cdot\mathbf{B}^{\alpha}u=f(t) \tag{1.1}$$

$$u(0)=u_0,\quad u'(0)=u_1. \tag{1.2}$$

We shall assume that

$$\text{the functions } a_{\alpha}(t) \text{ are } \textit{real analytic} \text{ on } [0,T] \tag{1.3}$$

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and that the following form is nonnegative:

$$\sum_{|\alpha|=2m} a_\alpha(t)\cdot\xi^\alpha \geq 0 \qquad \forall \xi \in \mathbf{R}^n. \tag{1.4}$$

Thus, equation (1.1) can be regarded as an abstract degenerate hyperbolic equation on H. Concerning the selfadjoint operators B_j, we shall assume that the resolvent operators $R(i, B_j)$ commute, i.e.,

$$R(i,B_j)R(i,B_k)u = R(i,B_k)R(i,B_j)u \qquad \forall j,k = 1\dots,n, \quad \forall u \in H. \tag{1.5}$$

Remark 1.1 Condition (1.5) is necessary in order to have a simultaneous diagonalization of the operators B_j. In the case of bounded operators it would have been sufficient to assume that $[B_j, B_k] = 0$; but in the unbounded case the concept of commuting operators is much more delicate. A handy substitute for (1.5) is the following assumption: there exists a subspace V dense in H such that, for all $j \neq k$, $V \subseteq D(B_k B_j)$ and

$$V_{jk} = (B_j - i)(B_k - i)(V) \text{ is dense in } H \text{ and } [B_j, B_k] = 0 \text{ on } V. \tag{1.6}$$

It is easy to prove that (1.6) implies (1.5); indeed, we have

$$B_j B_k v = B_k B_j v \qquad \forall v \in V$$

whence

$$(B_j - i)(B_k - i)v = (B_k - i)(B_j - i)v \qquad \forall v \in V.$$

Call w the vector $(B_j - i)(B_k - i)v$; we have then

$$R(i,B_j)R(i,B_k)w = R(i,B_k)R(i,B_j)w \qquad \forall w \in V_{jk},$$

but V_{jk} is dense and we obtain (1.5).

We can now state our main result:

Theorem 1 *Consider Problem* (1.1), (1.2) *under assumptions* (1.3)–(1.5). *Then, fixed $T > 0$, there exists an integer s_0 such that, for all $s \geq 2m$, for all data $u_0, u_1 \in H^{s+s_0}$ and $f(t) \in C([0,T]; H^{s+s_0})$ the problem has a unique solution $u(t) \in C^2([0,T]; H^s)$.*

In Section 2 we shall give a complete proof of our theorem, while Section 3 is devoted to the applications. We consider several examples: the mixed Cauchy-Dirichlet (or Neumann) problem in a rectangular set Ω for the equation

$$u_{tt} - \sum a_j(t)\partial_j^2 u = f(t,x)$$

where $a_j \geq 0$ are analytic; the mixed Cauchy-Dirichlet (or Neumann) problem in any open set $\Omega \subseteq \mathbf{R}^n$ with smooth boundary for

$$u_{tt} + \sum a_j(t)P_j(D)u = f(t,x)$$

($P_j(D)$ elliptic formally selfadjoint second order differential operator) which includes the equations of the form

$$u_{tt} - c(t)\sum b_j(t)\partial_j^2 u = f(t,x)$$

with $c(t) \geq 0$, $b_j(t) > 0$ analytic; the mixed Cauchy-Dirichlet (or Neumann) problem on a smooth Riemannian manifold with boundary for

$$u_{tt} - a(t)\Delta u = f(t,x),$$

Δ being the Laplace-Beltrami operator on the manifold and $a(t) \geq 0$; and finally, some problems on the whole $\mathbf{R}^n$, including

$$u_{tt} + \sum_{ij} a_{ij}(t)Y_iY_j u = f(t,x)$$

with Y_j commuting selfadjoint vector fields.

Remark 1.2 In the concrete case, problems of the form

$$u_{tt} - \sum_{ij} a_{ij}(t)\partial_i\partial_j u = f(t,x)$$

have been considered in [CJS], [O]; the semilinear case has been studied in [D]. Very few results exist for the degenerate hyperbolic mixed problem (with the exception of the constant coefficient case, see [S]); we mention [K], [Ku], and, in the semilinear case, [DR].

2. Proof of Theorem 1

We shall need an extension to the unbounded case of the well known spectral theorem for a finite number of bounded, commuting selfadjoint operators:

Theorem 2.1 *Let $A_1, \dots, A_n$ be bounded, selfadjoint, pairwise commuting operators on a Hilbert space H. Then there exist a measure space X with measure μ, a unitary map $W : H \longrightarrow L^2(X, d\mu)$, and real-valued functions $a_j \in L^\infty(X, d\mu)$ such that*

$$WA_jW^{-1}f(\xi) = a_j(\xi)f(\xi), \quad f \in L^2(X, d\mu), \quad 1 \le j \le n.$$

(see e.g. [KG] or [T]). In the unbounded case the commutativity assumption must be replaced by something stronger, owing to the difficulties with the domains; one possibility is to assume that the resolvent operators commute:

Theorem 2.2 *Let $B_1, \dots, B_n$ be selfadjoint operators on a Hilbert space H, satisfying (1.5), i.e., such that the resolvent operators $R(i, B_j)$ are pairwise commuting. Then there exist a measure space X with measure μ, a unitary map $W : H \longrightarrow L^2(X, d\mu)$, and measurable functions $b_j : X \to \mathbf{R}$ such that the following holds. Denoting with $M(b_j)$ the multiplication operator by b_j in $L^2(X, d\mu)$, with domain*

$$D(M(b_j)) = \{f \in L^2 \mid b_j f \in L^2\},$$

the diagrams

$$\begin{array}{ccc} D(B_j) & \xrightarrow{\ W\ } & D(M(b_j)) \\ {\scriptstyle B_j}\downarrow & & \downarrow{\scriptstyle M(b_j)} \\ H & \xrightarrow[\ W\ ]{} & L^2(X, d\mu) \end{array}$$

are commutative, i.e.,

$$WB_jW^{-1}f(\xi) = b_j(\xi)f(\xi) \quad \forall f \in D(M(b_j)), \quad 1 \le j \le n.$$

Proof. Consider the unitary operators

$$U_j = I + 2iR(i, B_j) = (B_j + i)(B_j - i)^{-1}$$

and define, for $j = 1, \dots, n$

$$A_j = \frac{1}{2}(U_j + U_j^*) \quad \text{and} \quad A_{n+j} = \frac{1}{2i}(U_j - U_j^*);$$

$A_1, \ldots, A_n, A_{n+1}, \ldots, A_{2n}$ are $2n$ selfadjoint, bounded commuting operators, hence we can apply Theorem 2.1. Thus we have a unitary operator W and a measure space $L^2(X, d\mu)$ such that

$$WA_jW^{-1}f(\xi) = a_j(\xi)f(\xi) \quad \forall j = 1, \ldots, 2n, \quad f \in L^2(X, d\mu)$$

with a_j real-valued functions in $L^\infty(X, d\mu)$. This gives

$$WU_jW^{-1}f(\xi) = u_j(\xi)f(\xi) \quad \forall j = 1, \ldots, n, \quad f \in L^2(X, d\mu)$$

with $u_j(\xi) = a_j(\xi) + ia_{n+j}(\xi)$. Notice that the complex-valued function u_j is a.e. different from 1 (otherwise one could construct an eigenvector v for U_j with eigenvalue 1, $U_jv = v$; by the definition of U_j this would imply that $(B_j - i)v = (B_j + i)v$ and hence $v = 0$). Moreover, $|u_j| = 1$ a.e. since U_j is a unitary operator. Thus the functions

$$b_j = i\frac{u_j + 1}{u_j - 1}$$

are finite and real valued a.e. By the definition of U_j we have

$$R(i, B_j) = \frac{i}{2}(I - U_j),$$

hence the domain of B_j coincides with the range of $U_j - I$,

$$D(B_j) = R(U_j - I),$$

and we can write

$$B_j = i(U_j + I)(U_j - I)^{-1}.$$

Now, W transforms $R(U_j - I)$ into the set

$$\{f \in L^2 \mid \exists g \in L^2 \ \text{ s.t. } f = (u_j - 1)g\}$$

which coincides with $D(M(b_j))$ as it is readily seen. Finally, it is clear that

$$WB_jW^{-1} = Wi(U_j + I)W^{-1}W(U_j - I)^{-1}W^{-1} = M(b_j)$$

on $D(M(b_j))$. □

We can now apply the transform W to Problem (1.1), (1.2) and we obtain the following ordinary differential equation with a parameter $\xi \in X$:

$$v'' + \sum_{|\alpha|=2m} a_\alpha(t) \cdot b(\xi)^\alpha v = g(t, \xi) \tag{2.1}$$

$$v(0) = v_0, \quad v'(0) = v_1 \tag{2.2}$$

where $v(t,\xi) = Wu(t)$, $v_0 = Wu_0$, $v_1 = Wu_1$, $g(t,\xi) = Wf(t)$ while $b(\xi)$ is the vector valued function $(b_1(\xi), \ldots, b_n(\xi))$.

Introducing the norms

$$\|v(t,\cdot)\|_s^2 = \sum_{|\alpha|\le s} \int_X (1 + |b(\xi)^\alpha|^2)^s |v|^2 d\mu(\xi),$$

we can consider the subspaces $\widehat{H}^s$ of $L^2(X, d\mu)$ defined as

$$\widehat{H}^s = \{v \in L^2(X, d\mu) \mid \|v\|_s < \infty\}.$$

These spaces carry an obvious Hilbert structure. Moreover, it is evident that W is an isomorphism between H^s and $\widehat{H}^s$.

Thus, the assumptions on the data are equivalent to the following:

$$v_0, v_1 \in \widehat{H}^{s+s_0}, \quad g(t,\xi) \in C([0,T]; \widehat{H}^{s+s_0}). \tag{2.3}$$

It is obvious that, for each $\xi \in X$, (2.1), (2.2) has a unique solution on $[0,T]$; we only need to give a suitable estimate of the solution $v(t,\xi)$ thus obtained in terms of the norms $\|\cdot\|_s$. This will allow us to define $u = W^{-1}v$ and will imply the thesis of Theorem 1.

Write for brevity

$$a(t,b) = \sum_{|\alpha|=2m} a_\alpha(t) \cdot b^\alpha$$

for $b \in \mathbf{R}^n$, and define the energy of order s of the solution $v(t,\xi)$ as

$$E_s(t) = \int_X k_s(t, b(\xi)) \left[|v'|^2 + (1 + a(t, b(\xi)))|v|^2\right] d\mu(\xi). \tag{2.4}$$

The function $k_s(t,b)$ appearing in this definition is a suitable weight function defined as follows: using the notation $\langle b \rangle = (1 + |b|^2)^{1/2}$, $b \in \mathbf{R}^n$, we set

$$k_s(t,b) = \langle b \rangle^{2s} \exp\left(-\int_0^t \frac{[\partial_t a(\tau,b)]^+}{a(\tau,b)+1} d\tau\right) \tag{2.5}$$

(here $[\lambda]^+ \equiv \lambda \vee 0$).

We shall need the following important property of $k(t,b)$:

Lemma *There exist an integer N_0 and a constant c_0, depending on T, s*

and the functions a_α only, such that

$$k_s(t,p+q) \le c_0\, k_s(t,p)\langle q\rangle^{2(N_0+s)} \tag{2.6}$$

for all $p,q \in \mathbf{R}^n$.

Proof. Since

$$\langle p+q\rangle \le \sqrt{2}\langle p\rangle\langle q\rangle, \tag{2.7}$$

it is sufficient to prove (2.6) for $s=0$:

$$k_0(t,p+q) \le c_0\, k_0(t,p)\langle q\rangle^{2N_0}. \tag{2.8}$$

We shall omit the index 0 in the following.

For each $p \in \mathbf{R}^n$, $|p|=1$, let $N(p)$ be the number of oscillations of the analytic function $a(t,p)$ on $[0,T]$; more precisely, if we consider the set

$$I^+(p) = \{t \in [0,T] \mid \partial_t a(t,p) \ge 0\},$$

we have for suitable points $s_j(p) \le t_j(p) \le s_{j+1}(p)$ in $[0,T]$

$$I^+(p) = [s_1(p), t_1(p)] \cup \ldots \cup [s_{N(p)}(p), t_{N(p)}(p)]. \tag{2.9}$$

Since $a(t,p)$ is analytic, the integer valued function $N(p)$ is locally bounded; since $a(t,p)$ is also homogeneous in p, we conclude

$$N = \max_{|p|=1} N(p) = \max_p N(p) < \infty.$$

Observing that on $[s_j(p), t_j(p)]$

$$\frac{[\partial_t a(t,p)]^+}{a(t,p)+1} = \partial_t[\log(a(t,p)+1)],$$

an integration on $[0,T]$ gives the explicit result

$$k(t,p) = \exp\left(-\int_0^t \frac{[\partial_t a(\tau,p)]^+}{a(\tau,p)+1}d\tau\right) = \prod_{j=1}^{N(p)} \frac{a(s_j(p),p)+1}{a(t_j(p),p)+1}. \tag{2.10}$$

Now we remark that

$$1 + a(t,p+q) \le C_0(1+a(t,p))\langle q\rangle^2$$

(with $C_0 = 2 + 2\sup_{[0,T]\times\{|p|=1\}} a(t,p)$), whence also

$$\frac{1}{1+a(t,p+q)} \le C_0 \frac{\langle q\rangle^2}{1+a(t,p)}.$$

This implies

$$k(t,p+q) \le C_0^{2N(p+q)} \langle q\rangle^{4N(p+q)} k(t,p)$$

and recalling that $N(p+q) \le N$ we obtain the thesis with $c_0(s) = C_0^{2N} 2^s$, $N_0 = 2N$. □

A consequence of the Lemma is

$$c_1 \langle b\rangle^{-2N_0} \le k_0(t,b) \le c_2 \langle b\rangle^{2N_0}$$

hence in general

$$c_1 \langle b\rangle^{2(s-N_0)} \le k_s(t,b) \le c_2 \langle b\rangle^{2(s+N_0)}. \tag{2.11}$$

We can now differentiate the energy $E_s(t)$ defined in (2.4) and we obtain

$$E_s'(t) = \int_X \Big\{ \partial_t[k_s(t,b(\xi))|v'|^2] + \partial_t[k_s(t,b(\xi))(1+a(t,b(\xi)))|v|^2] \Big\}\, d\mu(\xi).$$

We can write

$$\begin{aligned} \partial_t(k_s(t,b)|v'|^2) &= -\frac{[\partial_t a(t,b)]^+}{1+a(t,b)} k_s(t,b)|v'|^2 \\ &\quad + k_s(t,b)\cdot 2Re\bar{v}'(-a(t,b)v + g(t,\xi)), \end{aligned}$$

$$\begin{aligned} &\Big\{ \partial_t[k_s(t,b)(1+a(t,b))] \Big\} |v|^2 \\ &\quad = \Big\{ -[\partial_t a(t,b)]^+ \cdot k_s(t,b) + \partial_t a(t,b)\cdot k_s(t,b) \Big\} |v|^2 \le 0, \end{aligned}$$

$$k_s(t,b)(1+a(t,b))\partial_t |v|^2 = k_s(t,b)(a(t,b)+1)\cdot 2Re\bar{v}'v,$$

so that

$$\begin{aligned} E_s'(t) &\le \int_X k_s(t,b)\cdot 2Re\bar{v}'v\, d\mu(\xi) + \int_X k_s(t,b)\cdot 2Re\bar{v}'g(t,\xi)\, d\mu(\xi) \\ &\le 2E_s(t) + \int_X k_s(t,b(\xi))|g(t,\xi)|^2\, d\mu(\xi) \end{aligned}$$

by Schwartz' inequality. Using (2.11) we have

$$E_s'(t) \le 2E_s(t) + c_2 \int_X \langle b(\xi) \rangle^{2(s+N_0)} |g(t,\xi)|^2 d\mu(\xi),$$

which gives, by Gronwall's lemma,

$$E_s(t) \le C_s(T) \Big\{ E_s(0) + \int_0^t \|g(\tau,\cdot)\|_{s+N_0}^2 d\tau \Big\} \tag{2.12}$$

(we are using here the norms $\|\cdot\|_s$ of the spaces $\widehat{H}^s$). By the definition of $E_s(t)$ and by (2.11) we obtain immediately

$$c_1 \left[\|v'\|_{s-N_0}^2 + \|v\|_{s-N_0}^2 \right] \le E_s(t) \le c_2 \left[\|v'\|_{s+N_0}^2 + \|v\|_{s+N_0}^2 \right],$$

hence (2.12) implies the estimate

$$\|v'\|_s + \|v\|_s \le C(T) \left[\|v_0\|_{s+s_0} + \|v_1\|_{s+s_0} + \int_0^t \|g(\tau,\cdot)\|_{s+s_0} d\tau \right] \tag{2.13}$$

for a suitable s_0 not depending on s ($s_0 = 2N_0$).

Applying now the inverse transformation W^{-1} and recalling the correspondence between H^s and $\widehat{H}^s$, we conclude easily the proof. □

3. Applications

1) *Mixed problem on a rectangular set.* Let Ω be a rectangle in $\mathbf{R}^n$, i.e., a set of the form

$$\Omega = \prod_{j=1}^n I_j$$

where I_j is an open interval of $\mathbf{R}$. Consider the following mixed problem on $[0,T] \times \Omega$,

$$u_{tt} - \sum a_j(t) \partial_j^2 u = f(t,x) \tag{3.1}$$

$$u(0,x) = u_0(x), \; u_t(0,x) = u_1(x) \tag{3.2}$$

$$u(t,x) = 0 \quad \text{for} \quad x \in \partial\Omega \tag{3.3}$$

where

$$a_j(t) \ge 0 \;\; \text{are real analytic functions on } [0,T]. \tag{3.4}$$

We choose $H = L^2(\Omega)$, and for $j = 1, \dots, n$ we consider the operators

$$A_j = -\partial_j^2$$

with domain $C_0^\infty(\Omega)$. These are positive symmetric operators and can be extended to selfadjoint operators by the Friedrichs method; we shall denote the extensions again by A_j. These extensions coincide with the derivative in the distribution sense, since this is true for the adjoint operators A_j^*. The domain $D(A_j)$ can be easily proved to be the set of all $f \in L^2(\Omega)$ with $\partial_j^2 f \in L^2(\Omega)$ and belonging to the closure of $C_0^\infty(\Omega)$ in the norm $\|\phi\|_{L^2} + \|\partial_j \phi\|_{L^2}$. We also recall that the ordinary differential operator $-d^2/dx^2 + i$ is a bijection from $H^2(a, b) \cap H_0^1(a, b)$ onto $L^2(a, b)$, (a, b) any interval. Now consider the set V_0 of functions

$$u(x) = u_1(x_1) \cdot \dots \cdot u_n(x_n) \tag{3.5}$$

with $u_j(s) \in H^2(I_j) \cap H_0^1(I_j)$; clearly $V_0 \subseteq D(A_j)$ for all j, and the image of V_0 through $T_{jk} = (A_j - i)(A_k - i)$, by what observed above, is the set of all functions of the form (3.5), with u_j and u_k any function of $L^2(I_j)$, while the other u_h are unchanged, hence they are all possible functions in $H^2(I_h) \cap H_0^1(I_h)$. Thus it is clear that $T_{jk}(V_0)$ contains e.g. all functions of the form (3.5) with all u_j smooth and compactly supported in I_j. We can now apply Remark 1.1 by choosing as V the vector space generated by V_0, and this proves that condition (1.5) is satisfied.

Theorem 2.2 thus can be applied to the A_j, and we obtain the unitary map W which transforms each A_j in a multiplication operator by a nonnegative function $a_j(\xi)$. We set now

$$B_j = A_j^{1/2} = \sqrt{-\partial_j^2}$$

and the same map W transforms B_j in the multiplication operator by the function $a_j^{1/2}$. We also remark that, since

$$-\Delta = \sum A_j = \sum B_j^2$$

by elliptic regularization we have that the space $H^{2j} = D(\mathbf{B}^{2j})$ contains (at least) $H_0^{2j}(\Omega)$, and is contained in $H^{2j}(\Omega) \cap H_0^1(\Omega)$.

Finally, we can write Equation (3.1) as

$$u'' + \sum a_j(t) B_j^2 u = f(t)$$

and apply Theorem 1, which gives directly

Theorem 3.1 *Consider Problem* (3.1)–(3.3) *under assumption* (3.4). *There exists an integer* s_0 *such that, for all* $s \geq 2$ *and data* $u_0, u_1 \in H_0^{s+s_0}(\Omega)$, $f \in C([0,T]; H_0^{s+s_0}(\Omega))$, *there exists a unique solution* $u \in C^2([0,T]; H^{s+s_0}(\Omega) \cap H_0^1(\Omega))$.

We mention that in a similar way we can consider Neumann boundary conditions.

2) *Mixed problem on a smooth domain.* Let Ω be now any bounded open subset of $\mathbf{R}^n$ with smooth boundary, assume that

$$a_j(t) \geq 0 \text{ are real analytic functions on } [0,T] \tag{3.6}$$

and let $P_j(D)$ be formally selfadjiont, strictly elliptic, second order differential operators such that $P_j(D) \geq 0$. Consider the mixed Cauchy problem on $[0,T] \times \Omega$

$$u_{tt} + \sum a_j(t) P_j(D) u = f(t,x) \tag{3.7}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{3.8}$$

$$u(t,x) = 0 \quad \text{for} \quad x \in \partial\Omega. \tag{3.9}$$

We extend the P_j by Friedrich's method to selfadjoint operators, which we denote by the same symbols; as in [T, p.82 ff.], it is easy to see that their domain is $H^2(\Omega) \cap H_0^1(\Omega)$. Moreover, the space

$$V = C_c^\infty(\Omega)$$

satisfies (1.6); indeed $P_j - i$ is onto L^2, hence, fixed $u \in L^2$, we can find $v \in H^4$ such that $(P_j - i)(P_k - i)v = u$, and approximating v by a sequence $\phi_\ell \in V$ in the H^4 norm we get $(P_j - i)(P_k - i)\phi_\ell \to u$ as required. Thus we can apply Theorem 2.2 and transform the P_j into multiplication operators by nonnegative functions $a_j(\xi)$ on a suitable $L^2(X) = L^2(X, d\mu)$, through a unitary map $W : L^2(\Omega) \to L^2(X)$. The operators

$$B_j = P_j(D)^{1/2}$$

will be represented as multiplication operators by $a_j(\xi)^{1/2}$, and their resolvents $R(i, B_j)$ as multiplication by $(i - a_j(\xi)^{1/2})^{-1}$; it is clear that they commute and (1.5) is satisfied.

Thus Equation (3.7) can be written as

$$u'' + \sum a_j(t) B_j^2 u = f(t)$$

and we can apply Theorem 1, obtaining a result identical to Theorem 3.1; notice that we have the equivalence of norms

$$\|B_j u\|_{L^2}^2 + \|u\|_{L^2}^2 = (P_j u, u)_{L^2} + \|u\|_{L^2}^2 \sim \|u\|_{H^1}^2,$$

for $u \in C_c^\infty$ and hence for $u \in H_0^1$.

Notice also that any equation like

$$u_{tt} - c(t) \sum b_j(t) \partial_j^2 u = f(t,x)$$

with c analytic nonnegative, $b_j(t)$ analytic and strictly positive, can be put in the form (3.7).

3) *Mixed problem on a Riemannian manifold.* Let $\overline{\Omega}$ be a compact Riemannian manifold with smooth boundary, and consider the Cauchy problem

$$u_{tt} - a(t)\Delta u = f(t,x) \tag{3.10}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{3.11}$$

$$u(t,x) = 0 \quad \text{for} \quad x \in \partial\Omega. \tag{3.12}$$

where Δ is the Laplace operator on $\overline{\Omega}$ and $a(t)$ is analytic nonnegative; we obtain a result similar to 3.1. Alternatively, Neumann conditions can be treated.

In case the manifold has no boundary, we may drop (3.12) and we obtain

Theorem 3.2 *Consider Problem* (3.10), (3.11) *under the assumption that $a(t)$ is analytic and nonnegative. There exists an integer s_0 such that, for all $s \geq 2$ and data $u_0, u_1 \in H^{s+s_0}(\overline{\Omega})$, $f \in C([0,T]; H^{s+s_0}(\overline{\Omega}))$, there exists a unique solution $u \in C^2([0,T]; H^{s+s_0}(\overline{\Omega}))$.*

4) *Cauchy Problems on* $\mathbf{R}^n$. We may consider Cauchy Problems of the form

$$u_{tt} + \sum_{|\alpha|=2m} a_\alpha(t) \mathbf{P}(x,D)^\alpha u = f(t,x) \tag{3.13}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \tag{3.14}$$

where $\mathbf{P}(x,D) = (P_1(x,D), \ldots, P_n(x,D))$ is a n-tuple of selfadjoint commuting operators on $L^2(\mathbf{R}^n)$. To see the simplest case, we can consider the equation

$$u_{tt} + \sum_{ij} a_{ij}(t) Y_i Y_j u = f(t,x) \tag{3.15}$$

where Y_j are commuting vector fields with smooth bounded coefficients on $\mathbf{R}^n$; e.g.,

$$Y_1 = i\partial_1, \quad Y_j = i\partial_j + i\partial_j \phi(x')\partial_1 \tag{3.16}$$

where $\phi(x') = \phi(x_2, \ldots, x_n)$ is a smooth function not depending on x_1. We write

$$H^s = \{f \in L^2 : Y_1^{j_1} \cdots Y_n^{j_n} f \in L^2 \quad \forall j_1 + \cdots + j_n \leq s\}.$$

We obtain in a straightforward way

Theorem 3.3 *Consider Problem* (3.15), (3.14) *under the assumption that* $\sum a_{ij}(t)\xi_i\xi_j$ *is analytic and nonnegative. Exists an integer* s_0 *such that, for all* $s \geq 2$ *and data* $u_0, u_1 \in H^{s+s_0}(\overline{\Omega})$, $f \in C([0,T]; H^{s+s_0}(\overline{\Omega}))$, *there exists a unique solution* $u \in C^2([0,T]; H^{s+s_0}(\overline{\Omega}))$.

The only difficulty in interpreting this result is to make the spaces H^s explicit; for instance, in the example (3.16), it is not difficult to see that H^s is the usual Sobolev space $H^s(\mathbf{R}^n)$.

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Mariagrazia Di Flaviano
Dipartimento di Matematica Pura ed Applicata
Università degli Studi di L'Aquila
Via Vetoio - Loc. Coppito
67010 L'Aquila, Italy
E-mail: diflavia@univaq.it

Piero D'Ancona
Dipartimento di Matematica
Università "La Sapienza" di Roma
Piazzale Aldo Moro, 2
I-00185 Roma, Italy
E-mail: dancona@mat.uniroma1.it