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Weighted sharing and a result of Ozawa

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Abstract. We prove a uniqueness theorem for meromorphic functions sharing three values with unit weight which improves a result of Ozawa.

Key words: weigted sharing, uniqueness, meromorphic function.

1. Introduction, Definitions and Results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane C. If for some $a\in C\cup\{\infty\}$ the zeros of $f-a$ and $g-a$ coincide in locations and multiplicities we say that f and g share the value a CM (counting multiplicities) and if coincide in locations only we say that f and g share a IM (ignoring multiplicities).

We do not explain the standard notations and definitions of the value distribution theory as these are available in [2]. However, we explain some notations and definitions which will be needed in the sequel. Throughout the paper we denote by f and g two nonconstant meromorphic functions defined on C unless otherwise stated.

Definition 1 [4] We denote by $N(r, a; f\mid=1)$ the counting function of simple a-points of f.

Definition 2 [4] We denote by $\overline{N}(r, a; f\mid\geq 2)$ the counting function of multiple a-points of f, where each a-point is counted only once.

Definition 3 [10] We define $\delta_{2}(a;f)=1-\limsup_{r\longrightarrow\infty}\frac{N_{2}(r,a;f)}{T(r,f)}$, where $N_{2}(r, a;f)=\overline{N}(r, a;f)+\overline{N}(r, a;f\mid\geq 2)$.

Clearly $0\leq\delta(a;f)\leq\delta_{2}(a;f)\leq\Theta(a;f)\leq 1$.

In order to investigate the influence of the distribution of zeros on the uniqueness of entire functions M. Ozawa [5] proved the following theorem.

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Theorem A *Let f and g be entire functions of finite order sharing* 0, 1 *CM. If $\delta(0;f) > \frac{1}{2}$ then either $f \equiv g$ or $fg \equiv 1$.*

H. Ueda [6, 7] extended *Theorem* A to meromorphic functions and removed the order restriction on f and g. Later on H.X. Yi [8, 9] further worked on the theorem of Ozawa and improved the same by relaxing the restriction on the deficiency. Now it appears that the problem of relaxing the nature of sharing of values in *Theorem* A is of interest to investigate. However some attempt has been made in [3, 4] but there relaxation of the nature of sharing of only one value (i.e. the value 0) was possible. In the present paper we further investigate this problem and make a twofold improvement of *Theorem* A: firstly by relaxing the nature of sharing of all the values and secondly by relaxing the restriction on deficiency.

To this end we explain the idea of weighted sharing as introduced in [4].

Definition 4 [4] Let k be a nonnegative integer or infinity. For $a \in C \cup \{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$.

If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f - a$ with multiplicity m $(\leq k)$ if and only if z_0 is a zero of $g - a$ with multiplicity m $(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity m $(> k)$ if and only if z_0 is a zero of $g - a$ with multiplicity n $(> k)$ where m is not necessarily equal to n.

We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Now we state the main result of the paper.

Theorem 1 *If f, g share $(0,1)$, $(1,1)$, $(\infty,1)$ and*

$$2\delta_2(0;f) + 2\delta_2(\infty;f) + \min\left\{ \sum_{a\neq 0,1,\infty} \delta_2(a;f),\ \sum_{a\neq 0,1,\infty} \delta_2(a;g) \right\} > 3$$

then either $f \equiv g$ or $fg \equiv 1$. If f has at least one zero or pole, the case $fg \equiv 1$ does not occur.

Considering $f = \exp(z) - 1$ and $g = 2 - 2/\exp(z)$ one can easily verify that *Theorem* 1 is sharp. However, the author does not know whether in *Theorem* 1 the weight of sharing of values can be reduced from unity to zero and so it remains an open problem.

2. Lemmas

In this section we discuss some lemmas which will be needed in the sequel.

Lemma 1 [1] *If* f, g *share* $(0,0)$, $(1,0)$, $(\infty,0)$ *then outside a set of finite linear measure*

$$\limsup_{r\longrightarrow\infty}\frac{T(r,f)}{T(r,g)} \le 3 \quad \textit{and} \quad \limsup_{r\longrightarrow\infty}\frac{T(r,g)}{T(r,f)} \le 3.$$

Lemma 2 *Let* f, g *share* $(1,0)$ *and* $H = \frac{f'}{f} - \frac{g'}{g}$. *If* $\overline{N}(r,1;f) \ne S(r,f)$ *and* $H \equiv 0$ *then* $f \equiv g$.

Proof. Since $H \equiv 0$, it follows that $f \equiv cg$, where c is a constant. Since f, g share $(1,0)$ and $\overline{N}(r,1;f) \ne S(r,f)$, there exists $z_o \in C$ such that $f(z_o) = g(z_o) = 1$ so that $c = 1$. Therefore $f \equiv g$. This proves the lemma. □

Henceforth we shall denote by h a meromorphic function defined by

$$h = \left(\frac{f''}{f'} - \frac{2f'}{f-1}\right) - \left(\frac{g''}{g'} - \frac{2g'}{g-1}\right).$$

Lemma 3 *If* f, g *share* $(1,1)$ *and* $h \not\equiv 0$ *then*

$$N(r,1;f \mid = 1) = N(r,1;g \mid = 1) \le N(r,h) + S(r,f) + S(r,g).$$

Proof. Since f, g share $(1,1)$, it follows that a simple 1-point of f is a simple 1-point of g and conversely. If z_o is a simple 1-point of f and g, then by a simple calculation we see that in some neighbourhood of z_o

$$h(z) = (z - z_o)\psi(z),$$

where ψ is analytic at z_o.

Hence

$$\begin{aligned} N(r,1;f \mid = 1) &\le N(r,0;h) \\ &\le T(r,h) + O(1) \end{aligned}$$

$$= N(r,h) + m(r,h) + O(1)$$
$$= N(r,h) + S(r,f) + S(r,g),$$

by the first fundamental theorem and Milloux theorem {p.55 [2]}.

Since $N(r,1;f\mid=1) = N(r,1;g\mid=1)$, the lemma is proved. □

Lemma 4 *Let f, g share $(0,1)$, $(1,1)$, $(\infty,1)$ and $a_1, a_2, \ldots, a_n$ be pairwise distinct complex numbers such that $a_i \neq 0, 1, \infty$ $(i = 1, 2, \ldots, n)$. Then*

$$\begin{aligned} N(r,h) \leq\ & \overline{N}(r,0;f\mid\geq 2) + \overline{N}(r,\infty;f\mid\geq 2) + \overline{N}(r,1;f\mid\geq 2) \\ & + \sum_{i=1}^{n} \overline{N}(r,a_i;f\mid\geq 2) + \sum_{i=1}^{n} \overline{N}(r,a_i;g\mid\geq 2) \\ & + \overline{N}_o(r,0;f') + \overline{N}_o(r,0;g'), \end{aligned}$$

where $\overline{N}_o(r,0;f')$ is the reduced counting function of the zeros of f' which are not the zeros of $f(f-1)\prod_{i=1}^{n}(f-a_i)$.

The proof is omitted.

Lemma 5 *If f, g share $(0,1)$, $(1,1)$, $(\infty,1)$ and $f \not\equiv g$ then*

(i) $\overline{N}(r,1;f\mid\geq 2) \leq \overline{N}(r,0;f\mid\geq 2) + \overline{N}(r,\infty;f\mid\geq 2) + S(r,f)$,

(ii) $\overline{N}(r,1;g\mid\geq 2) \leq \overline{N}(r,0;f\mid\geq 2) + \overline{N}(r,\infty;f\mid\geq 2) + S(r,f)$.

Proof. Since $\overline{N}(r,1;f) = \overline{N}(r,1;g)$, the lemma is obvious if $\overline{N}(r,1;f) = S(r,f)$. So we suppose that $\overline{N}(r,1;f) \neq S(r,f)$. Let

$$H = \frac{f'}{f} - \frac{g'}{g}.$$

Since $f \not\equiv g$, it follows from *Lemma* 2 that $H \not\equiv 0$. Also since f, g share $(1,1)$, a multiple 1-point of f is a multiple 1-point of g and vice-versa and so it is a zero of H. Hence

$$\begin{aligned} \overline{N}(r,1;f\mid\geq 2) &\leq N(r,0;H) \\ &\leq N(r,H) + m(r,H) + O(1) \\ &= N(r,H) + S(r,f), \end{aligned} \tag{1}$$

by Milloux theorem {p.55 [2]} and *Lemma* 1.

The possible poles of H occur at the zeros and poles of f, g. Clearly if z_o is a zero or a pole of f and g with the same multiplicity then z_o is not a pole of H. Since all the poles of H are simple and f, g share $(0,1)$, $(\infty,1)$, it follows that

$$N(r,H)=\overline{N}(r,H)\leq \overline{N}(r,0;f\mid\geq 2)+\overline{N}(r,\infty;f\mid\geq 2). \tag{2}$$

Now (i) follows from (1) and (2). Also (ii) follows from (i) because f, g share $(1,1)$ so that $\overline{N}(r,1;f\mid\geq 2)=\overline{N}(r,1;g\mid\geq 2)$. This proves the lemma. □

Lemma 6 *Let $a_1,a_2,\ldots,a_n$ be pairwise distinct complex numbers such that $a_i\neq 0,1,\infty$ $(i=1,2,\ldots,n)$ and $\overline{N}_o(r,0;f')$ be defined as in Lemma 4. Then*

$$\begin{aligned}&\overline{N}_o(r,0;f')+\sum_{i=1}^{n}\overline{N}(r,a_i;f\mid\geq 2)+\overline{N}(r,1;f\mid\geq 2)\\&\quad\leq \overline{N}(r,f)+\overline{N}(r,0;f)+S(r,f).\end{aligned}$$

Proof. From the definition of $\overline{N}_o(r,0;f')$ we see that

$$\begin{aligned}&\overline{N}_o(r,0;f')+\sum_{i=1}^{n}\overline{N}(r,a_i;f\mid\geq 2)+N(r,0;f)-\overline{N}(r,0;f)\\&\quad+\overline{N}(r,1;f\mid\geq 2)\\&\quad\leq N(r,0;f')\leq N\left(r,0;\frac{f'}{f}\right)+N(r,0;f)-\overline{N}(r,0;f)\\&\quad\leq N\left(r,\frac{f'}{f}\right)+N(r,0;f)-\overline{N}(r,0;f)+S(r,f)\\&\quad\leq \overline{N}(r,0;f)+\overline{N}(r,f)+N(r,0;f)-\overline{N}(r,0;f)+S(r,f).\end{aligned}$$

i.e.

$$\begin{aligned}&\overline{N}_o(r,0;f')+\sum_{i=1}^{n}\overline{N}(r,a_i;f\mid\geq 2)+\overline{N}(r,1;f\mid\geq 2)\\&\quad\leq \overline{N}(r,0;f)+\overline{N}(r,f)+S(r,f).\end{aligned}$$

This proves the lemma. □

3. Proof of Theorem 1

Let $f\not\equiv g$. We shall show that $fg\equiv 1$.

First we suppose that $h\not\equiv 0$. Let $a_1,a_2,\ldots,a_n$ be pairwise distinct complex numbers such that $a_i \neq 0, 1, \infty$ for $i = 1, 2, \ldots, n$. By the second fundamental theorem we get

$$(n+1)T(r,f) \leq \overline{N}(r,1;f) + \overline{N}(r,0;f) + \overline{N}(r,f) + \sum_{i=1}^{n} \overline{N}(r,a_i;f) - N_o(r,0;f') + S(r,f), \tag{3}$$

where $N_o(r,0;f')$ is the counting function of those zeros of f' which are not the zeros of $f(f-1)y\prod_{i=1}^{n}(f-a_i)$.

Now by *Lemma* 1, *Lemma* 3, *Lemma* 4, *Lemma* 5 and *Lemma* 6 we obtain because f, g share $(0,1)$, $(1,1)$, $(\infty,1)$

$$\begin{aligned}\overline{N}(r,1;f) &= N(r,1;f \mid = 1) + \overline{N}(r,1;f \mid \geq 2)\\ &\leq N(r,h) + \overline{N}(r,1;f \mid \geq 2) + S(r,f)\\ &\leq \overline{N}(r,0;f \mid \geq 2) + \overline{N}(r,\infty;f \mid \geq 2) + \overline{N}(r,1;f \mid \geq 2)\\ &\quad + \sum_{i=1}^{n} \overline{N}(r,a_i;f \mid \geq 2) + \sum_{i=1}^{n} \overline{N}(r,a_i;g \mid \geq 2) + \overline{N}_o(r,0;f')\\ &\quad + \overline{N}_o(r,0;g') + \overline{N}(r,1;f \mid \geq 2) + S(r,f)\\ &\leq 2\overline{N}(r,0;f \mid \geq 2) + 2\overline{N}(r,\infty;f \mid \geq 2) + \sum_{i=1}^{n} \overline{N}(r,a_i;f \mid \geq 2)\\ &\quad + \overline{N}(r,0;g) + \overline{N}(r,g) + \overline{N}_o(r,0;f') + S(r,f). \end{aligned} \tag{4}$$

From (3) and (4) we get

$$(n+1)T(r,f) \leq 2N_2(r,0;f) + 2N_2(r,\infty;f) + \sum_{i=1}^{n} N_2(r,a_i;f) + S(r,f). \tag{5}$$

Similarly we can prove that

$$(n+1)T(r,g) \leq 2N_2(r,0;f) + 2N_2(r,\infty;f) + \sum_{i=1}^{n} N_2(r,a_i;g) + S(r,f). \tag{6}$$

Combining (5) and (6) we get

$$(n+1)T(r) \leq 2N_2(r,0;f) + 2N_2(r,\infty;f) + \max\left\{\sum_{i=1}^{n} N_2(r,a_i;f), \sum_{i=1}^{n} N_2(r,a_i;g)\right\} + S(r,f), \tag{7}$$

where $T(r) = \max\{T(r,f), T(r,g)\}$.

Let $S = \{a : a \in \mathcal{C},\ a \neq 0,1,\infty \text{ and } \delta_2(a;f) + \delta_2(a;g) > 0\}$. Since S is countable, we suppose that $S = \{a_i : i \in N_+\}$, where N_+ is a set of positive integers.

Let $\sum_{a\neq 0,1,\infty} \delta_2(a;f) < \sum_{a\neq 0,1,\infty} \delta_2(a;g)$. Then there exists a positive integer n_o such that

$$\sum_{i=1}^{n_o} \delta_2(a_i;f) \leq \sum_{i=1}^{n_o} \delta_2(a_i;g) \quad \text{and} \quad \sum_{i=1}^{n_o} \delta_2(a_i;f) > \sum_{a\neq 0,1,\infty} \delta_2(a;f) - \varepsilon,$$

where ε (> 0) is given.

Then from (7) we get

$$n_o + 1 < 4 + n_o - 2\delta_2(0;f) - 2\delta_2(\infty;f) - \sum_{a\neq 0,1,\infty} \delta_2(a;f) + \varepsilon$$

i.e.

$$2\delta_2(0;f) + 2\delta_2(\infty;f) + \sum_{a\neq 0,1,\infty} \delta_2(a;f) < 3 + \varepsilon.$$

Since ε (> 0) is arbitrary, it follows that

$$2\delta_2(0;f) + 2\delta_2(\infty;f) + \sum_{a\neq 0,1,\infty} \delta_2(a;f) \leq 3. \tag{8}$$

If $\sum_{a\neq 0,1,\infty} \delta_2(a;g) < \sum_{a\neq 0,1,\infty} \delta_2(a;f)$, similarly we can prove that

$$2\delta_2(0;f) + 2\delta_2(\infty;f) + \sum_{a\neq 0,1,\infty} \delta_2(a;g) \leq 3. \tag{9}$$

If $\sum_{a\neq 0,1,\infty} \delta(a;f) = \sum_{a\neq 0,1,\infty} \delta(a;g)$ then from (5) we obtain (8).

Now (8) and (9) contradict the given condition. Therefore $h \equiv 0$ and so

$$f \equiv \frac{Ag+B}{Cg+D}, \tag{10}$$

where A, B, C, D are constants and $AD - BC \neq 0$.

Let $C = 0$. Then from (10) we get

$$f \equiv ag + b, \tag{11}$$

where $a = \frac{A}{D}$, $b = \frac{B}{D}$ and $AD \neq 0$.

Let 0, ∞ be Picard's exceptional values (e.v.P.) of f and so of g. Then from (11) we see that b is also an e.v.P. of f which is impossible unless $b = 0$. So from (11) we get $f \equiv ag$, Since $f \not\equiv g$, it follows that $a \neq 1$ and so 1 becomes an e.v.P. of f because f, g share $(1, 1)$. This is again impossible.

Let ∞ be an e.v.P. of f and g but 0 be not an e.v.P. of f and g. Since f, g share $(0, 1)$, from (11) we get $b = 0$ and so $f \equiv ag$. Since $f \not\equiv g$, $a \neq 1$ and so 1 becomes an e.v.P. of f and g. Hence $\sum_{t\neq 1,\infty} \delta_2(t; f) = 0$. This contradicts the given condition.

Let 0 be an e.v.P. of f and g but ∞ be not. If 1 is an e.v.P. of f then $\sum_{t\neq 0,1} \delta_2(t; f) = 0$, which contradicts the given condition. Hence there exists $z_o \in \mathcal{C}$ such that $f(z_o) = g(z_o) = 1$ and so from (11) we get $a + b = 1$ and so

$$f \equiv ag + 1 - a. \tag{12}$$

Since f, g share $(0, 1)$ and 0 is an e.v.P. of f and g, it follows from (12) that $1 - a$ is an e.v.P. of f and $(a - 1)/a$ is an e.v.P. of g. Since $f \not\equiv g$, from (12) we see that $a \neq 1$ and it follows that $\delta_2(\infty; f) = 0$ and $\sum_{t\neq 0,1,\infty} \delta_2(t; g) = 1$, which contradicts the given condition.

Let 0, ∞ be not e.v.P. of f and so of g. Then from (11) we get $f \equiv ag$ because f, g share $(0, 1)$. Since $f \not\equiv g$, it follows that $a \neq 1$ and so 1 becomes an e.v.P. of f and g because f, g share $(1, 1)$. Hence by the deficiency relation we get

$$\delta_2(0; f) + \delta_2(\infty; f) + \sum_{t\neq 0,1,\infty} \delta_2(t; f) \leq 1$$

and so

$$2\delta_2(0; f) + 2\delta_2(\infty; f) + \sum_{t\neq 0,1,\infty} \delta_2(t; f)$$
$$\leq 1 + \delta_2(0; f) + \delta_2(\infty; f) \leq 2,$$

a contradiction to the given condition.

Let $C \neq 0$. From (10) we get

$$f - \frac{A}{C} \equiv \frac{B - \frac{AD}{C}}{Cg + D}. \tag{13}$$

Since f, g share $(\infty, 1)$, it follows from (13) that $\frac{A}{C}$, ∞ are e.v.P. of f and $-\frac{D}{C}$, ∞ are e.v.P. of g.

Let $A = 0$. Then from (13) we get

$$f \equiv \frac{1}{\alpha g + \beta}, \tag{14}$$

where $\alpha = \frac{C}{B}$, $\beta = \frac{D}{B}$ and $B \neq 0$.

Since 0, ∞ are e.v.P. of f and f, g share $(1, 1)$, it follows that there exists $z_o \in \mathcal{C}$ such that $f(z_o) = g(z_o) = 1$. So from (14) we get $\alpha + \beta = 1$ and hence

$$f \equiv \frac{1}{\alpha g + 1 - \alpha}. \tag{15}$$

Since f, g share $(0, 1)$, $(\infty, 1)$ and 0, ∞ are e.v.P. of f, it follows from (15) that 0, ∞, $\frac{\alpha - 1}{\alpha}$ are e.v.P. of g, which is impossible unless $\alpha = 1$. Hence from (15) we get $fg \equiv 1$.

Let $A \neq 0$. Since $\frac{A}{C}$, ∞ are e.v.P. of f, it follows that $\delta_2(0; f) = 0$ and $\sum_{t \neq 0,1,\infty} \delta_2(t; f) \leq 1$, which contradicts the given condition. This proves the theorem. □

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