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On isomorphy of pure hulls of purifiable torsion-free subgroups

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Abstract. A subgroup A of an arbitrary abelian group G is said to be *purifiable* in G if there exists a pure subgroup H of G containing A which is minimal among the pure subgroups of G that contain A. Such a subgroup H is said to be a *pure hull* of A in G. In general, not all pure hulls of purifiable subgroups of arbitrary abelian groups are isomorphic. We show that if A is a purifiable torsion-free subgroup of an arbitrary abelian group, then all pure hulls of A are isomorphic and for all pure hulls H of A, the quotients H/A are isomorphic.

Key words: purifiable subgroup, pure hull, almost-dense subgroup.

All groups considered are arbitrary abelian groups. The terminologies and notations not expressly introduced here follow the usage of [6]. Throughout this note, p denotes a prime integer, G_p the p-primary subgroup and T the maximal torsion subgroup of the abelian group G.

Definition 1 A subgroup A of an abelian group G is said to be *purifiable* in G if, among the pure subgroups of G containing A, there exists a minimal one. Such a minimal pure subgroup is called a *pure hull* of A.

Hill and Megibben [8] established properties of pure hulls of p-groups and characterized the p-groups for which all subgroups are purifiable.

Later, Benabdallah and Irwin [2] introduced the concept of almost-dense subgroups of p-groups and used it to characterize pure hulls of purifiable subgroups of p-groups.

Furthermore, Benabdallah and Okuyama [3] introduced new invariants, the so-called *n-th overhangs* of a subgroup of a p-group, which are related to the n-th relative Ulm-Kaplansky invariants. Using them, they obtained a necessary condition for subgroups of p-groups to be purifiable.

Benabdallah, Charles, and Mader [1] introduced the concept of maximal vertical subgroups supported by a given subsocle of a p-group and

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characterized the p-groups for which the necessary condition given in [3] is also sufficient. Other results about purifiable subgroups of p-groups are contained in [4], [5], [9], and [10].

Recently, in [13], we extended the concept of almost-dense subgroup from p-groups to arbitrary abelian groups and began to study purifiable subgroups of arbitrary abelian groups. We determined the groups for which all subgroups are purifiable and characterized purifiable torsion-free rank-one subgroups of an arbitrary abelian group. The characterization of purifiable subgroups in arbitrary abelian groups is an open question even if the subgroup is torsion-free.

In this note, we consider the isomorphy of pure hulls of purifiable torsion-free subgroups of arbitrary abelian groups. In a p-group G, it is well known that if G is a direct sum of cyclic groups, then every subsocle is purifiable and all pure hulls of a subsocle are isomorphic. However, in torsion-complete groups, every subsocle is purifiable, but not all pure hulls of a subsocle are isomorphic (see [7], [6, 66, Exercise 8]). In [11, Corollary 4.4], we proved that if a subsocle S of a torsion-complete group G is closed, then S is purifiable and all pure hulls of S are isomorphic. Moreover, we proved in [3, Theorem 3.2] that if a subgroup A of a p-group G is purifiable, then all residual subgroups determined by G and A are isomorphic. In [11, Theorem 3.1], we showed that if some pure hull of a purifiable subgroup A of a p-group G is a direct summand of G, then all pure hulls of A are isomorphic.

As for isomorphy of pure hulls of torsion-free purifiable subgroups of arbitrary abelian groups, we proved in [14, Theorem 3.4] that if a torsion-free rank-one subgroup N of an arbitrary abelian group is purifiable, then all pure hulls of N are isomorphic. Moreover, in [14, Theorem 4.1], we showed that if a $T(X)$-high subgroup N of an arbitrary abelian group X is purifiable, then all pure hulls of N are isomorphic.

Our goal in this note is to prove that if A is a purifiable torsion-free subgroup of an arbitrary abelian group G, then all pure hulls of A are isomorphic and for all pure hulls H of A the quotients H/A is isomorphic.

First, we recall crucial definitions and properties mentioned in [13].

Definition 2 Let G be an abelian group. A subgroup A is said to be *p-almost-dense* in G if, for every p-pure subgroup K of G containing A, the torsion part of G/K is p-divisible. Moreover, A is said to be *almost-dense* in G if A is p-almost-dense in G for every prime p.

Proposition 3 [13, Proposition 1.4] *For an abelian group G and a subgroup A of G, the following properties are equivalent:*
(1) *A is almost-dense in G;*
(2) *for all integers $n \geqq 0$ and all primes p, $A + p^{n+1}G \supseteq p^n G[p]$;*

Proposition 4 [13, Theorem 1.11] *Let G be an abelian group and A a subgroup of G. Suppose that A is purifiable in G. Then a pure subgroup H of G containing A is a pure hull of A in G if and only if the following three conditions are satisfied:*
(1) *A is almost-dense in H;*
(2) *H/A is torsion;*
(3) *for every prime p, there exists a nonnegative integer m_p such that*

$$p^{m_p} H[p] \subseteq A.$$

Let G, A, and H be as in Proposition 4. If A is torsion-free, then H_p is bounded for every prime p.

Proposition 5 [13, Theorem 4.1 (2)] *Let G be a p-group and A a subgroup of G. If $A \cap p^m G$ is p-purifiable in $p^m G$ for some $m \geqq 0$, then A is purifiable in G.*

Note that if G is an abelian group and A is a subgroup of G, then $A+T$ is purifiable in G and has a unique pure hull M given by the stipulation that M/T is the pure hull of $(A+T)/T$ in the torsion-free group G/T.

Furthermore, the subgroup M has the following property.

Proposition 6 *Let G be an abelian group, A a subgroup of G, and M/T the pure hull of $(A+T)/T$ in G/T. Then*

$$M/A = T(G/A).$$

Proof. By Proposition 4, $M/(A+T)$ is torsion. Hence $M/A \subseteq T(G/A)$. Let $g + A \in T(G/A)$ with $g \in G$. Then there exists an integer m such that $mg \in A \subset M$. Since M is pure in G, there exists $x \in M$ such that $mg = mx$. Hence $g - x \in T \subset M$. □

Definition 7 Let G be an abelian group and A a subgroup of G. For every nonnegative integer n, we define the *n-th p-overhang* of A in G to be the verctor space

$$V_{p,n}(G,A)=\frac{(A+p^{n+1}G)\cap p^nG[p]}{(A\cap p^nG)[p]+p^{n+1}G[p]}.$$

It is convenient to use the following notations for the numerator and the denominator of $V_{p,n}(G,A)$:

$$A_G^n(p)=(A+p^{n+1}G)\cap p^nG[p]=((A\cap p^nG)+p^{n+1}G)[p]$$

and

$$A_n^G(p)=(A\cap p^nG)[p]+p^{n+1}G[p].$$

If A is p-almost-dense in G, then $A+p^{n+1}G\supseteq p^nG[p]$, so $A_G^n(p)=p^nG[p]$. If A is torsion-free, then $A_n^G(p)=p^{n+1}G[p]$. Thus, if A is torsion-free and p-almost-dense in G, then

$$V_{p,n}(G,A)=\frac{p^nG[p]}{p^{n+1}G[p]},$$

the nth Ulm-Kaplansky invariant of G_p.

Proposition 8 [13, Proposition 2.2] *Let G be an abelian group and A a subgroup of G. For every p-pure subgroup K of G containing A,*

$$V_{p,n}(G,A)\cong V_{p,n}(K,A)$$

for all $n\geqq 0$.

Let G and A be as in Proposition 8. Suppose that A is torsion-free purifiable in G and let H and K be pure hulls of A in G. By the comment after Proposition 4, H_p and K_p are bounded for every prime p. By Proposition 8 and the comment after Definition 7, for every prime p, $H_p\cong K_p$.

Proposition 9 [13, Theorem 2.3] *Let G be an abelian group and A a subgroup of G. If A is purifiable in G, then, for every prime p, there exists a nonnegative integer m_p such that $V_{p,n}(G,A)=0$ for all $n\geqq m_p$.*

Let G and A be as in Proposition 9. If A is purifiable in G, then, by [13, Proposition 2.4], for every pure hull H of A in G and every prime p, the least integer m_p such that $V_{p,n}(G,A)=0$ for all $n\geqq m_p$ is equal to the least one such that $p^{m_p}H[p] \subseteq A$ in Proposirion 4 (3). Hence, if A is torsion-free purifiable in G, then, for every pure hull H of G and every prime p, the least integer m_p such that $V_{p,n}(G, A) = 0$ for all $n \geqq m_p$ is equal to the least one such that $p^{m_p}H_p = 0$.

Before proving the main theorem, we give a useful lemma.

Lemma 10 *Let H be a pure subgroup of an abelian group G containing some T-high subgroup of G. If, for each prime p, U_p is a subgroup of G such that $G_p = H_p \oplus U_p$, then $G = H \oplus U$ where $U = \oplus_p U_p$.*

Proof. Let $ng \in H \oplus U$ with $g \in G$ and $n \in \mathbf{Z}$. Then we have $mng \in H$ for some integer m. Since H is pure in G, there exists $h \in H$ such that $mng = mnh$. Then $g - h \in T \subset H \oplus U$ and so $H \oplus U$ is pure in G. Since $H \oplus U$ is essential in G, $G = H \oplus U$. □

Theorem *Let G be an arbitrary abelian group and A a torsion-free subgroup of G. Suppose that A is purifiable in G. Then all pure hulls of A are isomorphic and for all pure hulls H of A, the groups H/A are isomorphic.*

Proof. By the comment before Proposition 6, $A + T$ is purifiable in G and has a unique pure hull M of G. Let H be any pure hull of A in G. By Proposition 4, H/A is torsion. By Proposition 6, all pure hulls of A are included in M.

Note that A is purifiable in M and H is a pure hull of A in M. Fix a prime p. After the comment of Proposition 9, there exists the least integer m_p such that $A_M^n(p) = A_n^M(p)$ for all $n \geqq m_p$.

For integer $n \geqq 0$, let $p^n g + A \in p^n(M/A)[p]$. Since $p^{n+1}g \in H \cap p^{n+1}M = p^{n+1}H$, there exists $h \in H$ such that $p^{n+1}g = p^{n+1}h$. Since $p^n g - p^n h \in p^n M[p]$, we have $p^n(M/A)[p] = p^n(H/A)[p] + (p^n M[p] + A)/A$. Let $x \in A_M^n(p)$. Then we can write $x = a + p^{n+1}g'$ for some $a \in A$ and $g' \in M$. Since $x + A \in p^{n+1}(M/A)[p] = p^{n+1}(H/A)[p] + (p^{n+1}M[p] + A)/A$, there exist $a' \in A$, $h' \in H$, and $p^{n+1}g_0 \in p^{n+1}M[p]$ such that $x = a + p^{n+1}g' = a' + p^{n+1}h' + p^{n+1}g_0$. Since $h_p(a) \geqq n$, $h_p(a') \geqq n$. Hence $A_M^n(p) = A_H^n(p) + A_n^M(p)$. By Proposition 4 (1), A is almost-dense in H. By the comment after Proposition 7, $A_H^n(p) = p^n H[p]$ and $A_n^M(p) = p^{n+1}M[p]$. Hence, for all $n \geqq 0$, there exist subsocles S_n and H_n of M such that

$$\begin{aligned} p^n M[p] &= A_M^n(p) \oplus S_n = (p^n H[p] + p^{n+1}M[p]) \oplus S_n \\ &= H_n \oplus p^{n+1}M[p] \oplus S_n. \end{aligned}$$

By the comment after Proposition 9, the integer m_p is the least one such that $p^{m_p}H[p] = 0$. Inductively, we have

$$\begin{aligned} M[p] &= H_0 \oplus pM[p] \oplus S_0 \\ &= H_0 \oplus H_1 \oplus p^2M[p] \oplus S_0 \oplus S_1 \\ &= \cdots \\ &= \left(\bigoplus_{i=1}^{m_p-1} H_i\right) \oplus p^{m_p}M[p] \oplus \left(\bigoplus_{i=1}^{m_p-1} S_i\right) \\ &= H[p] \oplus p^{m_p}M[p] \oplus S^{(p)} \end{aligned}$$

where $S^{(p)} = \left(\bigoplus_{i=1}^{m_p-1} S_i\right)$.

If K is another pure hull of A in G, then we have similarly

$$M[p] = K[p] \oplus p^{m_p}M[p] \oplus S^{(p)}.$$

Hence, for every prime p, there exist a nonnegative integer m_p and a subsocle $S^{(p)}$ of M such that

$$M[p] = H[p] \oplus p^{m_p}M[p] \oplus S^{(p)} = K[p] \oplus p^{m_p}M[p] \oplus S^{(p)}.$$

Note that $(S_p \oplus p^{m_p}M[p]) \cap p^{m_p}M_p = (S_p \cap p^{m_p}M_p) \oplus p^{m_p}M[p] = p^{m_p}M[p]$. By Proposition 5, $(S_p \oplus p^{m_p}M[p])$ is purifiable in M_p. Then there exists a pure hull L_p of $(S_p \oplus p^{m_p}M[p])$ in M_p. It is immediate that $M_p \supseteq H_p \oplus L_p$ for every prime p. Now we prove that $M_p = H_p \oplus L_p$ for every prime p.

Let $h \in H[p]$ and $x \in L_p[p]$. We prove that $h_p(h+x) = \min\{h_p(h), h_p(x)\}$. We can write $x = p^{m_p}t + s$, where $p^{m_p}t \in p^{m_p}M$ and $s \in S^{(p)}$. Then $h_p(h+x) = h_p(h+s)$ unless $h = s = 0$. If $h = s = 0$, then we have $h_p(h+x) = \min\{h_p(h), h_p(x)\}$. Suppose that $n = h_p(x) = h_p(s)$. Then $n \leqq m_p - 1$. Without loss of generality, we may assume that $h \in H_n$ and $s \in S_n$. By the definitions of H_n and S_n, we have $h_p(h+x) = n$. Hence, by [9, Lemma 4], $M_p = H_p \oplus L_p$. Similarly, $M_p = K_p \oplus L_p$. Let $T_1 = \oplus_p L_p$. Note that all $T(H)$-high subgroups are $T(M)$-high subgroups. By Lemma 10, we have

$$M = H \oplus T_1 = K \oplus T_1.$$

Hence $H \cong K$. Furthermore, since $M/A = H/A \oplus (T_1 \oplus A)/A = K/A \oplus (T_1 \oplus A)/A$, $H/A \cong K/A$. □

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